Approximation of oscillatory Bessel integral transforms*

by

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Abstract

The numerical treatment of oscillatory integrals is a demanding problem in applied sciences, particularly for large-scale problems. The main concern of this work is on the approximation of oscillatory integrals having Bessel-type kernels with high frequency and large interpolation points. For this purpose, a modified meshless method with compactly supported radial basis functions is implemented in the Levin formulation. The method associates a sparse system matrix even for high frequency values and large data points, and approximates the integrals accurately. The method is efficient and stable than its counterpart methods. Error bounds are derived theoretically and verified with several numerical experiments.

Keywords: Highly oscillatory Bessel integral transformsCompactly supported radial basis functionsStable algorithmsLevin methodHybrid functions

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1. Introduction

The efficient and stable evaluation of oscillatory integrals with Bessel kernel is a demanding problem in applied and computational science like acoustics, electromagnetics, optics, seismology image processing, and astronomy [1, 7, 10, 18, 19]. For instance, a time-harmonic acoustic plane wave with the total acoustic field satisfies the oscillatory Helmholtz equation:

$$\Delta v + \omega^2 v = 0, \qquad R^d \setminus \Xi, \qquad d = 2, 3, \tag{1.1}$$

where is a scattered field satisfy the Sommerfeld radiation condition in the far field. Using Green's theorem [6], the solution of Eq. (1.1) take the form

$$v(x) = v^i(x) - \int_{\delta\Xi} G(x, y)\, u(y)\, \mathrm{d}s(y), x \in \Xi,$$

where represents the standard solution and is defined as [3]

$$G(x, y) = \begin{cases} \frac{i}{4} H_0^{(1)}(\omega |x - y|), & d = 2 \\ \frac{1}{4\pi} \frac{e^{i\omega|x-y|}}{|x-y|}, & d = 3. \end{cases}$$

Here *H(x) = J (x) + i Y(x)* represents Hankel function of order and first kind.

In case of *d=2* and *d=3*, one encounter with the integrals given by

$$I_1[h] = \int_c^d h(x)\, H_0^{(1)}(\omega g(x))\, \mathrm{d}x, \qquad I_2[h] = \int_c^d \int_e^f h(x, y)\, e^{i\omega g(x,y)}\, \mathrm{d}x\mathrm{d}y,$$

respectively.

Moreover, Helmholtz equation appears in many model problems like natural convection flows [24], aerospace research [27], failure of sheet metals [2], and optical wave guides [21].

In the current work, a family of oscillatory integrals

$$I[h, \omega] = \int_c^d \sum_{i=1}^n h_i(x)\, \phi_\omega^{[i]}(x)\, \mathrm{d}x, \tag{1.2}$$

has been considered, where are smooth and non-oscillatory functions and are linearly independent highly oscillatory functions and . The function represents Hankel, Bessel or Airy functions. This work concentrates on the approximation of the integrals given by

$$I[h, \omega] = \int_c^d h(x)\, J_\mu(\omega x)\, \mathrm{d}x, \qquad \text{and} \tag{1.3}$$
$$I[h, \omega_1, \omega_2] = \int_c^d h(x)\, J_\mu(\omega_2 x) \cos(\omega_1 x)\, \mathrm{d}x,$$

where J represents oscillatory Bessel function of order and the frequencies . These integrals are associated with as Bessel function of the first kind and is the real part of Hankel function.

For higher values of , these integrals are challenging to be evaluated by the classical quadratures like Gaussian and Simpson quadratures, etc. Some special techniques are needed to tackle these integrals. For this purpose, several methods exists in the literature and implemented to treat these integrals accurately and efficiently. These methods include, Filon method [7], Levin method [16,18, 19, 25, 34, 35, 36, 38], complex line integration method [20, 33, 36] and Clenshaw–Curtis–Filon type method [9, 29, 30].

The author [29] presented a new shape of the Levin method for evaluation of Bessel integrals and obtained spectacular results. Whereas the authors in [30] considered the Clenshaw–Curtis–Filon method with special Hermite interpolation and obtained accurate results to compute oscillatory Bessel integrals. The method is extended to evaluate Volterra integral equations with Bessel kernels. Further, the authors [31] presented asymptotic method and Levin method to compute Bessel oscillatory integrals. In [4], the author have transformed the integrals on, and have used the Gauss–Laguerre quadrature to get efficient analysis of the Bessel integrals. Moreover, the authors [5] focused on analysis of Bessel transforms with multiple integrals.

The authors in [32], have applied an improved steepest descent method for computing the Bessel oscillatory integrals. The method derives an asymptotic expansion formula using integration by parts techniques. In [22], the authors have applied the convolution quadrature rule to oscillatory integrals with Bessel functions and extended its applicability to Volterra integral equation with Bessel kernels. The particular emphasis is on the convergence rate of the convolution quadrature. In [23], the author presented a complex integration method to compute the highly oscillatory and Bessel oscillatory integrals. In [12], the authors have performed an asymptotic analysis and developed a quadrature based on Filon-type methods for a class of improper oscillatory Bessel integrals. The authors concerns in [13] are on the fast computation and error analysis, they have studied proper and improper Bessel oscillatory integrals and transformed the Bessel function while using its asymptotic expansion. The idea of the compactly supported radial basis function in the context of highly oscillatory integrals (HOIs) has been implemented in [17], to evaluate the Fredholm integral equation with oscillatory kernels in [14], and the method is extended to evaluate boundary integral equation formulations in [15].

Levin theory coupled with multiquadric radial basis function (MQ-RBF) has been reported for HOIs [11, 16, 37] and highly oscillatory Bessel integrals [34, 35]. Despite the better performance of the MQ-RBF in the Levin method, accuracy is much affected by choosing an appropriate value of the shape parameter. As it is known that MQ-RBF is a globally defined basis function, which produces a dense and ill-conditioned matrix, specifically for large nodal points. Due to this characteristic, the method loses efficiency and stability for large interpolation points. In the stated literature, a regularization method TSVD is used to handle the presence of ill-conditioning of the system matrix for larger data points but the accuracy of the method is then affected.

In the current work, another class of RBFs, known as compactly supported RBFs (CS-RBFs) are used in the meshless collocation method. In this method, a coupled system of ODEs is approximated numerically. Levin's approach is then used to approximate the oscillatory Bessel integrals. A well-posed system matrix is obtained from the coupled system of ODEs even for large interpolation points, and executed efficient and stable results. The case of singularity at in the domain can be tackled by an adaptive splitting technique. Error analysis has been performed in the inverse power of frequency. In numerical section accuracy and stability are verified.

