PRINCIPAL AND NONPRINCIPAL SOLUTIONS OF IMPULSIVE DYNAMIC EQUATIONS: LEIGHTON AND WONG TYPE OSCILLATION THEOREMS*

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Abstract. Principal and nonprincipal solutions of differential equations play a critical role in studying the qualitative behavior of solutions in numerous related differential equations. The existence of such solutions and their applications are already documented in the literature for differential equations, difference equations, dynamic equations, and impulsive differential equations. In this paper, we make a contribution to this field by examining impulsive dynamic equations and proving the existence of such solutions for second-order impulsive dynamic equations. As an illustration, we prove the famous Leighton and Wong oscillation theorems for impulsive dynamic equations. Furthermore, we provide supporting examples to demonstrate the relevance and effectiveness of the results.

1. Preliminaries and Introduction

Differential equations and difference equations are commonly used to model physical processes involving continuous and discrete data, respectively. However, natural phenomena encountered in various fields may require the use of both data types [1, 9, 10, 13], necessitating the simultaneous use of differential and difference equations. Stefan Hilger first proposed the time scales theory in 1988 [17, 18] as a way to bring these two types of equations together. This theory has since been heavily used for modeling by researchers in different areas. In addition, evolutionary phenomena often display jumping behaviors in their states, requiring the addition of impulses to the differential equations. Impulsive differential equations defined on a time scale are referred to as impulsive dynamic equations, which have garnered significant attention from researchers. Nonetheless, the notion of principal and nonprincipal solutions for impulsive dynamic equations has yet to be introduced, and their applications remain to be discovered.

The motivation behind investigating Wong and Leighton-type oscillation theorems for impulsive dynamic equations is indeed substantial. These theorems hold significant importance in oscillation theory and have various applications. Their simplicity and precision make them highly applicable in practice. Utilizing these theorems allows researchers to easily determine whether solutions are oscillatory. This provides valuable assistance to researchers working with such equations, as it enables them to quickly assess the nature of the solutions before delving into further analyses or investigations. By exploring and applying Wong and Leighton-type oscillation theorems to impulsive dynamic equations, researchers can streamline their studies and gain crucial insights into the behavior of these equations. This, in turn, facilitates a more efficient and informed approach to research in this field.

The present paper makes two significant contributions: (1) the introduction of principal and nonprincipal solutions for impulsive dynamic equations, and (2) the derivation of two renowned oscillation theorems for impulsive dynamic equations as applications, specifically the Leighton and Wong theorems, that extend and enhance the earlier results [5, 6, 8, 21–23, 26, 28]. Principal and nonprincipal solutions of differential equations are frequently used in studying the asymptotic behavior and oscillation of solutions. Excellent sources on the properties of the principal and nonprincipal solutions of ordinary differential

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equations can be found in books such as [16, 20]. The existence of such solutions for impulsive differential equations with continuous solutions was initially introduced in [26], with the counterparts for impulsive differential equations with discontinuous solutions identified in [3, 5]. Moreover, the existence of principal and nonprincipal solutions for second-order dynamic equations was established in [29]. Some related results for impulsive dynamic equations can be found in [23]. It is worth mentioning that principal and nonprincipal solutions are also useful tools to study asymptotic integration problems [2–4, 7, 12, 14, 27] as well as oscillation theory [15, 24, 25]. Ongoing research is being conducted to explore additional applications in this area, and the potential outcomes will be reported in due course.

The paper is structured as follows: Section 1 offers a concise overview of time scales, highlighting the main features, and introduces the problem being addressed. Section 2 establishes the existence of principal and nonprincipal solutions for a specific class of impulsive dynamic equations. Section 3 presents new oscillation theorems, based on nonprincipal solutions, with a focus on Leighton and Wong type. Lastly, Section 4 provides illustrative examples to demonstrate the findings.

1.1. **Time Scale Essentials.** We introduce some basic concepts and terminology from [9] that we will use throughout the paper. A time scale $\mathbb{T}$ is any closed subset of the real numbers. The forward and backward jump operators $\sigma, \rho\colon \mathbb{T} \to \mathbb{T}$ are defined, respectively as $\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\}$ and $\rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$. A point $t \in \mathbb{T}$ is left dense, right dense, left scattered, right scattered if $\rho(t) = t$, $\sigma(t) = t$, $\rho(t) < t$, $\sigma(t) > t$, respectively. Let $\mathbb{K}$ be a scalar field, either $\mathbb{R}$ or $\mathbb{C}$, and $\varphi : \mathbb{T} \to \mathbb{K}$. Define $\varphi^\sigma(t) = \varphi(\sigma(t))$. The function φ is called regulated if it has a finite right limit at all right dense points of $\mathbb{T} \setminus \{\sup \mathbb{T}\}$ and finite left limit at all left dense points of $\mathbb{T} \setminus \{\inf \mathbb{T}\}$. It is called regressive if $1 + \mu(t)\varphi(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$, where $\mu(t) = \sigma(t) - t$ is called the graininess function and $\mathbb{T}^\kappa$ is the set of points $t \in \mathbb{T}$ such that t is either left dense or non-maximal. We say that φ is rd-continuous if it is continuous at all right dense points of $\mathbb{T}$ and regulated. It is said to be Δ-differentiable at $t \in \mathbb{T}^\kappa$ if for all $\epsilon > 0$,

