STIELTJES CLASSES FOR KUMMER TYPE PROBABILITY DISTRIBUTIONS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of **Master of Science in Mathematics Department, Atılım University**.

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ABSTRACT

STIELTJES CLASSES FOR KUMMER TYPE PROBABILITY DISTRIBUTIONS

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The moment problem is one of the classical directions in Probability Theory, which studies whether or not a probability distribution is uniquely determined by its moments. The problem originated in XIX century and is still drawing attention of researches both in mathematics and applied disciplines. During the last decades, the subject of finding families of different probability distributions with the same moment sequences has gained a popularity and a large number of papers in this area has been published. Special classes of such families, called the Stieltjes classes, have become an area of intensive research.

In this thesis, after background information on the transform methods, a review of both classical and present-day results on the moment problems is presented. The review includes a general description of the moment problem, a list of checkable criteria for the moment (in)determinacy, and some methods to construct Stieltjes classes for probability densities. All notions and results are illustrated by examples. In addition, recently introduced power Lindley distribution has been studied and new Stieltjes classes for the power Lindley density has been constructed.

Keywords: Moment problem, Stieltjes classes, power Lindley distribution, probability generating function, moment generating function, characteristic function

ÖZ

KUMMER TİPİ OLASILIK DAĞILIMLARI İÇİN STIELTJES SINIFLARI

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Bir olasılık dağılımının momentleri yardımıyla tek olarak elde edilip edilemeyeciğini konu alan moment problemi Olasılık Teorisinin klasik problemlerinden biridir. Bu problem ilk olarak XIX. yüzyılda ele alınmış ve günümüzde de matematik ve uygulama alanlarındaki araştırmacılar tarafından yoğun bir şekilde çalışılmaktadır. Son yıllarda aynı moment dizisine sahip farklı olasılık dağılımları ailelerini bulmak popülarite kazanmış ve bu alanda çok sayıda makale yayımlanmıştır. Bu ailelerin özel sınıfı olan Stieljes sınıfı yoğun bir çalışma alanıdır.

Bu tezde dönüşüm metodları konusunda arka plan bilgisinden sonra, moment problemi hakkında hem klasik hem de güncel sonuçlar sunulmuştur. İnceleme şunları içermektedir: moment probleminin genel açıklaması, moment belirlilik/belirsizlik durumları için kontrol edilebilir kriterler listesi ve olasılık yoğunlukları için bazı Stieltjes sınıfları oluşturma yöntemleri. Bütün kavramlar ve sonuçlar örneklerle gösterilmiştir. Ayrıca, son zamanlarda tanıtılan kuvvet Lindley dağılımı çalışılmış ve kuvvet Lindley yoğunluğu için yeni Stieltjes sınıfları oluşturulmuştur.

Anahtar Kelimeler: Moment problemi, Stieltjes sınıfları, Lindley kuvvet dağılımı,

olasılık üreteç fonksiyonu, moment üreteç fonksiyonu, karakteristik fonksiyonu

To my country Libya

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LIST OF SYMBOLS

$\mathbb{N}$	:	the set of natural numbers
$\mathbb{N}_0$	:	the set of non-negative integers
$\mathbb{R}$	:	the set of real numbers
μ	:	mean value of a random variable
σ^2	:	variance of a random variable
σ	:	standard deviation of a random variable
$E[X]$	:	mathematical expectation of random variable X
$\mathrm{Var}\,(X)$	:	variance of random variable X
$X \sim P$	:	random variable X has distribution P
$\mathcal{N}(\mu, \sigma^2)$	:	normal (Gaussian) distribution
$\exp\,(\lambda)$	:	exponential distribution
$PL(\alpha, \theta)$	:	power Lindley distribution

CHAPTER 1

INRODUCTION

The problem of moments is one of the classical problems in mathematics. It originated in the works of Stieltjes (1856-1894), although some ideas in this direction can be traced to earlier studies attributed to P. L. Chebyshev (1821-1894). In his study of continued fractions [34], Stieltjes discovered that there exist functions of bounded variation such that $f(x) \not\equiv 0$ and at the same time

$$\int_0^\infty x^k f(x)\, dx = 0 \quad \text{for all} \quad k \in \mathbb{N}_0.$$

It has to be pointed out that simultaneously Stieltjes expanded the notion of Riemann integral and developed the concept of the integral called nowadays the Stieltjes or Riemann-Stieltjes integral (see, for example [32, Chapter 6]). Also, he noticed that, for the integral over bounded interval, the condition

$$\int_a^b x^k f(x)\, dx = 0 \quad \text{for all} \quad k \in \mathbb{N}_0$$

implies that $f(x) \equiv 0$. This phenomenon turns out to be crucial in the moment problem. More details on the history of the moment problem can be found in [20]. Further developments of the moment problem were achieved by a great number of researchers, including such outstanding scientists as N. I. Akhiezer, H. Hamburger, T. Carleman and M. G. Krein. At present, the interest to this subject has not faded and many significant results have been obtained during the last decades. Important contributions have been made by C. Berg, J. S. Christiansen and S. Khrushev. See, for example, [24]. The remarkable feature of the moment problem is its deep connections with various branches of mathematics, mainly with Probability Theory, Measure Theory, Real and Complex Analysis. Moreover, the role of moment indeterminate heavy-tailed probability distributions in applications to other disciplines, such as mathematical finance, software engineering, the quantum calculus and applied chemistry has increased. We refer to [12, 21, 36] just to provide a few papers, while this list can be easily extended. Generally speaking, the problem of moments can be split into two main parts: the existence of a measure with given sequence of moments and the uniqueness of a measure possessing the prescribed moments. In this thesis, we deal with the uniqueness of a solution in the class of probability measures. To be specific, the following question has been considered: given a random variable X with the probability distribution P_X, whose moment of order k is defined by

$$m_k = E[X^k], \quad k \in \mathbb{N}_0,$$

does there exist another random variable say, Y with $P_X \neq P_Y$ such that $E[X^k] = [Y^k]$ for all $k \in \mathbb{N}_0$? In other words, it is asked whether there exists a probability distribution P_Y different from P_X with the same sequence of moments $\{m_k\}_{k=0}^{\infty}$. Certainly, the problem is well-defined under the assumption that all of the mathematical expectations exist. In the case when $E[X^k] = E[Y^k]$ for all $k \in \mathbb{N}_0$ implies $P_X = P_Y$, the distribution P_X is said to be moment determinate. Otherwise, the distribution is moment indeterminate. Examples of moment determinate and moment indeterminate distributions along with the detailed explanations are given in Section 3.1. Clearly, if P_X and P_Y have the same sequences of moments $\{m_k\}_{k=0}^{\infty}$, then so does every probability distribution of the form $\alpha P_X + \beta P_Y$, where $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. Therefore, if a probability distribution P is moment indeterminate, then there exist infinitely many different distributions with same moment sequences as P. In fact, this result has a far-reaching generalization expressed by the following theorem: If a distribution P is moment indeterminate, then there are infinitely many probability distributions of each type an absolutely continuous, discrete and singular distributions having the same moment sequence as P. See [6, 7]. This result despite being of immense theoretical value does not indicate any practical way of finding different distributions with identical sequences of moments in the event of moment indeterminacy. To summarize, one can indicate the next steps generally performed with regard to the moment problem:

1. Classify a probability distribution P as moment determinate or moment indeterminate.

2. If it is moment indeterminate, find other distributions with the same sequences

of moments.

Both of these issues have been addressed in this thesis. More precisely, in Chapter 3, a number of so-called "checkable" conditions is discussed and their applications are demonstrated by examples dealing both with classical and recently emerged probability distributions. From the latter class, power Lindley distribution is selected and is described comprehensively in Chapter 4. This distribution was introduced by Ghitany et al. in 2013 as a generalization of the Lindley distribution, see [15]. The same authors have investigated some properties of this distribution, calculated its numerical characteristics, and demonstrated possible applications. In this thesis, the moment (in)determinacy has been identified for various values of parameters. The final part of the thesis is dedicated to Stieltjes classes. These are explicitly written families of probability densities or probability mass functions all possessing the same moment sequences. The Stieltjes classes de facto occurred in the works of Stieltjes [34] and similar ideas can be traced in the studies of Chebyshev, Markov, and Heyde [9, 19, 20, 33]. However, the name "Stieltjes class" was proposed by J. Stoyanov [37] who started their systematic investigation in 2004. Consequently, this subject can be viewed as quite recent and now it attracts a lot of interest. New Stieltjes classes are constantly coming out and new approaches to their construction are being proposed. See, for example, [24, 26, 29, 35, 39]. In this thesis, new Stieltjes classes for the power Lindley distribution in the case of moment indeterminacy are presented.

The contents of the thesis is structured according to the following breakdown: Chapter 2 presents an overview of the transform methods commonly used in Probability Theory. The transform methods form the main analytic tool to study probability distributions and, as such, they can be considered as a bridge between Probability Theory and Mathematical Analysis. Specifically, the definitions and properties of probability generating functions, moment generating functions, and characteristic functions with illustrative examples are given.

In Chapter 3, the necessary background on the moment problem is provided. It includes a general description of the problem and its two mostly studied special cases, namely the Hamburger and Stieltjes moment problems, examples of moment determinate and moment indeterminate distributions and a summary of some particular -

also known as “checkable” - conditions for the moment (in)determinacy, along with illustrations of their applications to both classical and newly emerged probability distributions.

Chapter 4 brings out a review of recent results related to Lindley distribution and its generalizations. Although this distribution was introduced by D. V. Lindley [23] in 1958, its newly found applications (see, for example, [13, 14, 15]) brought it to a spotlight during the last years. In this thesis, apart from the Lindley distribution itself, the attention is paid to one of its generalizations, called the power Lindley distribution. After presenting a survey of several publications on the subject, some new outcomes have been derived and they are exposed in Sections 4.1.2 and 4.2.3. It is shown that Lindley and power Lindley distributions are representatives of the family of p-Kummer distributions [27], and, for this reason, they are called Kummer-type distributions.

Finally, Chapter 5 deals with the main subject of this work, namely, Stieltjes classes. After a brief introduction and examples, two specific methods to construct Stieltjes classes are presented. Their usage is demonstrated by a few examples. In addition, they are used to find new Stieltjes classes for the power Lindley distribution.

CHAPTER 2

TRANSFORM METHODS

In this chapter, we describe several transform methods that are used as analytic tools to study probability distributions. We begin with probability generating function. Later, we study the concepts of moment generating function and the characteristic function. This chapter is written as a review of known results that can be found in many sources. We refer to [17, 25, 31, 38, 40].

2.1 Preliminaries

Let us recollect some notions and terminology that will be used in the sequel. Let X be a random variable considered on an underlying probability space $(\Omega, \mathcal{F}, P)$. The (probability) distribution of $X : \Omega \to \mathbb{R}$ is defined on Borel sets $E \subset \mathbb{R}$ as follows:

$$P_X(E) = P\{\omega \in \Omega : X(\omega) \in E\}.$$

Every probability distribution can be described by its distribution function.

Definition 2.1.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable. The distribution function of X is given by*

$$F_X(x) = P_X((-\infty, x]), \quad x \in \mathbb{R}.$$

A random variable X is said to be *discrete* if there exists a sequence $A = \{a_n\}_{n=1}^{\infty}$ such that $P_X(A) = 1$. In this case, the distribution of X can be described by the *probability mass function* of X defined by

$$P_X(x) = P(X = x), \quad x \in \mathbb{R}.$$

A random variable is said to be *absolutely continuous* if, for every Borel set E,

$$P_X(E) = \int_E f_X(t)dt$$

for some integrable function f_X called a *probability density* of X.

Notice that in the discrete case

$$F_X(x) = \sum_{x_i \leq x} p_X(x_i)$$

and in the absolutely continuous case

$$F_X(x) = \int_{-\infty}^{x} f_X(t)dt.$$

A random variable is said to have a *singular* distribution if

- $P(X = x) = 0$ for every $x \in \mathbb{R}$;
- there exists a set $A \subset \mathbb{R}$ with measure$(A) = 0$ such that $P(X \in A) = 1$.