2. Meshless procedure with Levin formulation

Here meshless procedure with CS-RBFs following the Levin formulation is discussed for approximation of Bessel oscillatory integrals (1.3). The procedure is applicable for every point *x=0* in the domain. The case of zero-singularity can be resolved by an adaptive splitting algorithm. The procedure is given as follows: Consider the HOIs given as follows

$$\int_c^d \sum_{i=1}^n h_i(x)\,\phi_\omega^{[i]}(x)\,\mathrm{d}x = \int_c^d \mathbf{H}(x)\,.\,\Phi_\omega(x)\,\mathrm{d}x, \qquad (2.1)$$

$$\mathbf{H}(x) = \left(h_1(x), h_2(x), \ldots, h_n(x)\right)^T$$

$$\Phi_\omega(x) = \left(\phi_\omega^{[1]}(x), \phi_\omega^{[2]}(x), \ldots, \phi_\omega^{[n]}(x)\right)^T$$

$$\Phi'_\omega(x) = \mathbf{C}(x)\,\Phi_\omega(x),$$

Where C(x) represents a square matrix with non oscillatory entries of the ore n x n.

IN the Levin formation theory, we look for and approimate function

$$\widetilde{\mathbf{S}}(x) = \sum_{j=1}^{M} \beta_j^{[k]} \varphi_j(x), k = 1, \ldots, n,$$

Following the first order ODE,

$$\mathbf{H}(x) = \mathbf{S}'(x) + \mathbf{C}^T(x)\,\mathbf{S}(x). \qquad (2.2)$$

On plugging the value of H(x) given in (2.2) into into (2.1), we obtain

$$Q_{\varphi_1}[h] = \int_c^d \left(\widetilde{\mathbf{S}}'(x) + \mathbf{C}^T(x)\,\widetilde{\mathbf{S}}(x)\right) . \Phi_\omega(x)\,\mathrm{d}x \qquad (2.3)$$
$$= \widetilde{\mathbf{S}}(d) . \Phi_\omega(d) - \widetilde{\mathbf{S}}(c) . \Phi_\omega(c).$$

In the present case, we assume a vector

$$\Phi_\omega(x) = \begin{pmatrix} J_{\mu-1}(\omega x) \\ J_\mu(\omega x) \end{pmatrix}$$

$$\Phi'_\omega(x) = \begin{pmatrix} \frac{\mu-1}{x} & -\omega \\ \omega & \frac{-\mu}{x} \end{pmatrix} \Phi_\omega(x) = C(x)\,\Phi_\omega(x), \qquad (2.4)$$

and

$$\mathbf{H}(x) = \begin{pmatrix} 0 \\ h(x) \end{pmatrix}.$$

According to the proposed procedure, an approximate solution

$$\widetilde{S}(x) = \begin{pmatrix} \sum_{j=1}^{M} \beta_j^{[1]} \varphi_j(x) \\ \sum_{j=1}^{M} \beta_j^{[2]} \varphi_j(x) \end{pmatrix}$$

with unknown coefficients and, is supposed. To obtain the unknowns of , the following interpolation condition be imposed on the ODE (2.2)

$$\mathbf{S}'(x_i) + \mathbf{C}^T(x_i)\,\mathbf{S}(x_i) = \mathbf{H}(x_i), \quad i = 1, \ldots, M, \quad (2.5)$$

which associates the system of equations as follows

$$\sum_{j=1}^{M} \beta_j^{[1]} \left(\varphi_j'(x_i) + \frac{\mu-1}{x_i}\varphi_j(x_i)\right) + \sum_{j=1}^{M} \beta_j^{[2]} \omega\varphi_j(x_i) = 0$$
$$-\sum_{j=1}^{M} \beta_j^{[1]} \omega\varphi_j(x_i) + \sum_{j=1}^{M} \beta_j^{[2]} \left(\varphi_j'(x_i) - \frac{\mu}{x_i}\varphi_j(x_i)\right) = h(x_i).$$

System (2.6) in matrix form can be given as

$$\begin{aligned} \beta^{[1]}\Psi_1 + \beta^{[2]}\Psi_2 &= \mathbf{0} \\ \beta^{[1]}\Psi_3 + \beta^{[2]}\Psi_4 &= \mathbf{h}. \end{aligned} \quad (2.7)$$

Each of k = 1, 2, 3, 4 represents an M x M square matrix whose entries are

$$\Psi_1^{ij} = \varphi_j'(x_i) + \frac{\mu-1}{x_i}\varphi_j(x_i), \qquad \Psi_2^{ij} = \omega\varphi_j(x_i),$$
$$\Psi_3^{ij} = -\omega\varphi_j(x_i), \qquad \Psi_4^{ij} = \varphi_j'(x_i) - \frac{\mu}{x_i}\varphi_j(x_i), \quad i, j = 1, 2, \ldots, M,$$

and 0, h are vectors of order M. Eq. (2.7) can also be written as

$$\mathbf{G}\beta = \mathbf{H},$$

where G is a block matrix of order 2M x 2M and H are block vectors of order 2M x 1.

To obtain a stable system of equations, we consider the following Wendland's CS-RBFs [28]

$$\varphi_1\left(\frac{r}{\alpha}\right) = \left(1 - \frac{r}{\alpha}\right)_+^3 \left(3\frac{r}{\alpha} + 1\right)$$

$$\varphi_2\left(\frac{r}{\alpha}\right) = \left(1 - \frac{r}{\alpha}\right)_+^4 \left(4\frac{r}{\alpha} + 1\right),$$

where

$$\left(1 - \frac{r}{\alpha}\right)_+ = \begin{cases} 1 - \frac{r}{\alpha}, & \text{if} \quad 0 \le r \le \alpha, \\ 0, & \text{if} \quad r > 0, \end{cases}$$

The role of scaling parameter in the case of CS-RBFs is much important. In the present work, we choose in the interval [0, 1]. For details about the scaling parameter, its effect on stability and accuracy, one should focus on the literature [8], [26] and references therein.

2.1. Integrals with a product of two oscillators

Integral of (1.3) contains a product of two types of oscillators (i.e. Fourier and Bessel oscillators) and can be expressed as

$$I[h, \omega_1, \omega_2] = Re \int_c^d h(x)\, e^{i\omega_1 x} J_\mu(\omega_2 x)\, \mathrm{d}x. \quad (2.8)$$

Accordingly, integral (2.8) can take the form

$$I\left[h,\omega_1,\omega_2\right]=Re\int_c^{\alpha}\mathbf{H}\left(x\right).\Phi\left(\omega_1,\omega_2,x\right)\mathrm{d}x, \qquad (2.9)$$

Where H(x) = (0, h(x)) and

$$\Phi\left(\omega_1,\omega_2,x\right)=\left(e^{i\omega_1 x}J_{\mu-1}\left(\omega_2 x\right),e^{i\omega_1 x}J_{\mu}\left(\omega_2 x\right)\right)^T.$$