$$|\varphi(\sigma(t)) - \varphi(\tau) - \varphi^\Delta(t)(\sigma(t) - \tau)| < \epsilon|\sigma(t) - \tau|.$$

The Δ-derivative of φ at t is then defined as

$$\varphi^\Delta(t) = \lim_{s \to t} \frac{\varphi(s) - \varphi(t)}{s - t} \quad \text{if} \quad \mu = 0, \quad \varphi^\Delta(t) = \frac{\varphi(\sigma(t)) - \varphi(t)}{\sigma(t) - t} \quad \text{if} \quad \mu > 0.$$

The set of all rd-continuous functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T})$, and the set of all regressive, rd-continuous functions is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T})$. For further details and advances on time scales calculus, we refer the reader to the books [9–11, 13, 19], and the references cited therein.

1.2. **Problem Statement.** Let $t_0 \in \mathbb{T}$ be fixed, and denote $[t_0, \infty)_\mathbb{T} := \mathbb{T} \cap [t_0, \infty)$. We define $\mathrm{PLC}_{rd}[t_0, \infty)_\mathbb{T}$ to be the set of functions $x : [t_0, \infty)_\mathbb{T} \to \mathbb{R}$ such that x is piecewise rd-continuous on $[t_0, \infty)_\mathbb{T}$. We consider the second-order dynamic impulsive equations of the form

$$\begin{cases} (a(t)x^\Delta)^\Delta + b(t)x^\sigma = 0, & t \neq t_i, \\ \Delta_0 x + a_i x = 0,\ \Delta_0(a(t)x^\Delta) + b_i x^\sigma + c_i x^\Delta = 0, & t = t_i \end{cases} \tag{1}$$

and

$$\begin{cases} (a(t)x^\Delta)^\Delta + b(t)x^\sigma = f(t), & t \neq t_i, \\ \Delta_0 x + a_i x = g_i,\ \Delta_0(a(t)x^\Delta) + b_i x^\sigma + c_i x^\Delta = h_i, & t = t_i, \end{cases} \tag{2}$$

where $\Delta_0\varphi(t_i)$ measures the size of the jump $\varphi(t_i+) - \varphi(t_i-)$ at $t = t_i$ with $\varphi(t_i\pm) = \lim_{t \to t_i\pm} \varphi(t)$, $a > 0, b, f$ belong to $\mathrm{PLC}_{rd}[t_0, \infty)_\mathbb{T}$, $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, $\{g_i\}$ and $\{h_i\}$, $i \in \mathbb{N}$ are sequences of real numbers. We assume that the impulse points $\{t_i\}$ are right dense and they form an increasing sequence.

Our aim is twofold: a) to show that impulsive dynamic equation (1) possesses principal and nonprincipal solutions and b) to obtain the Leighton and Wong type oscillation theorems, respectively, for (1) and (2) by employing a nonprincipal solution.

2. Principal and Nonprincipal Solutions

We assume that

$$w_i = (1 - a_i)(1 - \frac{c_i}{a(t_i)}) > 0. \tag{3}$$

Denote

$$\eta(t, s) = \prod_{i=\underline{n}(s)}^{\overline{n}(t)} w_i, \quad t \in [s, \infty)_{\mathbb{T}}, \tag{4}$$

and

$$\eta_1(t, s) = \prod_{i=\underline{n}(s)}^{\overline{n}(t)} \frac{(1 - a_i)^2}{w_i^2}, \quad t \in [s, \infty)_{\mathbb{T}}. \tag{5}$$

where

$$\underline{n}(t) := \inf\{i : \ t_i \geq t\} \text{ and } \overline{n}(t) := \sup\{i : \ t_i < t\}$$

Theorem 2.1. *Suppose* (1) *has a positive solution and* (3) *holds. Then,* (1) *has two linearly independent solutions u and v with the properties*

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0; \tag{6}$$

$$\int_{t_0}^{\infty} \frac{\eta(t, t_0)}{a(t)u(t)u^{\sigma}(t)}\Delta t = \infty, \quad \int_{t_0}^{\infty} \frac{\eta(t, t_0)}{a(t)v(t)v^{\sigma}(t)}\Delta t < \infty; \tag{7}$$

$$\frac{a(t)v^{\Delta}(t)}{v(t)} > \frac{a(t)u^{\Delta}(t)}{u(t)}, \quad t \in [t^*, \infty)_{\mathbb{T}}, \tag{8}$$

where $t^* \geq t_0$ is sufficiently large. The solutions u and v are called the principal and nonprincipal solutions, respectively.