By the Lebesque Decomposition Theorem, for any random variable X, there holds:

$$P_X = \alpha P_d + \beta P_{ac} + \gamma P_s$$

where $\alpha, \beta, \gamma \geq 0$, $\alpha + \beta + \gamma = 1$, while P_d, P_{ac} and P_s stand for a discrete, absolutely continuous, and a singular distribution. Such a representation is unique. In this thesis, only discrete and absolutely continuous distributions are handled.

Definition 2.1.2 *The k-th moment* ($k \in \mathbb{N}_0$) *of a random variable X, as denoted by* $E[X^k]$, *is defined by:*

$$E[X^k] = \int_{-\infty}^{\infty} x^k dF_X(x), \tag{2.1}$$

provided the Stieltjes integral converges.

2.2 Probability Generating Function (PGF)

In this section, an overview of the probability generating functions will be presented. It is important to mention that this transform method is applicable only to nonnegative integer-valued random variables. We start with the following definition.

Definition 2.2.1 *Let us have a nonnegative integer-valued random variable X whose probability mass function is given by:*

$$P_X(k) = p_k,\ k \in \mathbb{N}_0.$$

The probability generating function (PGF) is defined by:

$$G_X(t) = \sum_{k=0}^{\infty} t^k p_k, \quad t \in \mathbb{R}.$$

The series in the RHS absolutely converges for $t \in [-1, 1]$. However, in some cases, it can converge in a wider interval.

Example 2.2.2 *Let X have a binomial distribution with parameters n and p. Then*

$$G_X(t) = \sum_{k=0}^{\infty} t^k p_k = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} t^k = \sum_{k=0}^{n} \binom{n}{k} (pt)^k q^{n-k} = (pt+q)^n,$$

where $q = 1 - p$.

Example 2.2.3 *Let X have a geometric distribution with parameter p,* $0 < p < 1$. *Then*

$$G_X(t) = \sum_{k=1}^{\infty} pq^{k-1} t^k = pt \sum_{k=1}^{\infty} (qt)^{k-1} = pt \sum_{m=0}^{\infty} (qt)^m = \frac{pt}{1-qt},$$

where $-1/q < t < 1/q$ *and* $q = 1 - p$.

Example 2.2.4 *Let X have a Poisson distribution with parameter* $\lambda > 0$. *Then*

$$G_X(t) = \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} t^k = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} = e^{-\lambda} e^{\lambda t} = e^{\lambda(t-1)}, \quad -\infty < t < \infty.$$

The following theorem, corollary and proofs are provided in [17, p. 84]

Theorem 2.2.5 *Let X and Y be independent nonnegative integer-valued random variables. Then*

$$G_{X+Y}(t) = G_X(t) G_Y(t) \ \ for\ all\ \ t \in [-1, 1].$$

We can generalize this as follows:

Corollary 2.2.6 *Let $X_1, \ldots, X_n$ be independent nonnegative integer-valued random variables and $Z = X_1 + \cdots + X_n$. Then*

$$G_Z(t) = \prod_{i=1}^{n} G_{X_i}(t).$$

2.2.1 Properties of Probability Generating Function

Theorem 2.2.7 *Let X be a random variable whose probability generating function is $G_X(t)$. Then $G_X^{(k)}(1) = E[X(X-1)\cdots(X-k+1)]$.*

Proof. Clearly, $G_X(1) = \sum_{k=0}^{\infty} p_k = 1$. Next,

$$G'_X(t) = \frac{d}{dt}E[t^X] = E[Xt^{X-1}].$$

By setting $t = 1$, we have

$$G'_X(1) = E[X].$$

Likewise,

$$G_X^{(k)}(1) = \frac{d}{dt}E[t^X] = E[X(X-1)\cdots(X-k+1)t^{X-k}].$$

Plugging $t = 1$ yields the result. □

Observe that except for the first moment $E[X]$, the probability generating function does not generate pure moments as in (2.1). That is, $G_X^{(k)}(1) \neq E[X^k]$. Instead, it generates factorial moments.

Example 2.2.8 *Consider the probability generating function of a binomial distribution that we derived in Example 2.2.2:*

$$G_X(t) = (pt+q)^n.$$

By taking the first derivative we have

$$G'_X(t) = n(pt+q)^{n-1}p, \quad G'_X(1) = n(p+q)^{n-1}p = np,$$

which is the mathematical expectation of the binomial distribution.

2.3 Moment Generating Function (MGF)

In this section, we address the second transform method, which is called moment generating function (MGF). If the moment generating function exists for a given random variable, we can use it to obtain the moments for that random variable.

Definition 2.3.1 *Let X be a random variable, whose distribution function is $F_X(x)$. Then, the moment generating function $M_X(t)$ for X is defined as the following:*

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} dF_X(x),$$

provided the integral is convergent.

If the integral diverges for all $t \neq 0$, it is said that MGF does not exist. Notice that $M_X(t)$ represents the mathematical expectation of e^{tX}. Consequently, we can rewrite the definition as:

$$M_X(t) = E[e^{tx}].$$

Remark 2.3.2 *The existence of moment generating function for a random variable X implies that all moments of the random variable X also exist. That is, if $M_X(t)$ exists, then $E(X^k) < \infty$ for all k. The converse is not true, see Example 3.2.3.*

2.3.1 Special Cases of Moment Generating Function

2.3.1.1 Discrete Case

If X is a discrete random variable with probability mass function $P_X(x_k) = p_k$, $k \in \mathbb{N}_0$, then

$$M_X(t) = \sum_{k=0}^{\infty} e^{tx_k} p_k.$$

Example 2.3.3 *Let X be a random variable having a binomial distribution with parameters n and p. The moment generating function of X is*

$$M_X(t) = \sum_{k=0}^{n} e^{tk} \binom{n}{k} p^k q^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (pe^t)^k q^{n-k} = (pe^t + q)^n,$$

where $q = 1 - p$.

Example 2.3.4 *Let X have a geometric distribution with parameter p. The moment generating function of X is*

$$M_X(t) = \sum_{k=1}^{\infty} e^{tk} pq^{k-1} = pe^t \sum_{k=1}^{\infty} (e^t q)^{k-1} = \frac{pe^t}{(1 - qe^t)},$$

where $q = 1 - p$ and $t < \ln(1/q)$.

2.3.1.2 Absolutely Continuous Case

Let X have an absolutely continuous distribution with a density function $f_X(x)$. Then,

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\, dx$$

Example 2.3.5 *Let X be a random variable having an exponential distribution with parameter λ whose density is defined by:*

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

Then, by the definition of moment generating function we have

$$M_X(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx = \lambda \int_0^{\infty} e^{(t-\lambda)x}\, dx = \frac{\lambda}{t-\lambda} e^{(t-\lambda)x}\Big|_0^{\infty} = \frac{\lambda}{\lambda - t}, \quad t < \lambda.$$

2.3.2 Properties of Moment Generating Function

Below, we list some properties of moment generating function [31, p. 104]:

- MGF always exists at $t = 0$ and it equals 1 at this point.
- Let X and Y be independent random variables and $Z = X + Y$. Then,

  $$M_Z(t) = M_X(t) M_Y(t).$$

  In general, if $X_1, X_2, \ldots, X_n$ are independent random variables such that $Z = X_1 + X_2 + \cdots + X_n$, then,

  $$M_Z(t) = \prod_{i=1}^{n} M_{X_i}(t).$$

- Let X be a random variable with moment generating function $M_X(t)$. Then, if $Y = aX + b$, where a and b are constants, we have

$$M_Y(t) = e^{bt} M_X(at).$$

- **Uniqueness Theorem.** Let X and Y be two random variables, whose MGF's are $M_X(t)$ and $M_Y(t)$, respectively. If $M_X(t) = M_Y(t)$ for all $t \in (-a, a)$, then $F_X(x) = F_Y(x)$ for all $x \in \mathbb{R}$.

  In other words, let two random variables have the same MGF existing in some interval $(-a, a)$, $a > 0$. Then, these random variables have the same distribution functions. The inverse is also true. Therefore, if we could find MGF of a random variable, we have indeed determined its distribution.

2.3.3 Moments from Moment Generating Function

It is not difficult to notice that the existence of $M_X(t)$ for $-a < t < a$ where $a > 0$, implies that $M_X(t)$ is infinitely differentiable on the interval. In particular,

$$\begin{aligned} M'_X(t) &= \int_{-\infty}^{\infty} x e^{tx} dF_X(x), \\ M''_X(t) &= \int_{-\infty}^{\infty} x^2 e^{tx} dF_X(x), \end{aligned}$$

and, in general,

$$M_X^{(k)}(t) = \int_{-\infty}^{\infty} x^k e^{tx} dF_x(x), \quad k \in \mathbb{N}.$$

When we substitute $t = 0$ in the previous formulae, we obtain the following:

$$\begin{aligned} M'_X(0) &= \int_{-\infty}^{\infty} x dF_X(x) = E(X) \\ M''_X(0) &= \int_{-\infty}^{\infty} x^2 dF_X(x) = E(X^2) \\ &\vdots \\ M_X^{(k)}(0) &= \int_{-\infty}^{\infty} x^k dF_X(x) = E(X^k) \end{aligned}$$

We conclude that we can obtain the k-th moment of the random variable X, by evaluating the k-th derivative at $t = 0$.

Example 2.3.6 *Let* $M_X(t) = \frac{1}{(1-2t)^{10}}$ *be the MGF of a random variable X. Then,* $M'_X(t) = 20(1-2t)^{-11}$, $M''_X(t) = 440(1-2t)^{-12}$ *and* $M'''_X(t) = 10560(1-2t)^{-13}$. *By evaluating the last derivative at* $t = 0$, *we have*

$$E[X^3] = M'''_X(0) = 10560.$$

Example 2.3.7 *Let X have a Cauchy distribution whose density is defined by*

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty.$$

Then,

$$M_X(t) = \int_{-\infty}^{\infty} \frac{e^{tx}}{\pi(1+x^2)}\, dx.$$

The above integral is divergent for all $t \neq 0$. *Thus, Cauchy distribution does not have MGF, which shows that there are random variables for which the moment generating function does not exist on any real interval, only at* $t = 0$.

2.4 Characteristic Function (ChF)

In this section, we consider another transform method, which is called characteristic function. Unlike the previous methods as in Sections 2.2 and 2.3, a characteristic function exists for any random variable.

Definition 2.4.1 *Let X be a random variable. The characteristic function of X is defined by:*

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{itx} dF_X(x) = E[e^{itX}], \; -\infty < t < \infty$$

where i is the imaginary unit.

If X is a discrete random variable with probability mass function $P_X(x_k)$, then, its characteristic function is defined as the following:

$$\Phi_X(t) = \sum_{k=0}^{\infty} e^{itx_k} P_X(x_k).$$

If X is an absolutely continuous random variable with probability density function $f_X(x)$, then,

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx.$$

Example 2.4.2 *Let X be a discrete random variable with probability mass function*

$$P_X(x_k) = \begin{cases} \frac{1}{3}, & if \;\; k = 0 \\ \frac{1}{3}, & if \;\; k = 1 \\ \frac{1}{3}, & if \;\; k = 2 \\ 0, & otherwise \end{cases}$$

The characteristic function is

$$\Phi_X(t) = \sum_{k=0}^{2} e^{itx_k} P_X(x_k) = \frac{1}{3}[1 + e^{it} + e^{2it}].$$

Example 2.4.3 *Let X be a random variable exponentially distributed with parameter λ. Then,*

$$\Phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f_X(x)\, dx = \int_0^{\infty} e^{itx} e^{\lambda x}\, dx = - \left. \frac{e^{-(\lambda - it)x}}{\lambda - it} \right|_0^{\infty} = \frac{\lambda}{\lambda - it}.$$

2.4.1 Properties of Characteristic Function

Here, some basic properties of ChF are stated.