First derivative of is defined as

$$\Phi'\left(\omega_1,\omega_2,x\right)=\begin{pmatrix} i\omega_1+\frac{\mu-1}{x} & -\omega_2 \\ \omega_2 & i\omega_1-\frac{\mu}{x}\end{pmatrix}\begin{pmatrix} e^{i\omega_1 x}J_{\mu-1}\left(\omega_2 x\right) \\ e^{i\omega_1 x}J_{\mu}\left(\omega_2 x\right)\end{pmatrix} \qquad (2.10)$$
$$=C\left(\omega_1,\omega_2,x\right)\Phi\left(\omega_1,\omega_2,x\right).$$

In this case, the discretized form of the ODE (2.5) is given as

$$\left(\varphi'\left(x_j\right)+\left(i\omega_1+\frac{\mu-1}{x_j}\right)\varphi\left(x_j\right)\right)\beta^{[1]}+\omega_2\varphi\left(x_j\right)\beta^{[2]}=0_j$$
$$-\omega_2\varphi\left(x_j\right)\beta^{[1]}+\left(\varphi'\left(x_j\right)+\left(i\omega_1-\frac{\mu}{x_j}\right)\varphi\left(x_j\right)\right)\beta^{[2]}=h\left(x_j\right),$$

Where j = 1, ... M.

Eq. (2.11) can take the following matrix form

$$\begin{pmatrix}\chi_1 & \chi_2\\ \chi_3 & \chi_4\end{pmatrix}\begin{pmatrix}\beta^{[1]}\\ \beta^{[2]}\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \mathbf{h}\left(x\right)\end{pmatrix};$$

Or

$$\mathbf{A}\beta=\mathbf{H}, \qquad (2.12)$$

where A represents a block matrix with order 2M x 2M and, H are block 2M vectors. The same CS-RBFs for as mentioned in the above section, have been considered.

To get the unknowns from the system of equations (2.12), one has to look for an approximate radial basis function solution S(x). Consequently, the required solution of of (1.3) can be determined as

$$Q_{\varphi_1}[h] = Re\int_c^d \left(\widetilde{\mathbf{S}}'(x) + \mathbf{C}^T(x)\,\widetilde{\mathbf{S}}(x)\right).\Phi(\omega_1,\omega_2,x)\,\mathrm{d}x$$
$$= Re\left[\widetilde{\mathbf{S}}(d).\Phi(\omega_1,\omega_2,d) - \widetilde{\mathbf{S}}(c).\Phi(\omega_1,\omega_2,c)\right]. \tag{2.13}$$

When q is used as a basis function, the desired solution is denoted by $Q_1(h)$ and the approximate solution $Q_2(h)$ is obtained for q.

2.2. Case of singularity

As shown in (2.6), (2.11), the proposed procedures have a singularity at x = 0.In this regard, a splitting algorithm [11] is adopted to tackle the singularity. For this, we define a splitting parameter as. If c=0 a singular point, then the integral I of (1.3) can be split as

$$I[h,\omega] = \int_0^\eta h(x)\,J_\mu(\omega x)\,\mathrm{d}x + \int_\eta^d h(x)\,J_\mu(\omega x)\,\mathrm{d}x = I_1[h,\omega] + I_2[h,\omega], \tag{2.14}$$

where I has a singularity at the lower end point and can be approximated by the hybrid function based quadrature Q with N quadrature points) [11]. Fortunately, the quadrature Q can tackle the end point singularity. The integral I has no singular point and can be evaluated by the proposed procedures. Final value of the integral (1.3) can be obtained by the following formulae

$$Q_{\eta_2}[h] = Q_h[h] + Q_{\varphi_1}[h] \tag{2.15}$$
$$Q_{\eta_3}[h] = Q_h[h] + Q_{\varphi_2}[h].$$

The proposed procedures are compared in all problems with Q [34], where multi-quadric RBFs play the role of basis functions.

3. Convergence analysis

Here we theoretically derive some error estimates of the proposed methods, discussed in the above sections. Some related definitions and lemmas are stated as follows:

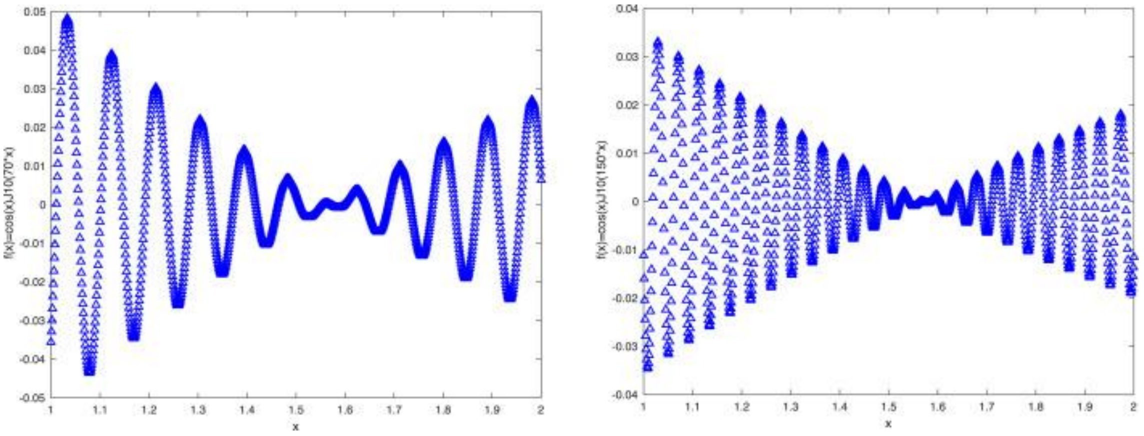

Fig. 1. Oscillatory behavior of the integral I with (left) w = 70, (right) w = 150.