Proof. Suppose that u is a positive solution of (1). First, we show that

$$v(t) = u(t) \int_{t_0}^{t} \frac{\eta(s, t_0)}{a(s)u(s)u^{\sigma}(s)}\Delta s \tag{9}$$

is a second solution of (1).

Let $t \neq t_l$. Then,

$$\begin{aligned} v^{\Delta}(t) =& u^{\Delta}(t) \int_{t_0}^{t} \frac{\eta(s, t_0)}{a(s)u(s)u^{\sigma}(s)}\Delta s + \frac{\eta(t, t_0)}{a(t)u(t)} \\ =& u^{\Delta}(t) \int_{t_0}^{\sigma(t)} \frac{\eta(s, t_0)}{a(s)u(s)u^{\sigma}(s)}\Delta s + \frac{\eta(t, t_0)}{a(t)u^{\sigma}(t),} \end{aligned} \tag{10}$$

and hence

$$(a(t)v^{\Delta}(t))^{\Delta} + b(t)v^{\sigma}(t) = (a(t)u^{\Delta}(t))^{\Delta} \int_{t_0}^{\sigma(t)} \frac{\eta(s,t_0)}{a(s)u(s)u^{\sigma}(s)}\Delta s + a(t)u^{\Delta}(t)\frac{\eta(t,t_0)}{a(t)u(t)u^{\sigma}(t)}$$
$$- \frac{\eta(t,t_0)u^{\Delta}(t)}{u(t)u^{\sigma}(t)} + b(t)u^{\sigma}(t)\int_{t_0}^{\sigma(t)} \frac{\eta(s,t_0)}{a(s)u(s)u^{\sigma}(s)}\Delta s = 0. \tag{11}$$

On the other hand,

$$\Delta_0 v|_{t=t_l} + a_l v(t_l) = \big[\Delta_0 u|_{t=t_l} + a_l u(t_l)\big] \int_{t_0}^{t_l} \frac{\eta(s,t_0)}{a(s)u(s)u^{\sigma}(s)}\Delta s = 0. \tag{12}$$

Since t_l is assumed to be right dense, we have

$$\Delta_0(pv^{\Delta})|_{t=t_l} + b_l v^{\sigma}(t_l) + c_l v^{\Delta}(t_l) = \big[\Delta_0(pu^{\Delta})|_{t=t_l} + b_l u^{\sigma}(t_l) + c_l u^{\Delta}(t_l)\big] \int_{t_0}^{\sigma(t_l)} \frac{\eta(s,t_0)}{a(s)u(s)u^{\sigma}(s)}\Delta s$$
$$+ w_l \frac{\eta(t_l,t_0)}{u^{\sigma}(t_l+)} - \frac{\eta(t_l,t_0)}{u^{\sigma}(t_l)} + c_l \frac{\eta(t_l,t_0)}{a(t_l)u^{\sigma}(t_l)} = 0. \tag{13}$$

From (11)-(13), we conclude that v is a solution of the dynamic impulsive equation (1).

Next, we will show that u and v satisfy the properties (6)-(8), i.e., they are the principal and nonprincipal solution of equation (1), respectively. Suppose that

$$\int_{t_0}^{\infty} \frac{\eta(s,t_0)}{a(s)u(s)u^{\sigma}(s)}\Delta s = \infty. \tag{14}$$

Then,

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = \lim_{t\to\infty} \left(\int_{t_0}^{t} \frac{\eta(s,t_0)}{a(s)u(s)u^{\sigma}(s)}\Delta s \right)^{-1} = 0. \tag{15}$$

In addition, for $t \neq t_i$, from (10) we can write

$$\frac{a(t)v^{\Delta}(t)}{v(t)} = \frac{a(t)u^{\Delta}(t)}{u(t)} + \frac{\eta(t,t_0)}{u^2(t)} \left(\int_{t_0}^{t} \frac{\eta(s,t_0)}{a(s)u(s)u^{\sigma}(s)}\Delta s \right)^{-1} > \frac{a(t)u^{\Delta}(t)}{u(t)}.$$