1. $\Phi_X(0) = 1$.
2. $|\Phi_X(t)| \leq 1$ for all $t \in \mathbb{R}$.
3. If X and Y are two independent random variables with characteristic functions $\Phi_X(t)$ and $\Phi_Y(t)$ respectively, and, $Z = X + Y$, then $\Phi_Z(t) = \Phi_X(t)\Phi_Y(t)$.
4. If X is a random variable with characteristic function $\Phi_X(t)$ and $Y = aX + b$, where a and b are constants, then $\Phi_Y(t) = e^{itb}\Phi_X(at)$.
5. **Uniqueness Theorem.** Let X and Y be two random variables such that $\Phi_X(t) = \Phi_Y(t)$ for all $t \in \mathbb{R}$. Then, X and Y have the same distributions.
6. If the moment generating function $M_X(t)$ of a random variable X exists on $(-a, a)$ where $a > 0$, and the characteristic function $\Phi_X(t)$ of that random variable is analytic on the same interval, then $\Phi_X(t) = M_X(it)$.

2.4.2 Moments From Characteristic Function

The following theorem is provided in [25, p. 111]

Theorem 2.4.4 *Let X be a random variable whose characteristic function is k times differentiable at $t = 0$, and $E(X^k) < \infty$. Then,*

$$E[X] = -i\frac{d}{dt}\Phi_X(t)\Big|_{t=0}.$$

More generally,

$$E[X^k] = (-i)^k\frac{d^k}{dt^k}\Phi_X(t)\Big|_{t=0}.$$

Therefore, the characteristic function provides a convenient tool to determine the moment of a random variable.

Example 2.4.5 *Consider the characteristic function of an exponential distribution: $\Phi_X(t) = \frac{\lambda}{\lambda - it}$. Taking the first derivative with respect to t, we get*

$$\frac{d}{dt}\Phi_X(t) = \frac{i\lambda}{(\lambda - it)^2}.$$

By evaluating this derivative at $t = 0$, we find the first moment of X:

$$E[X] = -i\frac{d}{dt}\Phi_X(t)\Big|_{t=0} = \frac{\lambda}{(\lambda - it)^2}\Big|_{t=0} = \frac{1}{\lambda}.$$

In this case, it is easy to obtain the k-th derivative of the characteristic function as:

$$\frac{d^k}{dt^k}\Phi_X(t) = \frac{i^k k!\lambda}{(\lambda - it)^{k+1}},$$

and the k-th moment as:

$$E[X^k] = (-i)^k\frac{d^k}{dt^k}\Phi_X(t)\Big|_{t=0} = \frac{k!\lambda}{(\lambda - it)^{k+1}}\Big|_{t=0} = \frac{k!}{\lambda^k}.$$

Remark 2.4.6 *It is possible to show that when k is even, the existence of $\frac{d^k}{dt^k}\Phi_X(t)\Big|_{t=0}$ implies the finiteness of the k-th moment $E[X^k]$. However, when k is odd, it possible that the k-th moment does not exist while $\frac{d^k}{dt^k}\Phi_X(t)\Big|_{t=0}$ exists. See [38, Examples 8.7 and 8.8].*

CHAPTER 3

THE MOMENT PROBLEM

In this chapter, we discuss the moment problem for probability distributions. Historically, as stated in Chapter 1, the classical moment problem first appeared in the work [34] of Stieltjes. Since then, many mathematicians have been investigating this topic. Let us begin with describing the moment problem.

3.1 Description and Examples

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \to \mathbb{R}$ be a random variable. By Definition 2.1.2 the moment of order k of X is given by:

$$m_k = E[X^k], \quad k \in \mathbb{N}_0.$$

While $m_0 = 1$ for all random variables, the other moments may or may not exist. In the case when moments of all orders exist, they form the moment sequence $\{m_k\}_{k=0}^{\infty}$ of random variable X, or, equivalently, of the corresponding probability distribution P_X. For probability distributions possessing moments of all orders, the next classical problem has gained great importance starting from the end of XIX century which can be stated as follows: given a sequence of moments $\{m_k\}_{k=0}^{\infty}$ of X, is P_X the only probability distribution with this moment sequence? If yes, the distribution P_X is said to be *moment determinate (M-determinate)*. Otherwise, P_X is said to be *moment indeterminate (M-indeterminate)*. This means that P_X is not unique in terms of the moments, in which case there must be at least one more probability distribution P_Y such that $P_X \neq P_Y$, while $E[X^k] = E[Y^k]$ for all $k \in \mathbb{N}_0$. In other words, the random variables X and Y have different distributions and coinciding moment sequences. In fact - as it was proved [6, 7] - if a distribution P is M-indeterminate, then there are infinitely many distributions of each type having the same moment sequences. It goes without saying that any distribution with finite moments of all orders is either moment determinate or moment indeterminate. Although the moment problem was introduced by Stieltjes in connection with his studies of continued fractions [34], two decades prior to this time, P. L. Chebyshev had handled this problem from a different perspective. For example, in one of his unpublished notes he stated that

$$\int_0^\infty x^k e^{-\sqrt[4]{x}} \sin(\sqrt[4]{x})\, dx = 0 \quad \text{for all} \quad k \in \mathbb{N}_0, \tag{3.1}$$

which shows that the probability distribution with the density $f(x) = \frac{1}{24} e^{-\sqrt[4]{x}}$, $x > 0$ is moment indeterminate, see Example 3.1.2. The moment problem appears to be interrelated with many branches of mathematics, and the pertinent research in this direction is still going on. A great number of results on the problem of moments is contained in fundamental works [1, 33]. Information on more recent developments is presented in [22, 24].

Let us provide several examples of distributions with different moment (in)determinacy properties.

Example 3.1.1 *If the moment generating function of X exists in some interval $(-a, a)$, $a > 0$, or equivalently, if the characteristic function of X is analytic in $|z| < a$, then P_X is M-determinate. This fact is known as Cramer's condition formulated in Theorem 3.2.1. This demonstrates immediately that such important distributions as normal, exponential, Poisson and binomial are M-determinate.*

Next, the example attributed to P. L. Chebyshev, which has been mentioned above is presented. Possibly, this is the first available example of a moment indeterminate distribution. The notes containing this example (prior to Stieltjes' works) had not been published at that time. See [9, page 172].

Example 3.1.2 *Let $X \geq 0$ be an absolutely continuous random variable whose density function is defined as follows:*

$$f(x) = \frac{1}{24} e^{-\sqrt[4]{x}}, \ x > 0.$$

Assume that we have another absolutely continuous random variable, say, $Y \geq 0$ with density function

$$g(x) = f(x)\,[1 + \sin(\sqrt[4]{x})], \; x > 0.$$

Observe that $P_X \neq P_Y$. As we will see, the two random variables X and Y have exactly the same sequence of moments $\{m_k\}$. That is, $E[X^k] = E[Y^k]$ for all $k \in \mathbb{N}_0$. To justify this, it suffices to show the validity of (3.1)*. The substitution $t = x^4$ followed by Euler's formula $e^{it} = \cos t + i \sin t$ yields:*

$$I_k := \int_0^\infty x^k e^{-\sqrt[4]{x}} \sin(\sqrt[4]{x})\,dx = 4\int_0^\infty t^{4k+3} e^{-t} \sin t\,dt = \operatorname{Im}\left(4\int_0^\infty t^{4k+3} e^{-t} e^{it} dt\right).$$

Making another substitution $z = (1-i)t$, one arrives at

$$I_k = \operatorname{Im}\left(\frac{4}{(1-i)^{4k+4}}\int_0^\infty z^{4k+3} e^{-z} dz\right) = \operatorname{Im}\left(\frac{-(4k+3)!}{(-4)^k}\right) = 0.$$

We conclude, therefore, that the moment problem is moment indeterminate because X and Y have the same sequence of moments, though they have different distributions $P_X \neq P_Y$.

Now, moment indeterminate discrete distributions will be presented. This will be achieved by means of the following key lemma.

Lemma 3.1.3 *[28, Theorem 2.1] Let X be an integer-valued random variable such that*

$$p\{X = j\} = p_j \quad j = 0, \pm 1, \pm 2, \cdots.$$

Assume that for some $q \in (0,1)$, there holds

$$p_j \geq C q^{j(j-1)/2} \quad \text{for all } j \geq 0. \tag{3.2}$$

Then, for all $a \in (0, q]$, the random variable $Y = a^{-X}$ is moment indeterminate provided that all moments of Y are finite.

Proof. The moments of Y are given by

$$m_k = \sum_{j=-\infty}^{\infty} a^{-jk} p_j, \quad k \in \mathbb{N}_0.$$

Let $\varphi(z) = \prod_{j=0}^{\infty}(1 - a^j z)$. By Euler's identity [8]:

$$\varphi(z) = \sum_{j=0}^{\infty}(-1)^j \frac{a^{j(j-1)/2}}{(a;a)_j} z^j \tag{3.3}$$

where $(a;q)_j$ is the q-shifted factorial defined by

$$(a;q)_0 := 1, \quad (a;q)_j = \prod_{s=0}^{j-1}(1 - q^s a), \quad j \in \mathbb{N}_0.$$

Obviously, $\varphi(a^{-k}) = 0$ for all $k \in \mathbb{N}_0$, that is,

$$\sum_{j=0}^{\infty}(-1)^j \frac{a^{j(j-1)/2}}{(a;a)_j} a^{-kj} = 0, \quad k \in \mathbb{N}_0. \tag{3.4}$$

Set

$$\tilde{h}_j = \begin{cases} 0 & if \;\; j < 0 \\ (-1)^j \dfrac{a^{j(j-1)/2}}{(a;a)_j p_j} & if \;\; j \geq 0. \end{cases}$$

By virtue of (3.2), as $a \leq q < 1$,

$$|\tilde{h}_j| \leq \frac{a^{j(j-1)/2}}{C(a;a)_j \, q^{j(j-1)/2}} \leq C_1.$$

Denoting $M_h := \sup |\tilde{h}_j|$, we set $h_j = \tilde{h}_j / M_h$. Clearly, $|h_j| \leq 1$ and, by (3.4),

$$\sum_{j=-\infty}^{\infty} a^{-kj} p_j h_j = 0, \quad k \in \mathbb{N}_0.$$

This means that

a) $\tilde{p}_j := p_j[1 + h_j] \geq 0$ for all $j \in \mathbb{Z}$;

b) $\sum_{j=-\infty}^{\infty} \tilde{p}_j = 1$;

c) $\sum_{j=-\infty}^{\infty} a^{-jk} \tilde{p}_j = m_k$ for all $k \in \mathbb{N}_0$.

The conditions a) and b) say that $\tilde{p}_j$ defines a probability mass function $\tilde{p}(a^{-j}) = \tilde{p}_j$. Since the sequence $\{h_j\}_{j=-\infty}^{\infty}$ is not identically zero, the probability mass functions $p(a^{-j})$ and $\tilde{p}(a^{-j})$ are different. Therefore, one obtains different distributions P and $\tilde{P}$ with the same moment sequence. □

Example 3.1.4 *Let X have Poisson distribution with parameter $\lambda > 0$. Then for any $\mu > 1$, the distribution of the random variable $Y = \mu^X$ is moment indeterminate. Indeed, by virtue of Lemma 3.1.3, it suffices to show that $\lambda^j / j! \geq C\mu^{-j(j-1)/2}$. Applying Stirling's formula*

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2n\pi}, \quad n \to \infty \tag{3.5}$$

one may write

$$\frac{\lambda^j}{j!} \sim C \exp\left\{ j \ln(e\lambda) - j \ln j - \frac{1}{2} \ln j \right\} \geq C_1 \exp\left\{ -\frac{j^2 \ln \mu}{2} + \frac{j}{2} \ln \mu \right\} = C_1 \mu^{-j(j-1)/2}$$

for j large enough.

Example 3.1.5 *Let X be a random variable such that*

$$p\{X = j\} = p_j = Ce^{-j^2}, \quad j = 0, \pm 1, \pm 2, \cdots$$

and $Y = e^{\alpha X}$, $\alpha > 2$. Notice that the case $\alpha = 8$ is considered in [38, 11.8]. The condition (3.2) is obviously satisfied since, when $\alpha > 2$, there is C_1 for which the inequality

$$e^{-j^2} \geq C_1 e^{-\alpha j(j-1)/2}$$

is valid for all $j \in \mathbb{Z}$. Applying the Lemma 3.1.3, one concludes that the distribution of Y is moment indeterminate.