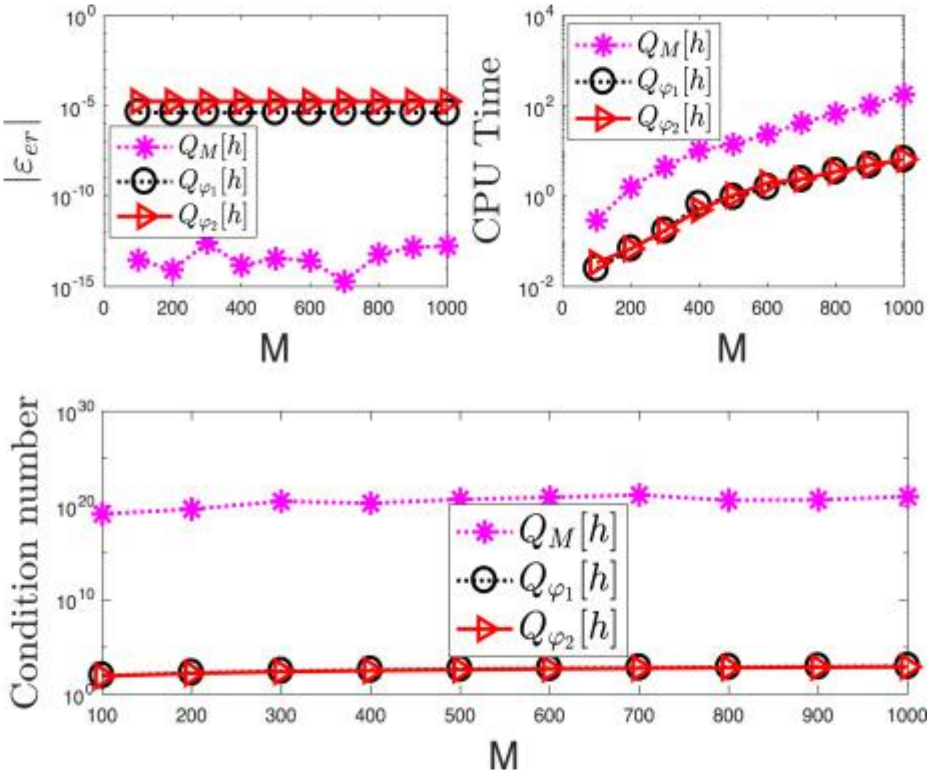

Fig. 2. (Upper left) ε, (Upper right) computational time, and (Lower) condition number analysis for w = 1000 to compute I.

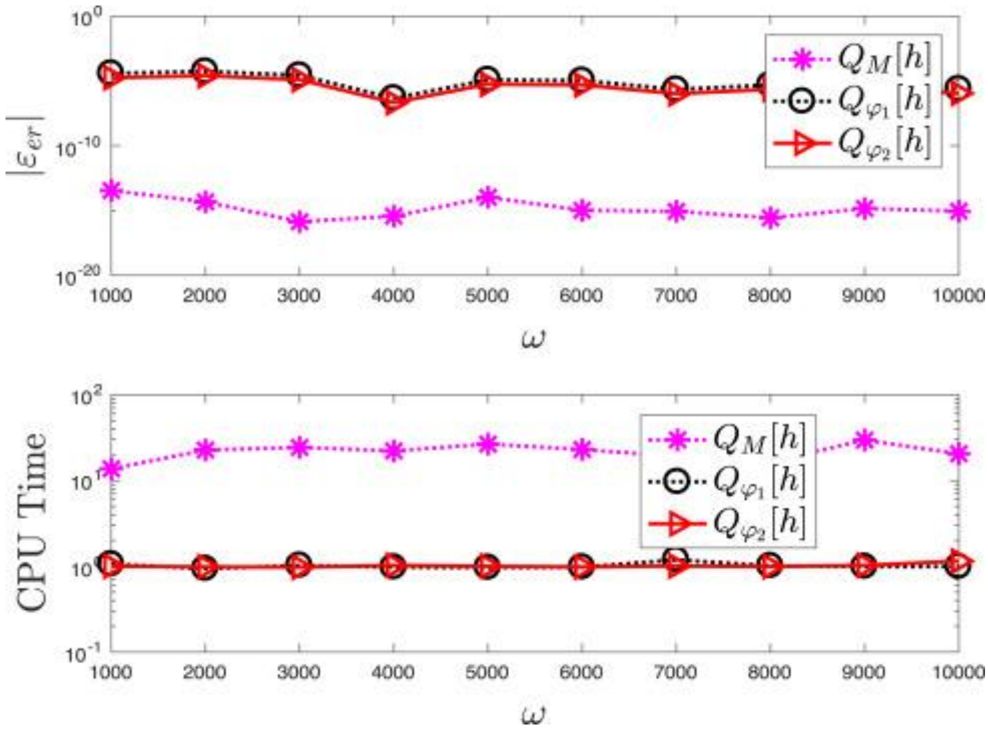

Fig. 3. (Upper) ε, (Lower) computational time with M=500 to compute I.

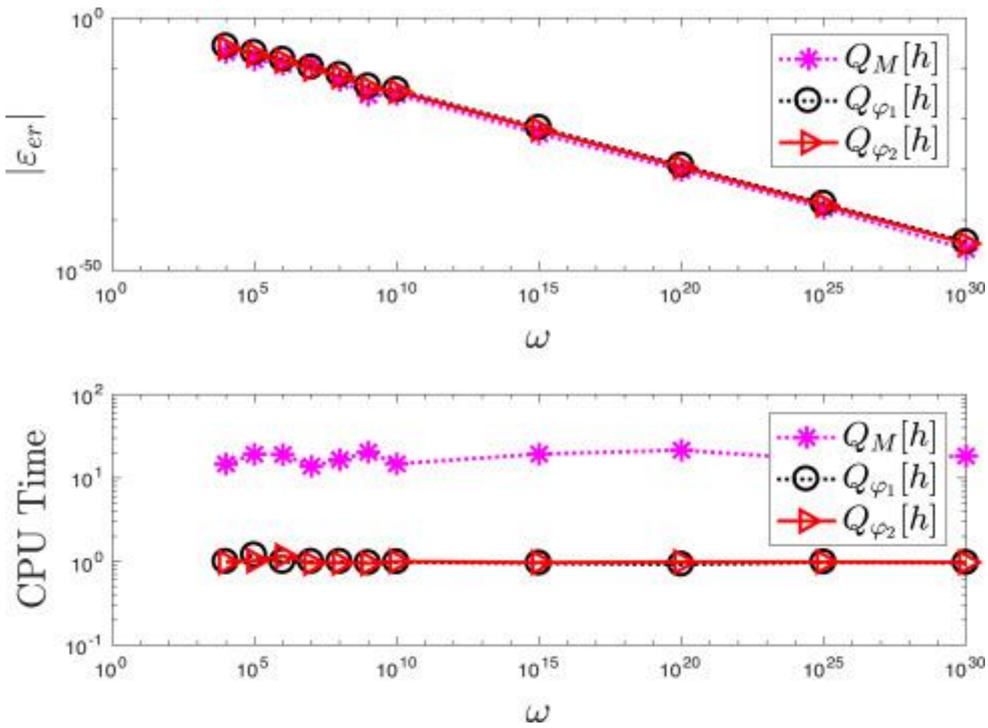

Fig. 4. (Upper) ε, (Lower) computational time with M=500 to compute I.