From the left continuity of the solutions, taking the limit of both sides as $t \to t_i-$ leads to

$$\frac{a(t_i)v^{\Delta}(t_i)}{v(t_i)} > \frac{a(t_i)u^{\Delta}(t_i)}{u(t_i)}.$$

Thus, for each $t \in [t_0,\infty)_{\mathbb{T}}$, we can write

$$\frac{a(t)v^{\Delta}(t)}{v(t)} > \frac{a(t)u^{\Delta}(t)}{u(t)}. \tag{16}$$

Finally, using the Wronskian, we have

$$W(v,u)(t) = v(t)u^{\Delta}(t) - u(t)v^{\Delta}(t) = -\frac{1}{a(t)}\eta(t,t_0)$$

and employing (15) it is seen that

$$\int_{t_0}^{\infty} \frac{\eta(s,t_0)}{a(s)v(s)v^{\sigma}(s)}\Delta s = -\int_{t_0}^{\infty} \frac{W(v,u)(s)}{v(s)v^{\sigma}(s)}\Delta s = -\int_{t_0}^{\infty} \left(\frac{u}{v}\right)^{\Delta}(s)\Delta s = \frac{u(t_0)}{v(t_0)} < \infty. \tag{17}$$

By (15), (16), and (17), we conclude that u and v are, respectively, the principal and nonprincipal solutions of equation (1).

If (14) does not hold, then we define the function

$$u_0(t) := u(t)\int_t^{\infty} \frac{1}{\eta(\infty,s)a(s)u(s)u^{\sigma}(s)}\Delta s.$$

Proceeding in a similar manner as in (11)-(13), we easily see that u_0 is a solution of (1). Since

$$\begin{aligned}\int_{t_0}^{\infty} \frac{1}{\eta(\infty,t)a(t)u_0(t)u_0^{\sigma}(t)}\Delta t &= \int_{t_0}^{\infty} \frac{1}{\eta(\infty,t)a(t)u(t)u^{\sigma}(t)}\left(\int_t^{\infty} \frac{1}{\eta(\infty,s)a(s)u(s)u^{\sigma}(s)}\Delta s\right)^{-2}\Delta t \\ &= \int_{t_0}^{\infty} \frac{\Delta}{\Delta t}\left(\int_t^{\infty} \frac{1}{\eta(\infty,s)a(s)u(s)u^{\sigma}(s)}\Delta s\right)^{-1}\Delta t = \infty,\end{aligned}$$

we may take

$$v_0(t) := u_0(t)\int_{t_0}^{t} \frac{\eta(s,t_0)}{a(s)u_0(s)u_0^{\sigma}(s)}\Delta s.$$

It follows that u_0 and v_0 are, respectively, the principal and nonprincipal solutions of equation (1). □

Remark 2.1. If u is the positive principal solution of equation (1), then the function v given by (9) is a nonprincipal solution. If $v > 0$ is a nonprincipal solution of (1), the principal solution u is

$$u(t) = v(t)\int_t^{\infty} \frac{1}{\eta(\infty,s)a(s)v(s)v^{\sigma}(s)}\Delta s.$$

3. Oscillation Criteria

Consider the homogeneous equation

$$(a(t)x')' + b(t)x = 0 \tag{18}$$

and the nonhomogeneous equation

$$(a(t)x')' + b(t)x = f(t), \tag{19}$$

where a, b and f be continuous on $[t_0,\infty)$, and $a(t) > 0$.

The classical theorems of Leighton and Wong are as follows.

Theorem A (Leighton [21])**.** *If*

$$\int_{t_0}^{\infty} \frac{1}{a(t)}\,\mathrm{d}t = \infty = \int_{t_0}^{\infty} b(t)\,\mathrm{d}t,$$

then (18) *is oscillatory.*

Theorem B (Wong [28]). *Let v be a nonprincipal solution of* (18). *If*

$$\limsup_{t\to\infty} \mathcal{H}_0(t) = -\liminf_{t\to\infty} \mathcal{H}_0(t) = \infty,$$

where

$$\mathcal{H}_0(t) := \int_{t_0}^{t} \frac{1}{a(s)v^2(s)} \int_{t_0}^{s} f(\tau)v(\tau)\,\mathrm{d}\tau\mathrm{d}s,$$

then (19) *is oscillatory.*

We may now state and prove the Leighton and Wong type oscillation theorems for impulsive dynamic equations (1) and (2), respectively, by making use of the existence of a nonprincipal solution of (1).