Traditionally, the following two versions of the moment problem have been mostly studied: the Hamburger moment problem and the Stieltjes moment problem. In the first case, a random variable X can attain values on the whole number line, whereas the latter case deals only with non-negative random variables. More specifically, in the Hamburger moment problem, given a random variable X with the moment sequence $\{m_k\}$, one searches whether there exists a random variable Y with $P_X \neq P_Y$ so that $\{m_k\}$ is the moment sequence for Y as well. In distinction, the Stieltjes moment problem starts with a non-negative random variable X, having distribution P_X with support in $[0, \infty)$, and searches whether there exists a non-negative random variable Y with the same moment sequence such that $P_X \neq P_Y$. Obviously, if X is M-indeterminate in the Stieltjes sense, then it is also M-indeterminate in the sense of the Hamburger moment problem, and also if P_X is Hamburger M-determinate, then, it is also M-determinate in the Stieltjes sense. In this connection, the following remarkable facts have to be mentioned. See [10].

(i) There exists $X \geq 0$ with moment sequence $\{m_k\}$ such that:

   (a) whenever Y is non-negative and has the same moment sequence $\{m_k\}$, then $P_X = P_Y$.

   (b) there is Y which is not non-negative, and has the same moment sequence $\{m_k\}$ along with $P_X \neq P_Y$.

(ii) The situation described in (b) may occur only if X possess a discrete distribution with $P\{X = 0\} \neq 0$.

It has to be emphasized that, despite intensive researches conducted on the moment problem for many decades, there is no unique "checkable" necessary and sufficient condition on the M-(in)determinacy. Instead, there is a great number of necessary conditions and a great number of sufficient conditions, which are based on various techniques such as moment growth estimates, convergence/divergence of pertinent series/integrals, tail estimates of the distribution function, etc. In the next section, some of those conditions will be provided together with illustrative examples.

3.2 Checkable Conditions for M-(in)determinacy

In this section, we give some well-known conditions that determine whether a certain distribution is moment determinate or indeterminate. There are many conditions, which occurred at different time and in various forms. Here, we represent the most commonly and widely used ones. We call these conditions "checkable", as they allow to check moment determinacy by evaluating certain asymptotic relations. It is important to mention that all the presented conditions are sufficient but not necessary for moment (in)determinacy. These conditions can be found in many sources. See, for example, [24, 38].

3.2.1 Cramer's Condition

Cramer's condition is considered as the most efficient condition for moment determinacy.

Theorem 3.2.1 (Cramer's Condition) *Let X be a random variable with distribution P_X. If its moment generating function exists on $(-a, a)$, $a > 0$, or equivalently, the characteristic function of X is analytic for $|t| < a$, then P_X is moment determinate.*

In the case when the moment generating does not exist, Cramer's condition does not indicate whether P_X is moment indeterminate. However, it is a marker to suspect the indeterminacy of P_X. Cramer's condition implies immediately that all probability distributions with bounded support are M-determinate.

Example 3.2.2 *Let X have a Nakagami distribution with density function*

$$f(x) = 2x\,e^{-x^2}, \quad x > 0.$$

Then, its moment generating function is:

$$E[e^{tX}] = \int_0^{\infty} 2x\,e^{tx-x^2}\,dx, \tag{3.6}$$

The integral in (3.6) *is convergent for all t. Therefore, Cramer's condition holds for this distribution, and we conclude that Nakagami distribution is moment determinate.*

Example 3.2.3 *Assume that we have a random variable X having power Nakagami distribution with density function:*

$$f(x) = \alpha x^{\alpha-1} e^{-x^{\alpha}}, \quad x, \alpha > 0.$$

Then the moment generating function of X is:

$$E[e^{tX}] = \alpha \int_0^{\infty} x^{\alpha-1} e^{tx-x^{\alpha}}\,dx. \tag{3.7}$$

Clearly, if $\alpha \geq 1$, then the moment generating function exists. Therefore, power Nakagami distribution with $\alpha \geq 1$ is moment determinate. When $\alpha < 1$, the integral in (3.7) *diverges for $t > 0$, so Carmer's condition fails to show the determinacy of this distribution. Later, in Example 3.2.12, it will be shown that it is moment indeterminate only for $\alpha < \frac{1}{2}$.*

3.2.2 Carleman's Condition

Theorem 3.2.4 (Carleman's Condition) *Let $\{m_k\}$ be moment sequence of the probability distribution P_X. Carleman's series, S, is defined as follows:*

$$S = \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{1}{(m_{2k})^{\frac{1}{2k}}} & \text{Hamburger case,} \\ \displaystyle\sum_{k=1}^{\infty} \frac{1}{(m_k)^{\frac{1}{2k}}} & \text{Stieltjes case.} \end{cases} \tag{3.8}$$

If the first series in (3.8) *is divergent, then P_X is uniquely determined by* $\{m_k\}$; *likewise, if the second series in* (3.8) *is divergent, then P_X is moment determinate in the Stieljes sense.*

Example 3.2.5 *Let X be a random variable having an exponential distribution with parameter λ. By Example 2.4.5 the k-th moment of X is*

$$m_k = \frac{k!}{\lambda^k}.$$

Obviously, $m_k \leq (k/\lambda)^k$ and $(m_k)^{-1/2k} \geq \sqrt{\lambda/k}$. Since $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ is divergent, it is clear that $\sum_{k=1}^{\infty}(m_k)^{-1/2k}$ is also divergent. Therefore, Carleman's condition holds and the exponential distribution is moment determinate.

Carleman's condition is also considered as one of the strongest checkable conditions for the uniqueness of a probability distribution in terms of its moment sequence. However, there exist moment determinate distributions such that the respective series in (3.8) converges. This is demonstrated by the next example. Before providing examples let us recall the following result (see [19]): Let X be a random variable whose distribution function is continuous at the origin and let $\{m_k\}$ be its moment sequence. If $\{m_k\}$ uniquely determines P_X in the Stieltjes sense, then it also determines P_X uniquely in the Hamburger sense.

Example 3.2.6 *[38, 11.10] Assume that X is a random variable with density function*

$$f(x) = \frac{\beta}{\Gamma(1/\beta)} \exp\left(-x^{\beta}\right), \quad x > 0,\ 0 < \beta < 1.$$

Then, the k-th moment of X is

$$m_k = \frac{\beta}{\Gamma(1/\beta)} \int_0^{\infty} x^k e^{-x^{\beta}} dx = \frac{\Gamma((k+1)/\beta)}{\Gamma(1/\beta)}, \quad k \in \mathbb{N}.$$

Therefore, $(m_k)^{\frac{1}{k}} \sim Ck^{\frac{1}{\beta}}$ where C is a constant. By applying Carleman's condition for Stieltjes case, we notice that the series $\sum_{k=1}^{\infty}(m_k)^{\frac{-1}{2k}}$ is divergent for $\frac{1}{2} \leq \beta < 1$, which implies that X is moment determinate for $\frac{1}{2} \leq \beta < 1$. Further, observe that the distribution of X has no discontinuity at the origin. It follows from the result mentioned above that X is also moment determinate in the Hamburger sense. However, when we apply Carleman's condition corresponding to Hamburger case to $\{m_k\}$, we find that the series $\sum_{k=1}^{\infty}(m_{2k})^{\frac{-1}{2k}}$ is convergent, which shows the fact that Carleman's condition is not necessary for a certain distribution to be determined by its sequence of moments.

3.2.3 Rate of Growth of Moments

Let X be a random variable with distribution P_X and moment sequence $\{m_k\}$. In the following statements, we provide some other checkable conditions for moment determinacy.

i) For the Hamburger moment problem, any of the conditions below is sufficient for distribution to be moment determinate.

- $\dfrac{m_{2(k+1)}}{m_{2k}} = O(k^2) \quad \text{as} \quad k \to \infty.$
- $m_{2k} \leq C^k (2k)!, \;\; k \in \mathbb{N}_0 \quad \text{for some constant } C > 0.$

ii) For the Stieltjes moment problem, any of the conditions below is sufficient for distribution to be moment determinate.

- $\dfrac{m_{k+1}}{m_k} = O(k^2) \quad \text{as} \quad k \to \infty.$
- $m_k = O(k^{2k}) \quad \text{as} \quad k \to \infty.$
- $m_k \leq C^k (2k)!, \;\; k \in \mathbb{N}_0 \quad \text{for some constant } C > 0.$

Example 3.2.7 *Let X have 2-stage Erlang distribution with density function*

$$f(x) = \alpha^2 x\, e^{-\alpha x}, \quad x \geq 0.$$

Its sequence of moments is:

$$m_k = E[X^k] = \frac{(k+1)!}{\alpha^k}.$$

Clearly,

$$m_k = \frac{(k+1)!}{\alpha^k} \leq \frac{(2k)!}{\alpha^k} = C^k (2k)! \quad \textit{for} \quad C = \frac{1}{\alpha}.$$

Therefore, 2-stage Erlang distribution is moment determinate.

Example 3.2.8 *Consider a random variable X with gamma distribution whose density function is*

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1}\, e^{-\beta x}, \quad x > 0, \beta, \alpha > 0.$$

The k-th moment of X can be found as

$$m_k = E[X^k] = \int_0^\infty \frac{\beta^\alpha}{\Gamma(\alpha)}\, x^{k+\alpha-1} e^{-\beta x}\, dx.$$

Upon substituting $y = \beta x$, one obtains

$$m_k = \frac{1}{\Gamma(\alpha)\beta^k} \int_0^\infty y^{k+\alpha-1}\, e^{-y} dy = \frac{\Gamma(k+\alpha)}{\Gamma(\alpha)\beta^k}.$$

Since

$$\frac{m_{k+1}}{m_k} = \frac{k+\alpha}{\beta} = O(k^2) \quad as \quad k \to \infty,$$

the condition ii) is satisfied, and hence gamma distribution is moment determinate.

3.2.4 Krein's Condition

Theorem 3.2.9 (Krein's Condition) *Let X be an absolutely continuous random variable whose distribution is P_X with density function $f(x)$. Krein's integral, K, is defined as follows:*

$$K = \begin{cases} \int_{-\infty}^{\infty} \frac{-\ln f(x)}{1+x^2}\, dx, & \text{Hamburger case,} \\ \int_a^{\infty} \frac{-\ln f(x^2)}{1+x^2}\, dx, & \text{Stieltjes case.} \end{cases} \tag{3.9}$$

Here a is some positive constant. If the integral in (3.9) is convergent, then P_X is moment indeterminate. Notice that there are some distributions for which K is divergent and they are moment indeterminate. For example, See [38, 11.11].

Corollary 3.2.10 *Let X be an absolutely continuous random variable whose density function is of the form:*

$$f(x) = C\, h(x)\, e^{-x^\delta}, \quad x > 0. \tag{3.10}$$

If $\delta < \frac{1}{2}$ and $\ln\{1/h(x^2)\} = O(x^{2\delta})$ as $x \to \infty$, then X is moment indeterminate.