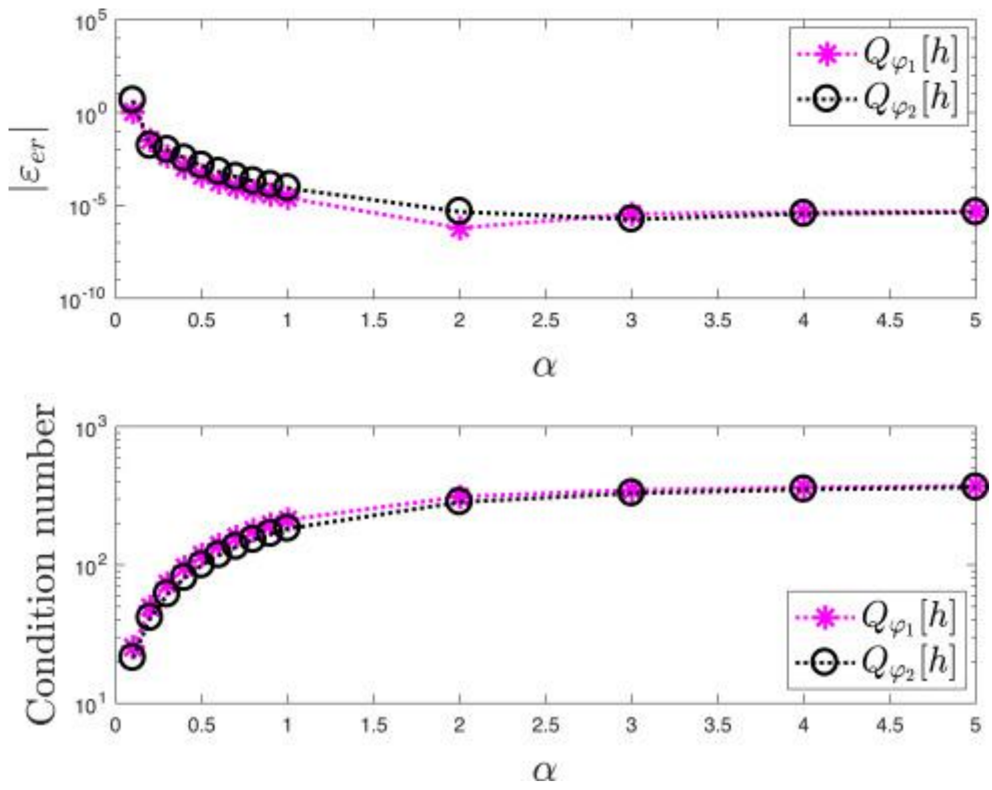

Fig. 5. (Upper) Effects of scaling parameter a on ε and (Lower) condition number for M=300,w=1000 to compute I.

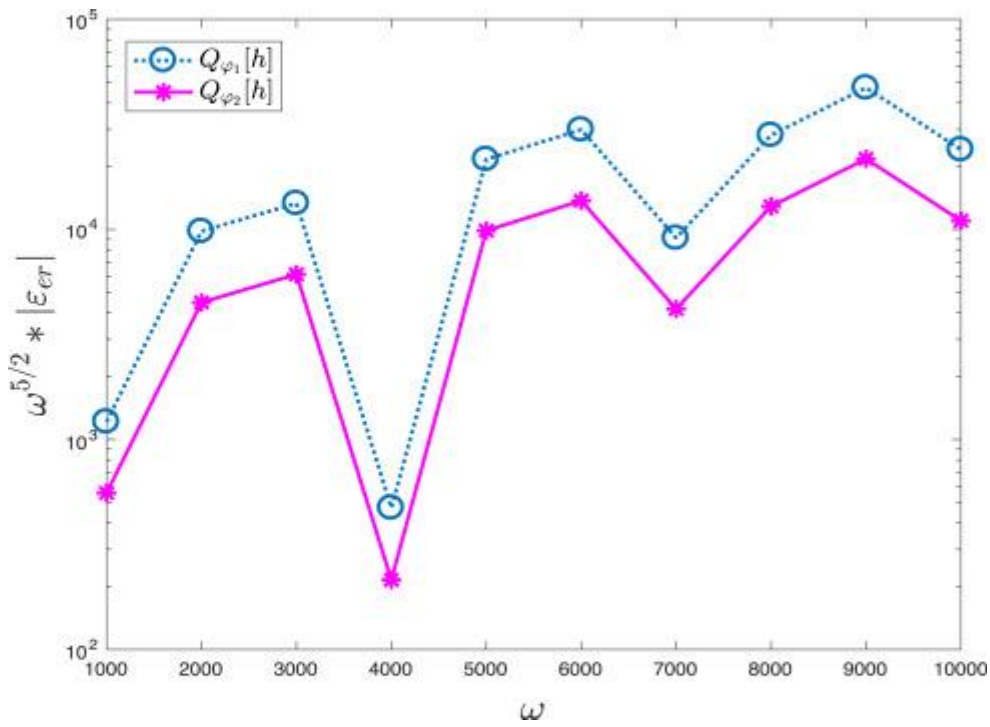

Fig. 6. ε scaled by w with M=500 for I.

4. Numerical assessments

This section demonstrates numerical verification of the proposed procedures. Few test examples have been considered from the literature. Mapel 2018 has been used to obtain the reference solution of the integrals. Absolute errors ε and scaled absolute errors are used to measure the accuracy of the new methods. Absolute error and errors scaled by are defined as

Scaled ablsolute error

For larger values of w = 10 ^{6}, Maple 18 fails to execute the reference solution in some problems, then the absolute errors are used by comparing the approximate values at lower and upper collocation points. All results are obtained using Maple 2017a, Apple MacBook Pro with 2.5 GHz Intel Core i5 and 16 GB 1600 MHz DDR3.