3.1. **Leighton's oscillation theorem.**

Theorem 3.1. *Suppose* (1) *has a positive solution and* (3) *holds. If*

$$\int_{t_0}^{\infty} \frac{\eta(s,t_0)}{a(s)}\,\Delta s = \limsup_{t\to\infty}\left[\int_{t_0}^{t} b(s)\eta_1(s,t_0)\,\Delta s + \sum_{i=\underline{n}(t_0)}^{\overline{n}(t)} \frac{b_i}{1-c_i/a(t_i)}\eta_1(t_i,t_0)\right] = \infty, \tag{20}$$

where η and η_1 are as in (4) *and* (5), *then equation* (1) *is oscillatory.*

Proof. By Theorem 2.1, (1) has a nonprincipal solution $v(t)$ satisfying (7). Without loss of generality, assume that $v(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$, and define

$$y(t) = \frac{a(t)v^{\Delta}(t)}{v(t)}\eta_1(t,t_0), \quad t \in [t_0,\infty)_{\mathbb{T}}.$$

Using the fact that v is a solution of (1) we have

$$y^{\Delta}(t) = \frac{\eta_1(t,t_0)}{v(t)v^{\sigma}(t)}\Big[\big(a(t)v^{\Delta}(t)\big)^{\Delta}v(t) - a(t)\big(v^{\Delta}(t)\big)^2\Big] = -b(t)\eta_1(t,t_0) - \frac{v(t)}{a(t)\eta_1(t,t_0)v^{\sigma}(t)}y^2(t).$$

From

$$\eta_1(t_i+,t_0) = \frac{1-a_i}{1-\frac{c_i}{a(t_i)}}\eta_1(t_i,t_0)$$

we have

$$\begin{aligned}
\Delta_0 y =& \frac{a(t_i+)v^{\Delta}(t_i+)}{v(t_i+)}\eta_1(t_i+,t_0) - \frac{a(t_i)v^{\Delta}(t_i)}{v(t_i)}\eta_1(t_i,t_0)\\
=& \frac{a(t_i)v^{\Delta}(t_i) - b_iv^{\sigma}(t_i) - c_iv^{\Delta}(t_i)}{1-\frac{c_i}{a(t_i)}v(t_i)}\eta_1(t_i,t_0) - \frac{a(t_i)v^{\Delta}(t_i)}{v(t_i)}\eta_1(t_i,t_0)\\
=& -\frac{b_iv^{\sigma}(t_i)}{(1-c_i/a(t_i))v(t_i)}\eta_1(t_i,t_0).
\end{aligned}$$

Since the points t_i are right dense, we conclude that $y(t)$ is a solution of the dynamic impulsive equation

$$\begin{cases} y^{\Delta}(t) = -b(t)\eta_1(t,t_0) - \frac{1}{a(t)\eta_1(t,t_0)}y^2(t), & t \neq t_i\\ \Delta_0 y = -\frac{b_i}{1-c_i/a(t_i)}\eta_1(t_i,t_0), & t = t_i \end{cases}$$

and hence

$$\begin{cases} y^{\Delta}(t) \le -b(t)\eta_1(t,t_0), & t \neq t_i\\ \Delta_0 y = -\frac{b_i}{1-c_i/a(t_i)}\eta_1(t_i,t_0), & t = t_i. \end{cases} \tag{21}$$

Integrating both sides of (21) on $[t_0,t)_{\mathbb{T}}$ yields to

$$y(t)\le y(t_0)-\int_{t_0}^{t} b(s)\eta_1(s,t_0)\,\Delta s-\sum_{i=\underline{n}(t_0)}^{\overline{n}(t)}\frac{b_i}{1-c_i/a(t_i)}\eta_1(t_i,t_0).$$

Now, from the hypothesis (20) it is seen that

$$\limsup_{t\to\infty} y(t)=-\infty.$$

Choosing a sufficiently large $t_*\in[t_0,\infty)_{\mathbb{T}}$ so that $y(t)<0$ for $t\in[t_*,\infty)_{\mathbb{T}}$, it is seen that $v^{\Delta}(t)<0$ for $t\in[t_*,\infty)_{\mathbb{T}}$, i.e., v is decreasing on $[t_*,\infty)_{\mathbb{T}}$. Hence,

$$\int_{t_*}^{t}\frac{\eta(s,t_0)}{a(s)v(s)v^{\sigma}(s)}\,\Delta s\ge\frac{1}{v(t_*)v^{\sigma}(t_*)}\int_{t_*}^{t}\frac{\eta(s,t_0)}{a(s)}\,\Delta s.$$

In view of (20) we conclude that

$$\int_{t_*}^{\infty}\frac{\eta(s,t_0)}{a(s)v(s)v^{\sigma}(s)}\,\Delta s=\infty.$$

But this contradicts (7). Therefore, equation (1) cannot be non-oscillatory. □

3.2. **Wong's oscillation theorem.** Let

$$M_i=\frac{1}{w_i}\big(h_i(1-a_i)v(t_i)-g_i\big[(a(t_i)-c_i)v^{\Delta}(t_i)-b_iv^{\sigma}(t_i)\big]\big).$$