Proof. Using the direct comparison test, one obtains

$$\int_a^\infty \frac{-\ln h(x^2)}{1+x^2}\,dx \le C_1 \int_a^\infty \frac{x^{2\delta}dx}{1+x^2} \le C_1 \int_a^\infty \frac{dx}{x^{2-2\delta}}.$$

Now, if $\delta < \frac{1}{2}$, then the last integral converges as it is a p-integral, $p > 1$. Hence, Krein's condition gives us the stated result. □

Example 3.2.11 *Let X be a random variable possessing power r-stage Erlang distribution with density function*

$$f(x) = \frac{x^{\frac{r}{p}-1}}{p\,(r-1)!}\,e^{-x^{\frac{1}{p}}}, \quad x > 0,$$

which is of type (3.10) *with* $\delta = \frac{1}{p}$ *and* $h(x) = x^{\frac{r}{p}-1}$*. Therefore, by Corollary 3.2.10, power of r-stage Erlang distribution is moment indeterminate for* $p > 2$*.*

Example 3.2.12 *Consider again power Nakagami distribution with density*

$$f(x) = \alpha x^{\alpha-1} e^{-x^\alpha}, \quad x, \alpha > 0,$$

appeared in Example 3.2.3. Recall that for this distribution Cramer's condition is inconclusive. By applying Corollary 3.2.10 where in this case $\delta = \alpha$ *and* $h(x) = x^{\alpha-1}$*, one concludes that this distribution is moment indeterminate for* $\alpha < \frac{1}{2}$*. It was shown in Example 3.2.3 that power Nakagami distribution is moment determinate for* $\alpha \ge 1$*. Therefore, there is still to show the moment (in)determinacy for* $1 > \alpha \ge \frac{1}{2}$*. To show this, we will use the sufficient condition for moment determinacy in Stieltjes sense, which was given in Section 3.2.3*

$$\frac{m_{k+1}}{m_k} = O(k^2) \quad \text{as} \quad k \to \infty. \tag{3.11}$$

Here,

$$m_k = \int_0^\infty \alpha x^{\alpha+k-1} e^{-x^\alpha}\,dx,$$

using the substitution $y = x^\alpha$*, one has*

$$m_k = \int_0^\infty y^{\frac{k}{\alpha}} e^{-y}\,dy = \Gamma\!\left(\frac{k}{\alpha}+1\right).$$

Hence,

$$\frac{m_{k+1}}{m_k} = \frac{\Gamma(\frac{k}{\alpha}+\frac{1}{\alpha}+1)}{\Gamma(\frac{k}{\alpha}+1)}.$$

For $\alpha \geq \frac{1}{2}$, *observe that* $\Gamma\left(\frac{k}{\alpha} + \frac{1}{\alpha} + 1\right) \leq \Gamma\left(\frac{k}{\alpha} + 3\right)$. *Therefore,*

$$\frac{m_{k+1}}{m_k} \leq \left(\frac{k}{\alpha} + 2\right)\left(\frac{k}{\alpha} + 1\right) \leq Ck^2 \quad as \quad k \to \infty,$$

which signifies that the condition (3.11) *holds, as a result, power Nakagami distribution is moment determinate for* $\alpha \geq \frac{1}{2}$.

It has to be mentioned that power Nakagami distribution belongs to the family of p-Kummer distributions studied in [27]. More precisely, $X \sim \mathrm{Kum}_p(1, 1, 0)$ with $p = \frac{1}{\alpha}$. See formula (3) in [27].

CHAPTER 4

LINDLEY DISTRIBUTION AND GENERALIZATIONS

In this chapter, we will study in detail properties of Lindley and power Lindley distributions. More precisely, we will present their probability density functions, distribution functions, hazard functions, and also moments and the moment (in)determinacy.

4.1 Lindley distribution

Many real world phenomena can be described and predicted by using statistical distributions. Therefore, the study of probability distributions and their properties has become of interest to many researchers, and one of these distributions is Lindley distribution. Lindley distribution was suggested by an English statistician Dennis Lindley (1923-2013) to model failure time data, see [23]. Since then, this distribution has been used in a wide variety of fields, including biology, engineering and medicine [2, 13]. Recently, Ghitany et al. applied a two-parameter weighted Lindley distribution for modeling in mortality studies, see [14]. The advantage that appears in using Lindley distribution is that it is more flexible than exponential distribution in terms of modeling unimodal and bathtube shaped hazard rates [13]. Lately, Lindley distribution has been modified and extended into new classes of distributions such as Lindley-exponential distribution, discrete Lindley distribution, power Lindley distribution, etc. see [4, 11, 15].

Definition 4.1.1 *The probability distribution whose density is given by*

$$f(x) = \frac{\theta^2}{\theta + 1}\,(1 + x)\,e^{-\theta x}, \quad x,\ \theta > 0, \tag{4.1}$$

is called Lindley distribution with parameter θ.

It can be observed that Lindley distribution belongs to the family of Kummer distributions, see [27, Definition 1]. More precisely, Lindley distribution with parameter θ coincides with Kummer distribution with parameters $\alpha = 1, \beta = \theta, \gamma = -1, \delta = 1$. The density in (4.1) can be splitted into a sum of two densities with coefficients p and $1 - p$:

$$f(x) = p\,\xi_1(x) + (1-p)\,\xi_2(x), \tag{4.2}$$

where $p = \frac{\theta}{\theta+1}$,

$$\xi_1(x) = \theta\, e^{-\theta x}, \quad x > 0,$$
$$\xi_2(x) = \theta^2 x\, e^{-\theta x}, \quad x > 0.$$

Here, $\xi_1(x)$ is a density function of an exponential distribution and $\xi_2(x)$ is a density function of 2-stage Erlang distribution with parameter θ.

It can be observed that Lindley distribution is a special case of Kummer distribution, see [27, Definition 1]. More precisely, Lindley distribution with parameter θ coincides with the Kummer distribution having parameters $\alpha = 1, \beta = \theta, \gamma = -1$, and $\delta = 1$.

The cumulative distribution function of Lindley distribution can be found as follows

$$F(x) = P(X \le x) = \int_0^x f(t)\,dt = \frac{\theta^2}{\theta+1}\int_0^x (1+t)\,e^{-\theta t}dt.$$

Evaluating the last integral, we obtain

$$F(x) = 1 - \frac{\theta + 1 + \theta x}{\theta + 1}\, e^{-\theta x}, \quad x, \theta > 0.$$

The reliability function (survival function) of Lindley distribution is given by

$$R(x) = P(X > x) = 1 - F(x) = \frac{\theta + 1 + \theta x}{\theta + 1}\, e^{-\theta x}.$$

The failure rate (also called hazard function) of Lindley distribution is given by:

$$h(x) = \frac{-R'(x)}{R(x)} = \frac{\theta^2(1+x)}{\theta + 1 + \theta x}, \quad x > 0.$$

Observe that $h(0) = \frac{\theta^2}{\theta+1}$ and that $\lim\limits_{x\to\infty} h(x) = \theta$. Writing $h(x) = \theta\left(1 - \frac{1}{\theta+1+\theta x}\right)$, we notice that the distribution of X is increasing failure rate (IFR). Hence, the distribution of X

is new better than used (NBU).

The characteristic function of Lindley distribution, according to (4.2) can be expressed as

$$\phi_X(t) = p\,\phi_{\xi_1}(t) + (1-p)\,\phi_{\xi_2}(t) = p\,\frac{\theta}{\theta - it} + (1-p)\,\frac{\theta^2}{(\theta - it)^2}.$$

Since $p = \frac{\theta}{\theta+1}$, we derive (see [13]):

$$\phi_X(t) = \frac{\theta^2(\theta + 1 - it)}{(\theta+1)(\theta - it)^2}, \quad t \in \mathbb{R}.$$

Consequently, the moment generating function of Lindley distribution can be expressed as

$$M_X(t) = \phi_X(-it) = \frac{\theta^2(\theta + 1 - t)}{(\theta+1)(\theta - t)^2}, \quad t < \theta.$$

4.1.1 Moments and Related Quantities

In this section, outcomes of [13] are presented. From Definition 2.1.2 of moments, the k-th moment of Lindley distribution can be found as follows:

$$E[X^k] = p\,\theta \int_0^\infty x^k e^{-\theta x}\,dx + (1-p)\,\theta^2 \int_0^\infty x^{k+1} e^{-\theta x}\,dx,$$

For the simplicity of the calculations, the density is taken in the form (4.2). Using the substitution $t = \theta x$, one obtains

$$E[X^k] = \frac{p}{\theta^k}\int_0^\infty t^k e^{-t} dt + \frac{1-p}{\theta^k}\int_0^\infty t^{k+1} e^{-t} dt = \frac{k!(\theta + k + 1)}{\theta^k(\theta+1)}, \quad k \in \mathbb{N}_0. \qquad (4.3)$$

We can easily derive the first four moments by setting $k = 1, 2, 3, 4$,

$$E[X] = \mu = \frac{\theta+2}{\theta(\theta+1)}, \qquad E[X^3] = \frac{6(\theta+4)}{\theta^3(\theta+1)},$$

$$E[X^2] = \frac{2(\theta+3)}{\theta^2(\theta+1)}, \qquad E[X^4] = \frac{24(\theta+5)}{\theta^4(\theta+1)}.$$

From those, one gets the central moments

$$\mathrm{Var}\,(X) = \sigma^2 = E[X^2] - \mu^2 = \frac{\theta^2 + 4\theta + 2}{\theta^2(\theta+1)^2}.$$

$$\mu_3' = E[(X-\mu)^3] = \frac{2(\theta^3 + 6\theta^2 + 6\theta + 2)}{\theta^3(\theta+1)^3}.$$

$$\mu_4' = E[(X-\mu)^4] = \frac{3(3\theta^4 + 24\theta^3 + 44\theta^2 + 32\theta + 8)}{\theta^4(\theta+1)^4}.$$

The coefficient of variation (γ), skewness (γ_1) and the kurtosis (γ_2) of Lindley distribution can be obtained by the following relations

$$\gamma = \frac{\sigma}{\mu} = \frac{\sqrt{\theta^2 + 4\theta + 2}}{\theta + 2}.$$
$$\gamma_1 = \frac{\mu_3'}{\sigma^3} = \frac{2(\theta^3 + 6\theta^2 + 6\theta + 2)}{(\theta^2 + 4\theta + 2)^{3/2}}.$$
$$\gamma_2 = \frac{\mu_4'}{\sigma^4} = \frac{3(3\theta^4 + 24\theta^3 + 44\theta^2 + 32\theta + 8)}{(\theta^2 + 4\theta + 2)^2}.$$

Notice that the coefficient of skewness is positive for all θ, which means that we can not use Lindley distribution to model left skewed data set, and this led researchers to look for modifying and adjusting this distribution so that it can be more flexible in terms of its density's shape.

Formula (4.3) leads to the following Taylor expansion of $M_X(t)$:

$$M_X(t) = \sum_{k=0}^{\infty} \frac{m_k}{k!} t^k = \sum_{k=0}^{\infty} \frac{\theta + k + 1}{\theta^k(\theta + 1)} t^k, \quad |t| < \theta.$$

4.1.2 The Moment Problem for Lindley Distribution

It has been evaluated in (4.3) that the k-th moment for Lindley distribution is:

$$m_k = \frac{k!(\theta + k + 1)}{\theta^k(\theta + 1)}.$$

Now, we want to check whether Lindley distribution is moment determinate or indeterminate. This can be done by estimating the rate of growth of moments. The condition ii) in Section 3.2.3 related to the Stieltjes moment problem guarantees the moment determinacy under the condition

$$m_k = O(k^{2k}) \quad \text{as} \quad k \to \infty \tag{4.4}$$

In this case,

$$\frac{m_k}{k^{2k}} = \frac{k!(\theta + k + 1)}{k^{2k}\theta^k(\theta + 1)} \leq \frac{c}{(k\theta)^k} \to 0, \; k \to \infty,$$

that is, (4.4) holds. Thus, we conclude that Lindley distribution is M-determinate.

4.2 Power Lindley Distribution

Ghitany et al. [15] have proposed a two-parameter distribution called power Lindley distribution as an extension of Lindley distribution. This section gives a review of their results. It will be shown that power Lindley distribution gives more flexibility than ordinary Lindley distribution in terms of its density and hazard functions' shape.

Definition 4.2.1 *Let X have a Lindley distribution and $\alpha > 0$. The distribution of a random variable $Y = X^{\frac{1}{\alpha}}$ is called a power Lindley (PL) distribution.*

Observe that for $\alpha = 1$, one recovers the Lindley distribution with parameter θ.

Using the density (4.1) of Lindley distribution, one can find a density of power Lindley distribution as:

$$f(y) = \frac{\alpha\theta^2}{\theta + 1}(1 + y^\alpha)\, y^{\alpha-1}e^{-\theta y^\alpha}, \quad y > 0, \;\; \theta, \alpha > 0. \tag{4.5}$$

If Y has density (4.5), we write $Y \sim PL(\alpha, \theta)$.

It has already been stated that Lindley distribution is a Kummer probability distribution. Correspondingly, power Lindley distribution is a p-Kummer distribution [27, Definition 2] and, as such, can be regarded as Kummer-type distribution. Specifically, $X \sim PL(\alpha, \theta) \Leftrightarrow X \sim Kum_p(a, b, c)$, where $p = \frac{1}{\alpha}$, $a = 1$, $b = \theta$, $c = -1$.