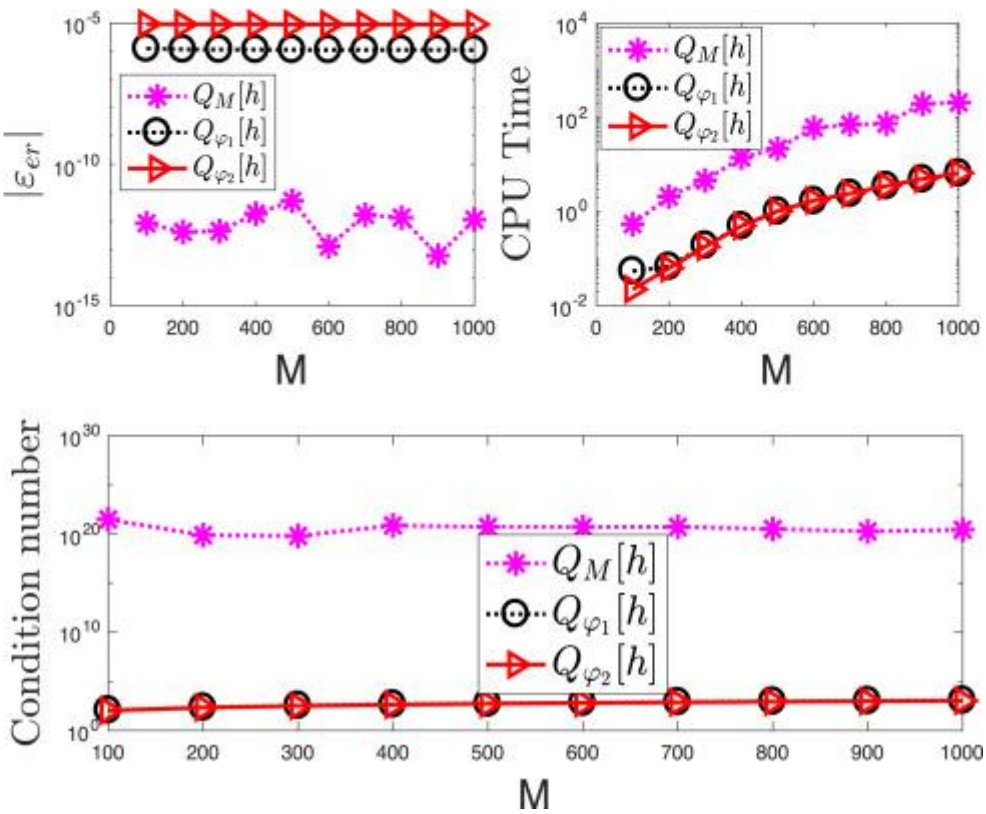

Fig. 7. (Upper left) ε, (Upper right) computational time, and (Lower) condition number analysis for w = 1000 to compute I.

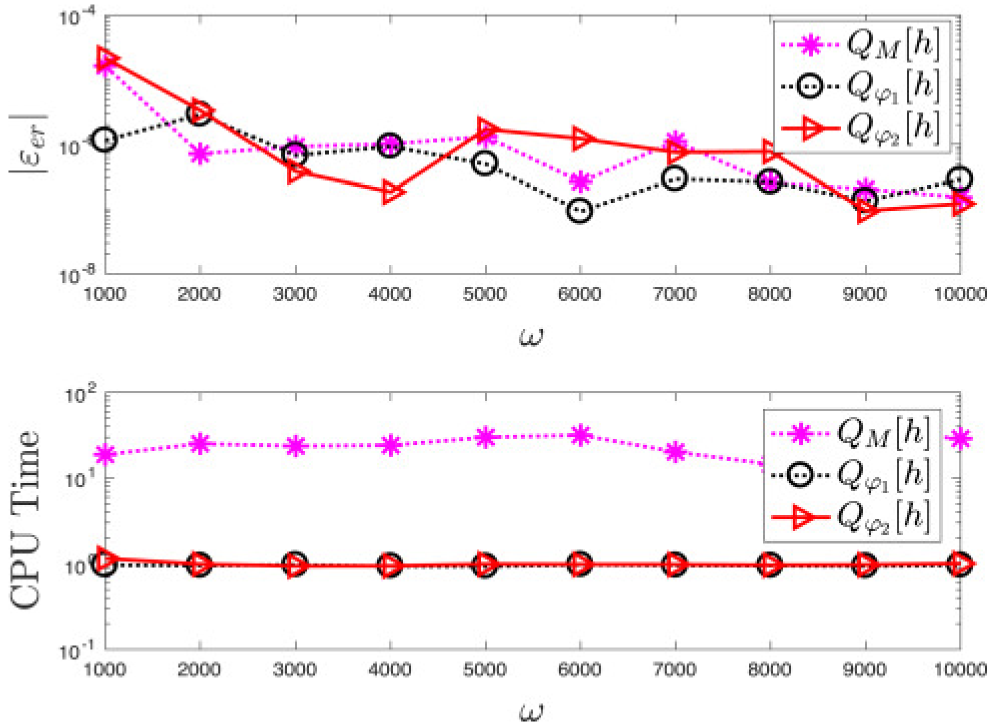

Fig. 8. (Upper) ε, (Lower) computational time with M=500 to compute I

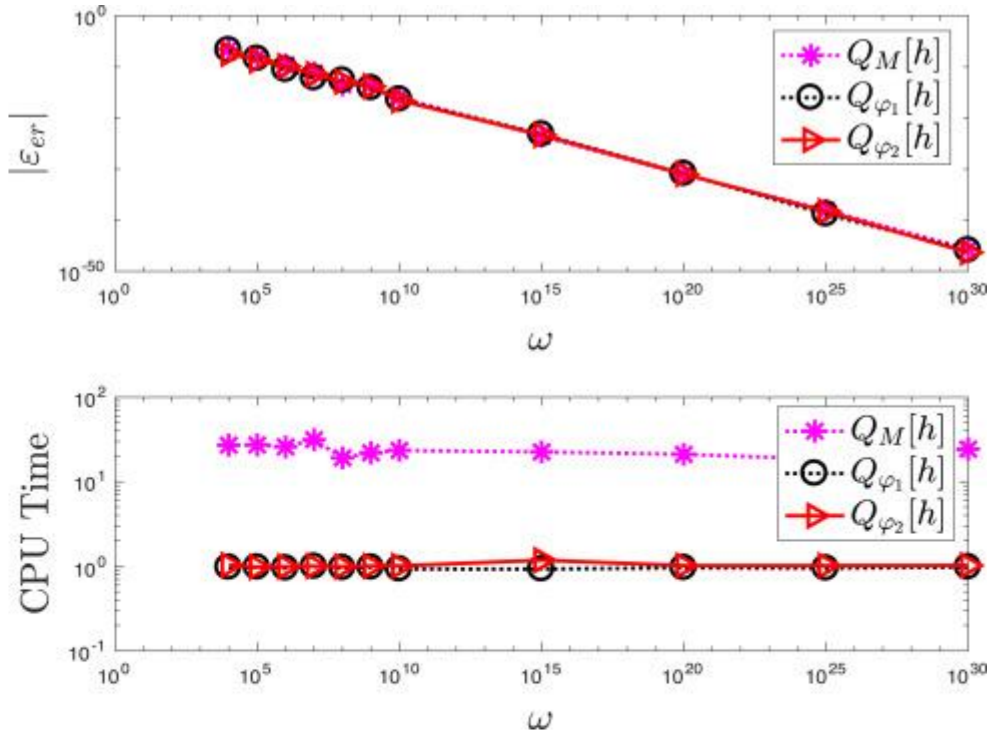

Fig. 9. (Upper) ε, (Lower) computational time with M=500 to compute I.