Theorem 3.2. *Let v be a nonprincipal solution of* (1). *If*

$$\limsup_{t\to\infty}\mathcal{H}(t)=-\liminf_{t\to\infty}\mathcal{H}(t)=+\infty,\tag{22}$$

where

$$\mathcal{H}(t):=\int_{t_0}^{t}\frac{1}{a(s)v(s)v^{\sigma}(s)}\bigg[\int_{t_0}^{s}\eta(s,\tau)f(\tau)v^{\sigma}(\tau)\,\Delta\tau+\sum_{i=\underline{n}(t_0)}^{\overline{n}(s)}\eta(s,t_i)M_i\bigg]\,\Delta s+\sum_{i=\underline{n}(t_0)}^{\overline{n}(t)}\frac{g_i}{(1-a_i)v(t_i)},$$

then equation (2) *is oscillatory.*

Proof. Let v be a nonprincipal solution of the homogeneous equation (1) associated with (2), and define $y(t)=v(t)[c+\mathcal{H}(t)]$, where $c\in\mathbb{R}$ is arbitrary. It is clearly seen from (22) that y is oscillatory. Thus, it suffices to prove that $y(t)$ is a solution of equation (2). For the sake of brevity, we define the operators

$$Lx=(a(t)x^{\Delta})^{\Delta}+b(t)x^{\sigma},\quad Ix=\Delta_0x+a_ix,\quad Jx=\Delta_0(a(t)x^{\Delta})+b_ix^{\sigma}+c_ix^{\Delta}.$$

For $t\ne t_l$, we can write

$$\begin{aligned}(v(t)\mathcal{H}(t))^{\Delta}=&v^{\Delta}(t)\mathcal{H}(t)+v^{\sigma}(t)\mathcal{H}^{\Delta}(t)\\=&v^{\Delta}(t)\mathcal{H}(t)+\frac{1}{a(t)v(t)}\bigg[\int_{t_0}^{t}\eta(t,\tau)f(\tau)v(\tau)\,\Delta\tau+\sum_{i=\underline{n}(t_0)}^{\overline{n}(t)}\eta(t,t_i)M_i\bigg].\end{aligned}\tag{23}$$

Then,

$$\begin{aligned}\big(a(t)(v(t)\mathcal{H}(t))^{\Delta}\big)^{\Delta} =&\big(a(t)v^{\Delta}(t)\big)^{\Delta}\mathcal{H}^{\sigma}(t)+a(t)v^{\Delta}(t)\mathcal{H}^{\Delta}(t)\\ &-\frac{v^{\Delta}(t)}{v(t)v^{\sigma}(t)}\Big[\int_{t_0}^{t}\eta(t,\tau)f(\tau)v^{\sigma}(\tau)\,\Delta\tau+\sum_{i=\underline{n}(t_0)}^{\overline{n}(t)}\eta(t,t_i)M_i\Big]+\frac{1}{v^{\sigma}(t)}\eta(t,t)f(t)v^{\sigma}(t)\\ =&-b(t)v^{\sigma}(t)\mathcal{H}^{\sigma}(t)+a(t)v^{\Delta}(t)\mathcal{H}^{\Delta}(t)-a(t)v^{\Delta}(t)\mathcal{H}^{\Delta}(t)+f(t).\end{aligned}$$

Therefore,

$$Ly=cL(v)+L(v\mathcal{H})=0+\big(a(t)(v\mathcal{H})^{\Delta}\big)^{\Delta}+b(t)v^{\sigma}\mathcal{H}^{\sigma}=f(t). \tag{24}$$

For $t=t_l$, we have

$$\Delta_0(v\mathcal{H})|_{t=t_l}=(1-a_l)v(t_l)\Big[\mathcal{H}(t_l)+\frac{g_l}{(1-a_l)v(t_l)}\Big]-v(t_l)\mathcal{H}(t_l)$$

which implies that

$$Iz=cI(v)+I(v\mathcal{H})=0+\Delta_0(v\mathcal{H})+a_lv\mathcal{H}=g_l. \tag{25}$$

Finally, by using (23) it can be seen that

$$J(v\mathcal{H})=\Delta_0a(t)(v\mathcal{H})^{\Delta}+b_i(v\mathcal{H})^{\sigma}+c_i(v\mathcal{H})^{\Delta}=h_l,$$

and so

$$Jy=0+cJ(v)+J(v\mathcal{H})=h_l. \tag{26}$$

Hence, from (24), (25) and (26) we conclude that $y(t)$ is a solution of (2).

□

4. Simulations

This section is devoted to examples demonstrating the efficiency of the results above.