Similar to (4.2), density of power Lindley can be expressed as the linear combination:

$$f(y) = p\, g_1(y) + (1 - p)\, g_2(y), \tag{4.6}$$

where $p = \frac{\theta}{\theta+1}$,

$$g_1(y) = \alpha\, \theta y^{\alpha-1}e^{-\theta y^\alpha}, \quad y > 0,$$
$$g_2(y) = \alpha\, \theta^2 y^{2\alpha-1}e^{-\theta y^\alpha}, \quad y > 0.$$

The first component $g_1(y)$ is the density of probability distribution, which is called Weibull distribution with parameters α and θ. As the second component $g_2(y)$ is a density corresponding to a distribution called generalized gamma distribution with parameters $(k = 2, \alpha, \theta)$. Changing the value of parameters θ and α can result in different shapes of the density curve, as demonstrated in the next three statements:

(a) $f(y)$ is decreasing when $\{\alpha = 1,\ \theta \geq 1\}$, $\{0 < \alpha \leq \frac{1}{2},\ \theta > 0\}$ or $\{\frac{1}{2} < \alpha < 1,\ \theta \geq \delta_1(\alpha)\}$, where $\delta_1(\alpha) = \dfrac{1 - 2\sqrt{\alpha(1-\alpha)}}{\alpha}$;

(b) $f(y)$ is increasing-decreasing when $\{\alpha = 1,\ 0 < \theta < 1\}$ or $\{\alpha > 1,\ \theta > 0\}$;

(c) $f(y)$ is decreasing-increasing-decreasing for $\{\frac{1}{2} < \alpha < 1,\ 0 < \theta < \delta_1(\alpha)\}$,

For the proof of the those statements see [15]. Notice that in case (a), there is no mode while there is one mode in case of the unimodal shape (b), and for (c), the density curve has one antimode and one mode.

The cumulative distribution function of power Lindley distribution can be evaluated as follows:

$$F_Y(y) = \frac{\alpha\theta^2}{\theta+1}\int_0^y (1+t^\alpha)t^{\alpha-1}e^{-\theta t^\alpha}\,dt = 1 - \left(1 + \frac{\theta}{\theta+1}y^\alpha\right)e^{-\theta y^\alpha}, \quad y > 0,\ \alpha,\theta > 0.$$

Hence, the reliability function of power Lindley distribution is given by:

$$R(y) = P(Y > y) = 1 - F(y) = \left(1 + \frac{\theta}{\theta+1}y^\alpha\right)e^{-\theta y^\alpha}, \quad y > 0,\ \alpha,\theta > 0.$$

The failure rate (hazard function) of power Lindley distribution equals:

$$h(y) = \frac{-R'(y)}{R(y)} = \alpha\,\theta^2\frac{(1+y^\alpha)\,y^{\alpha-1}}{\theta+1+\theta\,y^\alpha}, \quad y > 0,\ \alpha,\theta > 0.$$

The shape for the failure rate of power Lindley distribution can be changed depending on the range of the parameters, and this is illustrated in the following statements

(a) $h(y)$ is decreasing if $\{0 < \alpha \leq \frac{1}{2},\ \theta > 0\}$ or $\{\frac{1}{2} < \alpha < 1,\ \theta \geq \delta_2(\alpha)\}$, where $\delta_2(\alpha) = \dfrac{(2\alpha-1)^2}{4\alpha\,(1-\alpha)}$;

(b) $h(y)$ is increasing if $\{\alpha \geq 1,\ \theta > 0\}$;

(c) $h(y)$ is decreasing-increasing-decreasing if $\{\frac{1}{2} < \alpha < 1,\ 0 < \theta < \delta_2(\alpha)\}$.

Therefore, power Lindley distribution is DFR in case (a), and IFR in (b). As for (c) the distribution becomes a DFR eventually.

4.2.1 Moments and Related Quantities

Using the density in (4.6), one can find the k-th moment of power Lindley distribution by evaluating the integrals below:

$$E[Y^k] = p\,\alpha\,\theta \int_0^\infty y^{k+\alpha-1} e^{-\theta y^\alpha} dy + (1-p)\,\alpha\,\theta^2 \int_0^\infty y^{k+2\alpha-1} e^{-\theta y^\alpha} dy.$$

Making the substitution $u = \theta y^\alpha$, one obtains

$$E[Y^k] = p\,\frac{\Gamma(\frac{k}{\alpha}+1)}{\theta^{k/\theta}} + (1-p)\,\frac{\Gamma(\frac{k}{\alpha}+2)}{\theta^{k/\alpha}} = \frac{k\,\Gamma(\frac{k}{\alpha})[\alpha(\theta+1)+k]}{\alpha^2\theta^{k/\alpha}(\theta+1)}. \tag{4.7}$$

Clearly, for $\alpha = 1$, we have the k-th moment of Lindley distribution. Upon substituting $k = 1, 2, 3, 4$, we get the first four moments

$$E[Y] = \mu = \frac{\Gamma(\frac{1}{\alpha})[\alpha(\theta+1)+1]}{\alpha^2\theta^{\frac{1}{\alpha}}(\theta+1)}, \qquad E[Y^2] = \frac{2\,\Gamma(\frac{2}{\alpha})[\alpha(\theta+1)+2]}{\alpha^2\theta^{\frac{2}{\alpha}}(\theta+1)},$$
$$E[Y^3] = \frac{3\,\Gamma(\frac{3}{\alpha})[\alpha(\theta+1)+3]}{\alpha^2\theta^{\frac{3}{\alpha}}(\theta+1)}, \qquad E[Y^4] = \frac{4\,\Gamma(\frac{4}{\alpha})[\alpha(\theta+1)+4]}{\alpha^2\theta^{\frac{4}{\alpha}}(\theta+1)}.$$

It was shown by Ghitany et al. [15] that the coefficient of skewness can take negative values, which implies that unlike Lindley distribution, power Lindley distribution can have a left skewed density, therefore power Lindley distribution can be a good model for a left skewed data set.

4.2.2 Application of Power Lindley Distribution

In [15], power Lindley distribution was applied to model a real data set. The set comprised data collected by Bader and Priest [3] when testing the stretchy strength of 69 carbon fibers. According to findings in [15] power Lindley distribution provides better fit to model the pattern of the data in comparison to Gamma, Gompertz, Weibull, exponential and Lindley distributions.

4.2.3 The Moment Problem for Power Lindley Distribution

We have shown in Section 4.1.2 that Lindley distribution is moment determinate. Here, we consider the moment problem for power Lindley distribution. Assume that we have a random variable $X \sim PL(\alpha, \theta)$ with density function

$$f(x) = \frac{\alpha\theta^2}{\theta+1}(1+x^\alpha)x^{\alpha-1}e^{-\theta x^\alpha}, \quad x>0,\ \alpha,\theta>0.$$

Then by applying Corollary 3.2.10, where, in this case, $\delta = \alpha$ and $h(x) = (1+x^\alpha)x^{\alpha-1}$, we find that the distribution of X is moment indeterminate for $\alpha < \frac{1}{2}$. However, what about the case when $\alpha \geq \frac{1}{2}$?
Recall from (4.7) that the k-th moment for power Lindley distribution is

$$m_k = \frac{p\Gamma(\frac{k}{\alpha}+1) + (1-p)\Gamma(\frac{k}{\alpha}+2)}{\theta^{\frac{k}{\alpha}}}.$$

Now, let us try the sufficient condition for moment determinacy in the Stieltjes case, described in section 3.2.3:

$$\frac{m_{k+1}}{m_k} = O(k^2) \quad \text{as} \quad k \to \infty. \tag{4.8}$$

For power Lindley distribution,

$$m_{k+1} = \frac{p\,\Gamma(\frac{k}{\alpha}+\frac{1}{\alpha}+1) + (1-p)\,\Gamma(\frac{k}{\alpha}+\frac{1}{\alpha}+2)}{\theta^{\frac{k+1}{\alpha}}},$$

therefore,

$$\frac{m_{k+1}}{m_k} = \frac{\Gamma(\frac{k}{\alpha}+\frac{1}{\alpha}+1)\left[p+(1-p)\,(\frac{k+1}{\alpha}+1)\right]}{\Gamma(\frac{k}{\alpha}+1)\,\theta^{\frac{1}{\alpha}}\left[p+(1-p)\,(\frac{k}{\alpha}+1)\right]}.$$

For $\alpha \geq \frac{1}{2}$, notice that $\Gamma\left(\frac{k}{\alpha}+\frac{1}{\alpha}+1\right) \leq \Gamma\left(\frac{k}{\alpha}+3\right)$. In this case

$$\frac{m_{k+1}}{m_k} \leq C\left(\frac{k}{\alpha}+2\right)\left(\frac{k}{\alpha}+1\right) \leq C_1 k^2 \quad k \to \infty,$$

which means that the condition (4.8) is satisfied. Hence, we conclude that power Lindley distribution is moment determinate for $\alpha \geq \frac{1}{2}$.

CHAPTER 5

STIELTJES CLASSES

5.1 Definition and Examples

As it has already been mentioned in Chapter 1. in the case when a distribution P_X is M-indeterminate, the question of finding other distributions with exactly the same moments arises. For an absolutely continuous distribution P_X with density f_X, one may search for other probability densities f_Y such that $E[X^K] = E[Y^K]$ for all $k \in \mathbb{N}$. The ideas of Stieltjes and Chebyshev (the latter had not published them) were used by J.Stoyanov [37] to formulate the following two definitions.

Definition 5.1.1 *Let $f(x)$ be a density function of a random variable X with finite moments of all orders, and let $h(x)$ be an integrable function on $(-\infty, \infty)$ such that $M_h := \sup_{x\in\mathbb{R}} |h(x)| = 1$. If, for all $k \in \mathbb{N}_0$,*

$$\int_{\mathbb{R}} x^k h(x) f(x)\, dx = 0,$$

then $h(x)$ is called a perturbation function of the density $f(x)$.

In this case, it also can be said that the product $h(x)f(x)$ has its all moments vanishing.

Definition 5.1.2 *Let $f(x)$ be a density function and $h(x)$ be a perturbation function of $f(x)$. The set*

$$S = S(f, h) := \{f_\epsilon(x) : f_\epsilon(x) = f(x)[1 + \epsilon h(x)],\ x \in \mathbb{R},\ \epsilon \in [-1, 1]\}$$

is said to be a Stieltjes class for $f(x)$ based on $h(x)$.

Obviously, S is an infinite family of densities all having the same sequence of moments as $f(x)$. Observe that, for a density function $f(x)$, there are different Stieltjes classes based on various perturbation functions $h(x)$. Therefore, if $f(x)$ is a probability density of an M-indeterminate distribution, then in order to find an infinite collection of densities possessing the same moment sequences as the given distribution, it suffices to find a perturbation function $h(x)$. It seems likely that the first known perturbation function was exposed by P. L. Chebyshev, see Example 3.1.2 of Chapter 3. In fact, he constructed the Stieltjes class

$$S = S(f, h) := \{f_\epsilon(x) = f(x)[1 + \epsilon \sin \sqrt[4]{x}],\ x \in \mathbb{R},\ \epsilon \in [-1, 1]\}$$

for $f(x) = \frac{1}{24} e^{-\sqrt[4]{x}}$, $x > 0$. Let us consider some other known examples of Stieltjes classes.