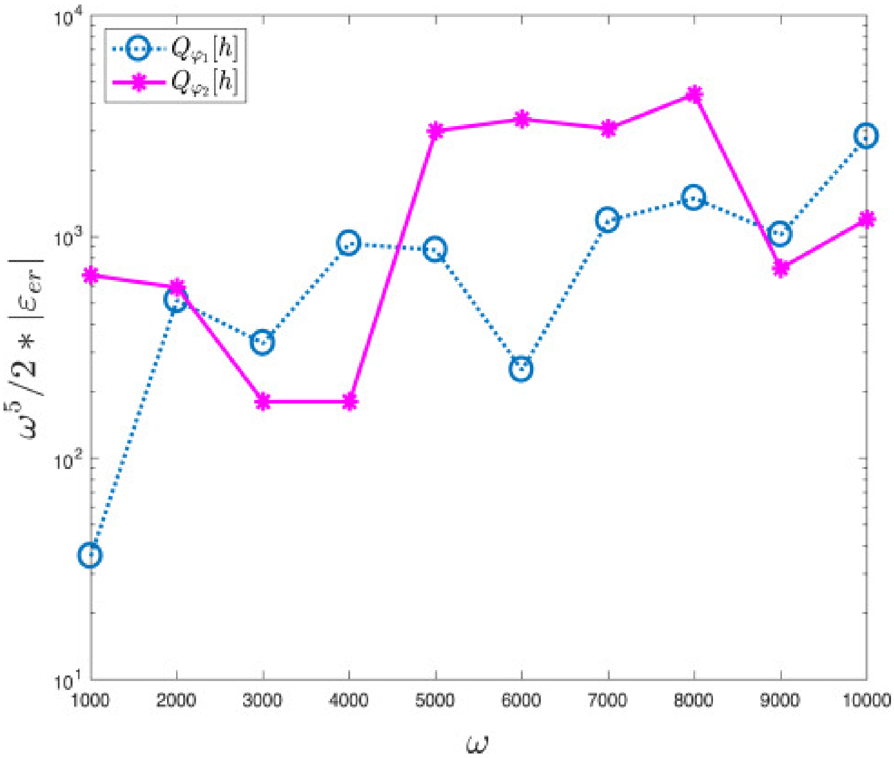

Fig. 10. ε scaled by w with M=500 for I.

Example 4.1

Test integral (4.1) is evaluated by the proposed algorithms. The oscillatory behavior of the integral for different frequencies is reported in Fig. 1. In Fig. 2, we consider computation for fixed frequency w and different nodes, and have presented absolute errors, computational time, and condition numbers. Furthermore, the analysis is carried out for fixed nodes and different frequencies as shown in Fig. 3. In both cases, the Levin method with MQ-RBF performs well and returns better accuracy but the worst computational time and condition number. The analysis with high frequencies w is performed and analyzed in Fig. 4, which confirms accurate analysis of both MQ-RBF and CS-RBFs but CS-RBFs exhibit efficient behavior. The effect of scaling parameter a on the accuracy and stability is shown in Fig. 5. Fig. 6 shows order of convergence, which confirms the theoretical error analysis of the described methods.

Example 4.2

We consider the following integral for computation by the new method. For fixed and varying nodes, the analysis is performed in Fig. 7, While for constant nodes and varying frequencies, the behavior of the proposed methods is demonstrated in Fig. 8. In these two cases, MQ-RBF performs better and returns better accuracy than CS-RBFs but worse in computational time and stability. To compute the problem with high frequencies, we have referred to Fig. 9, which retains the same behaviors as demonstrated in test Example 4.1. The asymptotic convergence of our method is confirmed by Fig. 10, which affirms the theoretical claims.

Example 4.3

We consider the following integral Fig. 11 presents the oscillatory nature of the integral. The new methods fail to evaluate the integral (4.3) because of the singularity at x=0 as mentioned in the theoretical section. For this purpose, the splitting procedure Q1, Q2 and Q3 are implemented to tackle the singularity. Results are obtained for varying nodes and fixed frequency, and are analyzed in Fig. 12. The analysis for fixed nodes and varying frequency is demonstrated in Fig. 13. In both cases, MQ-RBF and CS-RBFs perform well and return better accuracy but MQ-RBF are worst in efficiency and stability than CS-RBFs. Furthermore, the analysis for high frequencies is performed and is shown in Fig. 14, the same behaviors of both the RBFs are reflected in this case as well.

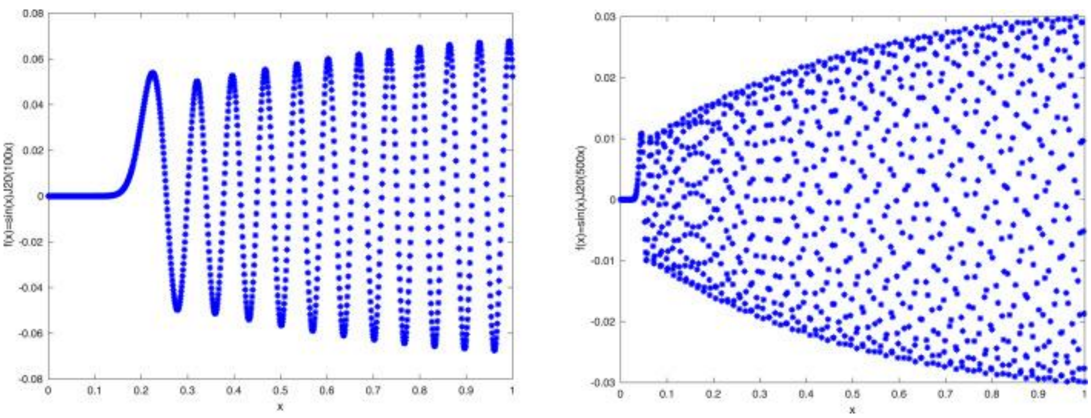

Fig. 11. Oscillatory behavior of the integral I with (left) w=100, (right) w=500.