Example 4.1. Consider the dynamic impulsive equation

$$\begin{cases}(tx^{\Delta})^{\Delta}-\frac{x^{\sigma}}{4t}=0, \quad t\neq 2i, \quad t\geq 2,\\ \Delta_0x-\dfrac{x}{i^2}=0, \quad \Delta_0(tx^{\Delta})+\dfrac{x^{\sigma}}{i}-\dfrac{2x^{\Delta}}{i}=0, \quad t=2i, \quad i>1\end{cases} \tag{27}$$

on the time scale $\mathbb{T}=\mathbb{P}_{1,1}=\cup_{n=0}^{\infty}[2n,2n+1]$.

Clearly, $t_i=2i$, $a(t)=t$, $b(t)=-1/4t$, $a_i=-1/i^2$, $b_i=1/i$, and $c_i=-2/i$. Let $t_0=2$, and say $s\in(2j-2,2j]_{\mathbb{T}}$. Then, $\overline{n}(s)=j$ and

$$\eta(s,2)=\prod_{i=2}^{\overline{n}(s)}\Big(1+\frac{1}{i^2}\Big)\Big(1+\frac{2}{2i^2}\Big)=\frac{(\overline{n}(s)+1)^4}{16}=\frac{(j+1)^4}{16}, \quad \eta_1(t,2)=\prod_{i=2}^{\overline{n}(t)}1=1.$$

Without loss of generality, we suppose that $t>2$ is an arbitrary integer in $\mathbb{P}_{1,1}$. Then

$$\int_2^t\frac{\eta(s,2)}{a(s)}\,\Delta s=\sum_{j=1}^{t}\int_{2j}^{2j+1}\frac{(j+1)^4}{16s}\,\mathrm{d}s+\sum_{j=1}^{t}\frac{(j+1)^4}{16(2j+1)}=\frac{1}{16}\sum_{j=1}^{t}(j+1)^4\Big[\ln\Big(1+\frac{1}{2j}\Big)+\frac{1}{(2j+1)}\Big]=\infty. \tag{28}$$

On the other hand, we have

$$\int_2^t b(s)\eta_1(s,2)\,\Delta s+\sum_{i=2}^{t}\frac{b_i}{1-c_i/a(i)}\eta_1(j,2)=-\sum_{j=1}^{t}\int_{2j}^{2j+1}\frac{1}{4s}\,\mathrm{d}s-\sum_{j=1}^{t}\frac{1}{4(2j+1)}+\sum_{i=2}^{t}\frac{i}{i^2+1}$$

$$=\frac{1}{4}\ln\Big(\frac{4}{5}\Big)+\frac{1}{4}\sum_{j=2}^{t}\Big[\ln\Big(\frac{2j}{2j+1}\Big)-\frac{1}{2j+1}+\frac{4j}{j^2+1}\Big]$$

which implies

$$\limsup_{t\to\infty}\Big[\int_2^t b(s)\eta_1(s,2)\,\Delta s+\sum_{i=2}^{t}\frac{b_i}{1-c_i/a(t_i)}\eta_1(i,2)\Big]=+\infty. \tag{29}$$

Thus, the hypothesis (20) holds in view of (28) and (29). Hence, by Theorem 3.1, (27) is oscillatory. The following figure supports our findings.

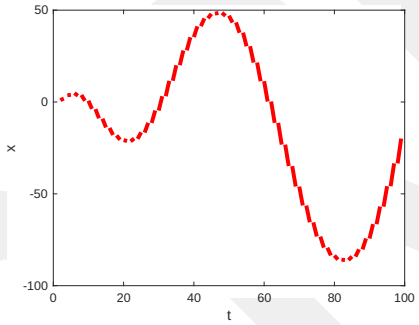

FIGURE 1. Graph of the solution of Eq. (27) for $t\in[2,100]_{\mathbb{T}}$.

Remark 4.1. Note that when the impulse conditions are dropped in (27), the equation becomes a nonoscillatory Cauchy-Euler type dynamic equation

$$t\sigma(t)x^{\Delta\Delta}+tx^{\Delta}-(1/4)x^{\sigma}=0,\qquad t\in\mathbb{P}_{1,1}.$$

In other words, the oscillatory behavior of solutions results from impulse actions.

Example 4.2. Consider the non-homogeneous dynamic impulsive equation

$$\begin{cases}(t^2x^{\Delta})^{\Delta}-\Big(1+\frac{t}{\sigma(t)}\Big)x^{\sigma}=\frac{1}{t}, & t\neq 2i,\quad t\geq 2,\\ \Delta_0 x+\dfrac{x}{i+1}=\dfrac{1}{i^5}, & \Delta_0(t^2x^{\Delta})+\dfrac{i}{i+1}x^{\sigma}+\dfrac{2i^2}{i+1}x^{\Delta}=(-1)^i i^4, \quad t=2i,\quad i>1\end{cases} \tag{30}$$

on the time scale $\mathbb{T}=\mathbb{P}_{1,1}$. Clearly, $t_i=2i$, $a(t)=t^2$, $b(t)=-1-t/(\sigma(t))$, $f(t)=1/t$ $a_i=1/(i+1)$, $g_i=1/i^5$, $b_i=i/(i+1)$, $c_i=2i^2/(i+1)$, and $h_i=(-1)^i i^4$. Let $s\in(2j-2,2j]$ and $\tau\in(2i-2,2i]$ for $j>i>2$. Then, $\underline{n}(\tau)=i-1$, $\overline{n}(s)=j$, and so

$$\eta(s,\tau)=\frac{\underline{n}(\tau)(2\underline{n}(\tau)+1)}{2(\overline{n}(s)+1)^2}=\frac{(i-1)(2i-1)}{2(j+1)^2}.$$