Example 5.1.3 *A Stieltjes class for the lognormal density*

$$f(x) = \frac{1}{\sqrt{2\pi}\, x} \exp\left\{-\frac{1}{2}(\ln x)^2\right\}, \quad x > 0, \tag{5.1}$$

was first presented by C. C. Heyde [18]:

$$S = S(f, h) := \{f_\epsilon(x) = f(x)[1 + \epsilon \sin(2\pi \ln x)],\ x > 0,\ \epsilon \in [-1, 1]\}.$$

To see that $h(x) = \sin(2\pi \ln x)$ *is a perturbation function, we have to show that*

$$\int_0^\infty x^k f(x) \sin(2\pi \ln x)\, dx = 0. \tag{5.2}$$

By substituting $\ln x = u$, *the integral* (5.2) *becomes*

$$C \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2 + ku} \sin(2\pi u)\, du = Ce^{\frac{1}{2}k^2} \int_{-\infty}^{\infty} e^{-\frac{(u-k)^2}{2}} \sin(2\pi u)\, du,$$

and if we let $u - k = y$, *the last integral can be reduced to*

$$C_1 \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} \sin(2\pi y)\, dy. \tag{5.3}$$

Due to the fact that the integral (5.3) *converges, the integrand is an odd function, and that* $(-\infty, \infty)$ *is a symmetric interval around zero, the integral* (5.3) *is equal to zero for all* $k \in \mathbb{N}_0$. *Clearly,* $M_h = \sup_{x \in \mathbb{R}} |h(x)| = 1$. *Therefore,* $h(x)$ *is a perturbation function of the density* $f(x)$ *and the set* S *is a Stieltjes class for* $f(x)$.

Example 5.1.4 *Let Y be a random variable such that $Y \sim \exp(1)$. We have shown in Example 3.2.5 that the exponential distribution is moment determinate. Now, let us consider a new random variable $X = Y^p$, where p is any positive number. The density function of X equals*

$$f(x) = \frac{1}{p} x^{1/p-1} \exp\left\{-x^{1/p}\right\}, \quad x > 0.$$

From Corollary 3.2.10, it can be easily seen that X is M-indeterminate for $p > 2$. The following function is a perturbation of the density $f(x)$:

$$h(x) = \sin\left(\frac{\pi}{p} - \tan(\pi/p) x^{1/p}\right), \quad x > 0.$$

In order $h(x)$ to be a perturbation, the following must hold:

$$\int_0^\infty x^{(k+1/p)-1} e^{-x^{1/p}} \sin\left(\pi/p - \tan(\pi/p)\, x^{1/p}\right) dx = 0.$$

In fact, by substituting $y = x^{1/p}$, one has

$$I := \int_0^\infty y^{kp} e^{-y} \sin(\pi/p - \tan(\pi/p)\, y)\, dy, \tag{5.4}$$

which can be evaluated with the help of the following two equalities that are provided in [16, formulae 3.944.9 and 3.944.10]:

$$\int_0^\infty x^{r-1} e^{-qx} \sin(qx \tan(t))\, dx = \frac{1}{q^r} \Gamma(r) \cos^r(t) \sin(rt), \tag{5.5}$$

and

$$\int_0^\infty x^{r-1} e^{-qx} \cos(qx \tan(t))\, dx = \frac{1}{q^r} \Gamma(r) \cos^r(t) \cos(rt). \tag{5.6}$$

Multiplying both sides of (5.5) *by* $\cos(\alpha t)$ *and* (5.6) *by* $\sin(\alpha t)$ *and subtracting one from the other, one gets*

$$\int_0^\infty x^{r-1} e^{-qx} \sin(qx \tan(t) - \alpha t)\, dx = \frac{1}{q^r} \Gamma(r) \cos^r(t) \sin(rt - \alpha t). \tag{5.7}$$

Taking $r = kp + 1$, $t = \pi/p$ and $\alpha = 1$ in (5.7)*, one arrives at*

$$I = -\Gamma(kp+1) \cos^{kp+1}(\pi/p) \sin(k\pi),$$

which equals zero for all $k \in \mathbb{N}_0$. Also, it can be easily seen that $h(x)$ is bounded. As a result, a Stieltjes class for $f(x)$ can be written as

$$S = S(f, h) := \{f_\epsilon(x) : f_\epsilon(x) = f(x)[1 + \epsilon h(x)],\ x \in \mathbb{R},\ \epsilon \in [-1, 1]\}.$$

Example 5.1.5 *Assume that we have a random variable Y such that $Y \sim \mathcal{N}(\mu = 0, \sigma^2 = 1/2)$, and let $X = Y^3$. Then the density of the random variable X is*

$$f(x) = \frac{1}{3\sqrt{\pi}} x^{-2/3} e^{-x^{2/3}}, \quad x \in \mathbb{R}. \tag{5.8}$$

It was proven by C.Berg [5] that the distribution of X is M-indeterminate. In addition,

$$h(x) = \sin\left(\pi/6 - \sqrt{3}x^{2/3}\right), \quad x \in \mathbb{R}.$$

is a perturbation function for (5.8). *This can be exposed by proving that $h(x)$ is bounded, which is obvious, and that*

$$I_k := \int_{-\infty}^{\infty} x^{k-2/3} e^{-x^{2/3}} \sin\left(\pi/6 - \sqrt{3}x^{2/3}\right) dx = 0. \tag{5.9}$$

Here, we consider two cases depending on the parity of k. If k is odd, then since I_k is convergent, the integrand is an odd function and $(-\infty, \infty)$ is a symmetric interval around zero, I_k equals to zero for all odd k. If k is even, then we use the following steps to show I_k is also equal to zero. Upon the substitution $y = x^{2/3}$ and the symmetry of I_k, we have

$$I_k = \int_0^{\infty} y^{(3k-1)/2} e^{-y} \sin\left(\pi/6 - \sqrt{3}y\right) dy.$$

By using (5.7) *where $r = (3k + 1)/2$, $t = \pi/6$ and $\alpha = 1/2$,*

$$I_k = -\Gamma\left(\frac{3k+1}{2}\right) \cos^{(3k+1)/2}(\pi/3) \sin(k\pi/2). \tag{5.10}$$

Obviously, the term $\sin(k\pi/2)$ in (5.10) *is equal to zero for all even k's. Therefore, the equality* (5.9) *holds for all $k \in \mathbb{N}_0$. The Stieltjes class S for the density in* (5.8) *can be written as*

$$S = S(f, h) := \{f_\epsilon(x) : f_\epsilon(x) = f(x)[1 + \epsilon h(x)],\ x \in \mathbb{R},\ \epsilon \in [-1, 1]\}.$$

It has to be noticed that a perturbation function is not unique and using different perturbations leads to different Stieltjes classes. It may be useful to provide more than one perturbation function for the same density if it is possible. The following example illustrates the case in which there are different perturbation functions for a given density.

Example 5.1.6 *Consider a random varible $Y \sim \mathcal{N}(\mu = 0, \sigma^2 = 1/2)$ and let $X = Y^6$. The density function of X is*

$$f(x) = \frac{1}{3\sqrt{\pi}} x^{-5/6} \exp\left\{-x^{1/3}\right\}, \quad x > 0. \tag{5.11}$$

According to Berg [5]

$$h_1(x) = \frac{1}{2}\left[\cos\left(\sqrt{3}x^{1/3}\right) - \sqrt{3}\sin\left(\sqrt{3}x^{1/3}\right)\right], \quad x > 0,$$

is a perturbation function for the density (5.11). *Apart from that, J.Stoyanov in [37] has found another perturbation function for the same density* (5.8)*, namely,*

$$h_2(x) = C_1 x^{5/6} \sin\left[\left(2 + \sqrt{3}\right) x^{5/12}\right] e^{x^{1/3} - x^{5/12}}, \quad x > 0,$$

where C_1 *is a normalizing constant that makes* $M_{h_2} = 1$. *To show that* $h_2(x)$ *is a perturbation of* $f(x)$*, the following must be true:*

$$\int_0^\infty x^k e^{-x^{5/12}} \sin\left(\beta x^{5/12}\right) dx = 0, \quad k \in \mathbb{N}_0 \tag{5.12}$$

where $\beta = 2 + \sqrt{3}$. *Taking* $t = \beta x^{5/12}$, (5.12) *becomes*

$$\int_0^\infty t^{(12k+7)/5} e^{-t/\beta} \sin(t)\, dt = 0.$$

Using Euler's identity $e^{it} = \cos(t) + i\sin(t)$ *and* $\alpha = \frac{1}{\beta} - i$*, the last equality reads as*

$$\operatorname{Im}\left(\int_0^\infty t^{(12k+7)/5} e^{-\alpha t} dt\right) = 0, \tag{5.13}$$

and upon the substitution $z = \alpha t$*, one gets*

$$\operatorname{Im}\left(\frac{\Gamma(\frac{12k+12}{5})}{\alpha^{\frac{12k+12}{5}}}\right) = 0.$$

By knowing the fact that α^{12} *is a real number, one can see that* (5.12) *holds. Based on the density* $f(x)$ *and its perturbation functions* $h_1(x)$ *and* $h_2(x)$*, we can construct two Stieltjes classes* S_1 *and* S_2 :

$$S_1 = \{f_\epsilon(x) = f(x)[1 + \epsilon h_1(x)],\ \epsilon \in [-1, 1]\},$$

$$S_2 = \{g_\delta(x) = f(x)[1 + \delta h_2(x)],\ \delta \in [-1, 1]\}.$$

This means that for random variables $Y_\epsilon \sim f_\epsilon(x)$ *and* $Z_\delta \sim g_\delta(x)$*, we have* $E[Y_\epsilon^K] = E[Z_\delta^k] = E[X^k]$ *for all* $k \in \mathbb{N},\ \epsilon, \delta \in [-1, 1]$.

5.2 Methods to Construct Stieltjes Classes

As can be understood from the preceding reasoning, the main step in constructing Stieltjes classes is an exposition of a perturbation function for a given density. The following methods may be used to find such a perturbation function.

5.2.1 Method I: Using Integral Identities

Let $f(x)$ be a probability density such that $f(x) \neq 0$ on an interval I. Assume that $g(x) \neq 0$ is a function such that

$$\int_I x^k g(x)\,dx = 0 \quad \text{for all} \quad k \in \mathbb{N}_0. \tag{5.14}$$

If $\tilde{h}(x) := \frac{g(x)}{f(x)}$ is bounded on I, then

$$h(x) = \begin{cases} \dfrac{\tilde{h}(x)}{M_{\tilde{h}}}, & x \in I \\ 0, & x \notin I \end{cases}$$

is a perturbation for $f(x)$. In the next example, this will be used to find a perturbation function in $I = (0, \infty)$.

Example 5.2.1 *Let X be a random variable having p-Kummer distribution with density*

$$f(x) = C_p x^{\alpha/p-1}(1 + x^{1/p})^{-\gamma} \exp\left\{-\beta x^{1/p}\right\}, \quad x > 0,\ p, \alpha, \beta > 0,\ \gamma \in \mathbb{R},$$

where C_p is a normalizing constant. It follows from Corollary 3.2.10 that the distribution of X is M-indeterminate for $p > 2$.

To construct a Stieltjes class, integral identity (5.7) *will be applied. Now, consider the integral*

$$\int_0^\infty x^k g(x) dx = \int_0^\infty x^{k+\frac{\alpha}{p}-1} \exp\left\{-b x^{1/p}\right\} \sin\left(b \tan(\pi/p)\, x^{1/p} - \frac{\alpha\pi}{p}\right) dx. \tag{5.15}$$

In order to extract a perturbation from (5.15)*, we have to show first that* (5.15) *is zero for all $k \in \mathbb{N}_0$. For the sake of applying the formula* (5.7) *to evaluate* (5.15)*, we set $y = x^{1/p}$, then* (5.15) *becomes*

$$I_3 := \int_0^\infty p y^{pk+\alpha-1} e^{-by} \sin\left(b \tan(\pi/p)\, y - \frac{\alpha\pi}{p}\right) dy.$$

Using (5.7) *with $r = pk + \alpha$, $q = b$ and $t = \pi/p$, one has*

$$I_3 = \frac{p}{b^{pk+\alpha}} \Gamma(pk + \alpha) \cos^{pk+\alpha}(\pi/p) \sin(k\pi) = 0 \quad \textit{for all } k \in \mathbb{N}_0.$$

Therefore, the condition (5.14) *is satisfied. Now, we want to show that $\tilde{h}(x) := \frac{g(x)}{f(x)}$ is bounded on the interval* $(0, \infty)$.

$$\tilde{h}(x) := \frac{g(x)}{f(x)} = (1 + x^{1/p})^{\gamma} \exp\left\{-(b - \beta) x^{1/p}\right\} \sin\left(b \tan(\pi/p)\, x^{1/p} - \frac{\alpha\pi}{p}\right), \tag{5.16}$$

since $\tilde{h}(x)$ decreases exponentially for $b > \beta$, $\tilde{h}(x)$ is bounded. Hence, $h(x) = \dfrac{\tilde{h}(x)}{M_{\tilde{h}}}$ is a perturbation for $f(x)$. Then, the set

$$S = S(f,h) := \{f_\epsilon(x) : f_\epsilon(x) = f(x)[1 + \epsilon h(x)],\ x \in \mathbb{R},\ \epsilon \in [-1,1]\}$$

is a Stieltjes class for $f(x)$.