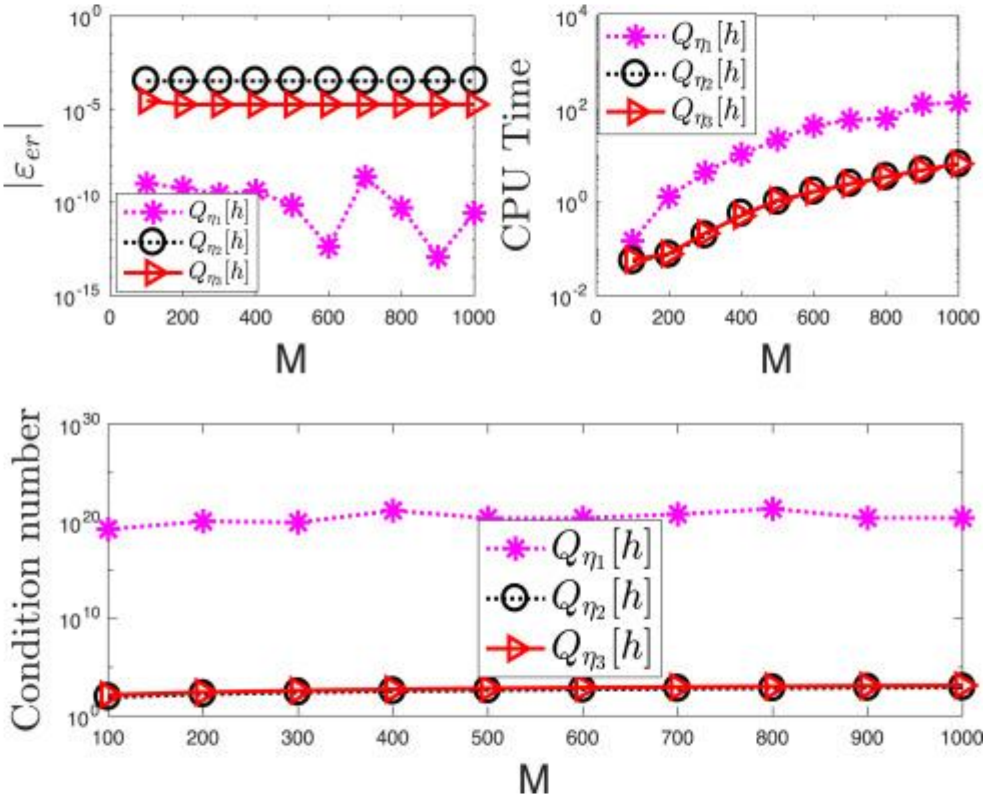

Fig. 12. (Upper left) ε, (Upper right) computational time, and (Lower) condition number analysis for w=1000 to compute I.

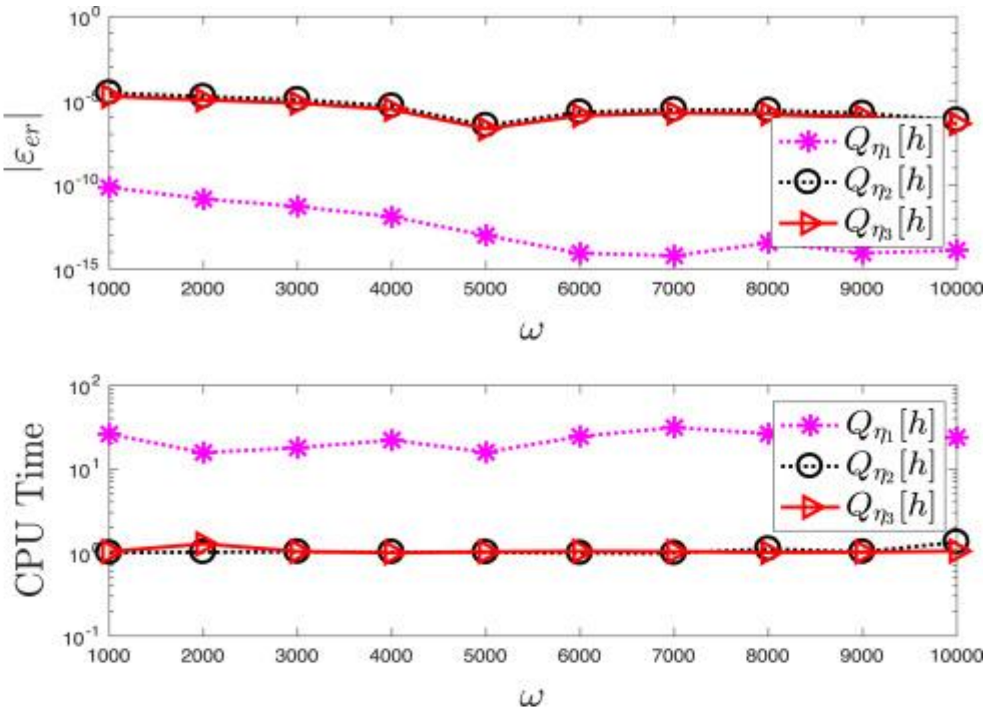

Fig. 13. (Upper) ε, (Lower) computational time with M=500 to compute I.

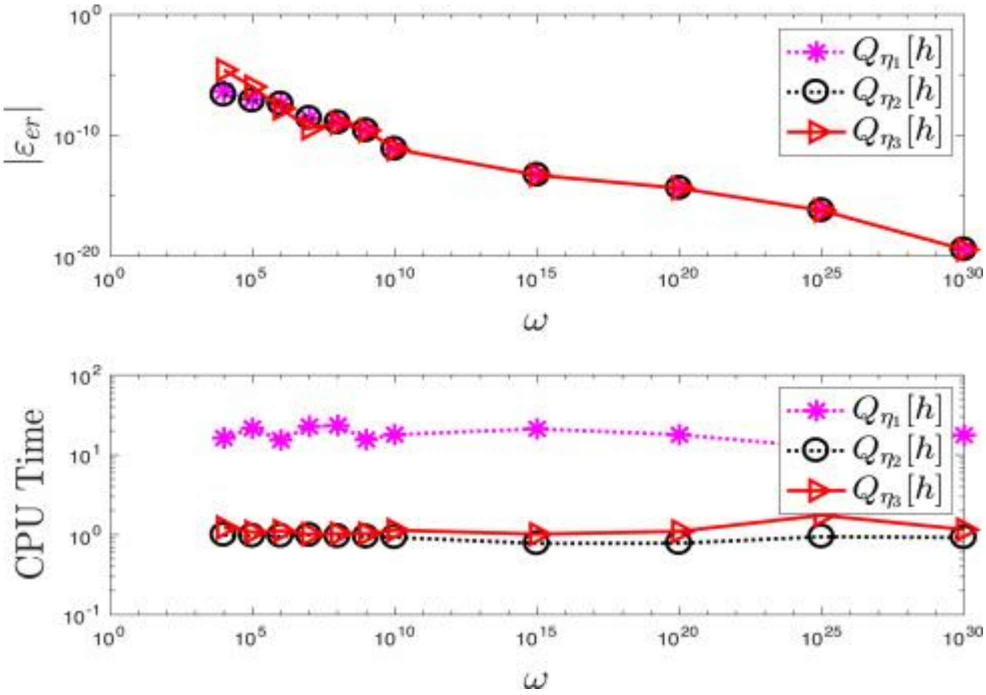

Fig. 14. (Upper) ε, (Lower) computational time with M=500 to compute I.

5. Conclusions

New algorithms have been implemented for the numerical treatment of oscillatory Bessel and Bessel-trigonometric integrals over the finite domain. The inherited singularity x=0 of the algorithm is tackled by a splitting procedure. The results are compared with the Levin approach based on MQ-RBF [34]. For fewer interpolation points and low frequencies, the Levin approach with MQ-RBF performs better than the proposed algorithms but as the nodal points increases or the parameter w gets larger, the proposed method gives the same or better accuracy, low CPU time, and well-posed system. The new method converges asymptotically with asymptotic convergence rate $O(w^{-5/2})$ for I[*h, w*] and $O(w^{-3/2})$ for I [h, w_1, w_2].

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