5.2.2 Method II: Contour Integration

Although both the probability density and a perturbation function for it are real valued, the complex integration comes in handy in finding perturbations.

Theorem 5.2.2 *[29] Let $X \geq 0$ be a random variable whose density function is $f(x)$ with finite moments of all orders. Assume that*

$$f(x) \geq A\exp\{-ax^\alpha\}, \quad x > 0, \tag{5.17}$$

where A is a positive constant and $\alpha \in (0, \frac{1}{2})$. Consider the complex valued function $g(z)$ which is analytic in $\{z : \mathrm{Im}(z) \geq 0\} \setminus \{0\}$, takes real values when $x > 0$, and satisfies the condition:

$$|g(z)| \leq B\exp\{-b|z|^\beta\}, \quad z \in \{z : \mathrm{Im}(z) \geq 0\} \setminus \{0\} \tag{5.18}$$

for some $B > 0$, $b > 0$ and $\beta \in [\alpha, \frac{1}{2})$.
Then, the following function

$$h(x) := \frac{\mathrm{Im}\, g(-x)}{f(x)}, \quad x \geq 0,$$

is bounded, while the product $f(x)h(x)$, $x > 0$ has all moments vanishing.

Proof. Let us take real numbers ρ and R, where $0 < \rho < R$ and consider the closed contour $L := l_1 \cup l_2 \cup l_3 \cup l_4$ in the upper half-plane, where $l_1 = [\rho, R]$ and $l_3 = [-\rho, -R]$ are two segments, while $l_2 = \{z : |z| = R, 0 < \arg z < \pi\}$ and $l_4 = \{z : |z| = \rho, 0 < \arg z < \pi\}$ are two semi-circles. It follows from the Cauchy Theorem that

$$\oint_L z^k g(z)\, dz = 0, \quad k \in \mathbb{N}_0.$$

By taking the counter clockwise direction of the path, we see that

$$\oint_L = \int_{l_1} + \int_{l_2} + \int_{l_3} + \int_{l_4} =: I_1 + I_2 + I_3 + I_4.$$

In the next step, it will be shown that for $\alpha < \frac{1}{2}$, I_2 and I_4 approach 0 as $\rho \to 0$ and $R \to \infty$. This can be done by estimating the upper bounds for moduli of both integrals I_2 and I_4. We observe that

$$|I_2| \leq \pi R^{k+1} B \exp\{-aR^\alpha\} \to 0 \text{ as } R \to \infty.$$

In a similar way, we have

$$|I_4| \leq \pi \rho^{k+1} B \exp\{-a\rho^\alpha\} \to 0 \text{ as } \rho \to 0.$$

Hence, after taking the limit as $R \to \infty$ and $\rho \to 0$, we obtain

$$\int_0^\infty x^k g(x)\, dx + (-1)^k \int_0^\infty x^k g(-x)\, dx = 0, \quad k \in \mathbb{N}. \tag{5.19}$$

This means that both the real and imaginary parts of (5.19) are equal to zero, thus we have

$$\int_0^\infty x^k\, [\text{Im} g(-x)]\, dx = 0, \quad k \in \mathbb{N}. \tag{5.20}$$

We take:

$$h(x) := \frac{\text{Im} g(-x)}{f(x)} \quad \text{for} \quad x > 0. \tag{5.21}$$

The integral (5.20) implicates that product $h(x)f(x)$ has all its moments vanishing. From (5.17), one concludes that

$$|h(x)| \leq \frac{B}{A}$$

This completes the proof. □

Example 5.2.3 *Assume we have the following density function*

$$f(x) = C \exp\{-ax^\alpha\}, \quad x > 0,\ a > 0, \alpha \in (0, 1/2). \tag{5.22}$$

Choose the complex-valued function $g(z) = \exp\{-\frac{a}{\cos(\pi\alpha)} z^\alpha\}$*. Obviously, for* $z = |z|e^{i\varphi}$, $0 \leq \varphi \leq \pi$*, we have*

$$|g(z)| = \exp\left\{-\frac{a}{\cos(\pi\alpha)} |z|^\alpha \cos(\varphi\alpha)\right\} \leq \exp\{-a|z|^\alpha\},$$

which satisfies the condition (5.18)*. And*

$$\text{Im} g(-x) = -\exp\{ax^\alpha\} \sin(ax^\alpha \sin(\pi\alpha)).$$

From (5.21)*, the perturbation function of* $f(x)$ *is*

$$h(x) = \sin(ax^\alpha \sin(\pi\alpha)).$$

5.3 New Stieltjes Classes for Power Lindley Distribution

Let X be a random variable having power Lindley distribution with parameters α and θ, i.e. $X \sim PL(\alpha, \theta)$. The density of X is given by (4.5) as

$$f(x) = \frac{\alpha\,\theta^2}{\theta + 1}\,(1 + x^\alpha)x^{\alpha-1}e^{-\theta x^\alpha}, \quad x > 0, \;\; \alpha, \theta > 0.$$

In Section 4.2.3, we have proved that the distribution of X is M-determinate for $\alpha \geq \frac{1}{2}$ and M-indeterminate for $\alpha < \frac{1}{2}$. Here, our aim is to construct Stieltjes classes for power Lindley distribution. That is, we are looking for a bounded function $h(x) \not\equiv 0$ satisfying the following equalities:

$$\int_0^\infty x^k f(x)h(x)\,dx = 0 \;\text{ for all }\; k \in \mathbb{N}_0. \tag{5.23}$$

Theorem 5.3.1 *Let $X \sim PL(\alpha, \theta)$ with $\alpha < \frac{1}{2}$. The function*

$$h(x) = \frac{\exp\{-x^\alpha b\cos(\pi\alpha) - \theta\}\Big[\sin\Big(bx^\alpha\sin(\pi\alpha) - \pi\alpha\Big) + x^\alpha\sin\Big(bx^\alpha\sin(\pi\alpha) - 2\pi\alpha\Big)\Big]}{1 + x^\alpha}$$

is a perturbation for the density $f(x)$, where $x > 0,\ \theta > 0,\ \alpha < \frac{1}{2},\ b \geq \frac{\theta}{\cos(\pi\alpha)}$. Then a Stieltjes class for power Lindley distribution is defined as the following

$$S = \{f_\epsilon(x) = f(x)[1 + \epsilon h(x)],\ x \in \mathbb{R},\ \epsilon \in [-1, 1]\}.$$

Proof. We are looking for a function $h(x)$ satisfying the following equalities:

$$\int_0^\infty x^k f(x)\,h(x)\,dx = 0 \;\text{ for all }\; k \in \mathbb{N}_0. \tag{5.24}$$

Also, $h(x)$ must be bounded such that $M_h = 1$. Consider the following complex-valued function:

$$H_k(z) = z^k z^{\alpha-1}(1 + z^\alpha)\exp\{-bz^\alpha\}, \quad b \geq \frac{\theta}{\cos(\pi\alpha)},\ \alpha, \theta > 0,\ k \in \mathbb{N}_0.$$

Observe that the functions $H_k(z),\ k \in \mathbb{N}_0$, are analytic everywhere except for $z = 0$, where they have a ramification point. Now, let us take real numbers ρ and R where $0 < \rho < R$ and consider the closed contour $L := l_1 \cup l_2 \cup l_3 \cup l_4$ in the upper half-plane, where $l_1 = [\rho, R]$ and $l_3 = [-\rho, -R]$ are two segments, while $l_2 = \{z : |z| = R, 0 <$

$\arg z < \pi\}$ and $l_4 = \{z : |z| = \rho, 0 < \arg z < \pi\}$ are two semi-circles. It follows from the Cauchy Theorem that

$$\oint_L H_k(z)\,dz = 0, \quad k \in \mathbb{N}_0.$$

By taking the counter clockwise direction of the path, we see that

$$\oint_L = \int_{l_1} + \int_{l_2} + \int_{l_3} + \int_{l_4} =: I_1 + I_2 + I_3 + I_4.$$

In the next step, we will show that, for $\alpha < \frac{1}{2}$, I_2 and I_4 approach 0 as $R \to \infty$ and $\rho \to 0$, respectively. This can be shown by estimating the upper bounds for moduli of both integrals I_2 and I_4. We notice that for any arbitrary $z \in L$, we have $\varphi = \arg z \in [0, \pi]$. With $\alpha < \frac{1}{2}$, one has $\cos(\varphi\alpha) \geq \cos(\pi\alpha) > 0$. Therefore, we get

$$|I_2| \leq \pi R^{k+\alpha}(1 + R^\alpha)\exp\{-bR^\alpha\} \to 0 \text{ as } R \to \infty.$$

In a similar way, we have

$$|I_4| \leq \pi \rho^{k+\alpha}(1 + \rho^\alpha)\exp\{-b\rho^\alpha\} \to 0 \text{ as } \rho \to 0.$$

Hence, after taking the limit as $R \to \infty$ and $\rho \to 0$, we obtain

$$\int_0^\infty H_k(x)\,dx + \int_0^\infty H_k(-x)\,dx = 0, \quad k \in \mathbb{N}_0. \tag{5.25}$$

This means that both the real and imaginary parts of (5.25) are equal to zero, thus we have

$$\int_0^\infty \mathrm{Im}\,(H_k(-x))\,dx = 0, \quad k \in \mathbb{N}_0.$$

By using the fact that $z^\beta = |z|^\beta e^{i\pi\beta}$ for any real number β, and that $e^{i\pi k} = (-1)^k$ for $k \in \mathbb{N}_0$, the function

$$H_k(-x) = (-1)^{k+1} e^{i\pi\alpha} x^{k+\alpha-1} \exp\{-bx^\alpha(\cos(\pi\alpha) + i\sin(\pi\alpha))\}\left[1 + e^{i\pi\alpha}x^\alpha\right].$$

Therefore,

$$\mathrm{Im}\,(H_k(-x)) = x^{k+\alpha-1}\exp\{-x^\alpha(b\cos(\pi\alpha) - \theta)\}\times \\ \left[\sin(bx^\alpha\sin(\pi\alpha) - \pi\alpha) + x^\alpha\sin(bx^\alpha\sin(\pi\alpha) - 2\pi\alpha)\right].$$

Hence, setting

$$\tilde{h}(x) := \frac{\exp\{-x^\alpha(b\cos(\pi\alpha) - \theta)\}\left[\sin(bx^\alpha\sin(\pi\alpha) - \pi\alpha) + x^\alpha\sin(bx^\alpha\sin(\pi\alpha) - 2\pi\alpha)\right]}{1 + x^\alpha},$$

one obtains,

$$\int_0^\infty x^k f(x)\tilde{h}(x)\,dx = \int_0^\infty \operatorname{Im}(H_k(-x))\;dx = 0 \;\text{ for all }\; k \in \mathbb{N}_0.$$

It is obvious that $|\tilde{h}(x)| \leq 1$ for $b \geq \frac{\theta}{\cos(\pi\alpha)}$. Therefore, we have found a perturbation function $h(x) = \tilde{h}(x)/M_{\tilde{h}}$ for power Lindley density $f(x)$. This completes the proof.

□

Corollary 5.3.2 *For* $b = \frac{\theta}{\cos(\pi\alpha)}$, *we have*

$$\tilde{h}(x) = \frac{\sin(\theta x^\alpha \tan(\pi\alpha) - \pi\alpha) + x^\alpha \sin(\theta x^\alpha \tan(\pi\alpha) - 2\pi\alpha)}{1 + x^\alpha}$$

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