It is not difficult to verify that $v(t)=t/i$, $t\in(2i,2i+2]_{\mathbb{T}}$ is a nonprincipal solution. Hence

$$M_i=\frac{4(i+1)}{2i+1}\Big[(-1)^i i^4-\frac{2}{i^4}\Big].$$

Let $t_0 = 2$. From

$$v^{\sigma}(\tau) = \frac{\sigma(\tau)}{i-1} = \begin{cases} \frac{\tau}{i-1}, & \tau \in (2i-2, 2i), \\ \frac{\tau+1}{i-1} = \frac{2i+1}{i-1}, & \tau = 2i \end{cases}$$

we have

$$\begin{aligned} F(s) := \int_2^s \eta(s,\tau) f(\tau) v^{\sigma}(\tau)\, \Delta\tau &= \sum_{i=2}^{j-1} \int_{2i-2}^{2i} \frac{\eta(s,\tau)}{i-1}\, \Delta\tau + \int_{2j-2}^{s} \frac{\eta(s,\tau)}{j-1}\, \Delta\tau \\ &= \frac{1}{2(j+1)^2} \Bigg(\sum_{i=2}^{j-1} (2i-1) \Big[\int_{2i-2}^{2i} \mathrm{d}\tau + (2j-1) \int_{2j-2}^{s} \mathrm{d}\tau \Bigg) \\ &= \frac{j(j-2)}{(j+1)^2} + O(1) = O(1), \quad j \to \infty. \end{aligned} \tag{31}$$

Now, by using $\eta(s, t_i) = i(4i+1)/(j+1)^2$, we can write that

$$\begin{aligned} G(s) := \sum_{i=2}^{\overline{n}(s)} \eta(s,t_i) M_i = \sum_{i=2}^{j} \eta(s,2i) M_i &= \frac{4}{(j+1)^2} \sum_{i=2}^{j} \frac{i(4i+1)(i+1)}{2i+1} \Big[(-1)^i (2i+1) i^4 - \frac{2}{i^4} \Big] \\ &= \frac{8(-1)^j j^7}{(j+1)^2} + + O(j^6), \quad j \to \infty. \end{aligned} \tag{32}$$

In view of

$$\frac{1}{a(s) v(s) v^{\sigma}(s)} = (j-1)^2 / (s^3 \sigma(s)), \quad s \in (2j-2, 2j],$$

from (31) and (32), we have

$$\int_2^t \frac{1}{a(s) v(s) v^{\sigma}(s)} \big[F(s) + G(s) \big]\, \Delta s = (-1)^i i^2 + O(i), \quad t \in (2i, 2i+2]_{\mathbb{T}}, \ \ i \to \infty. \tag{33}$$

It is not difficult to see also that

$$\sum_{i=2}^{\overline{n}(t)} \frac{g_i}{(1-a_i) v(t_i)} = \sum_{\eta_1=2}^{i} \frac{\eta_1 + 1}{2\eta_1^6} = O(1), \quad i \to \infty. \tag{34}$$

From (33) and (34), we can write

$$\mathcal{H}(t) = (-1)^i i^2 + O(i), \quad i \to \infty,$$

which shows that the condition (22) holds. Applying Theorem 3.2, we may deduce that impulsive dynamic equation (30) is oscillatory. In the following illustration, the solution is drawn in the interval $[2, 100]$ so that the oscillatory behavior is easily observed.

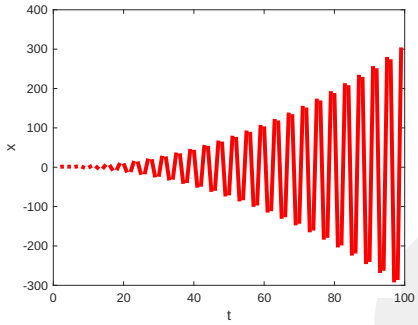

FIGURE 2. Graph of the solution of Eq. (30) for $t \in [2, 100]_{\mathbb{T}}$.

Remark 4.2. We should point out that the oscillation observed in the above example is due to the impulses. In the graph below, it can be seen that the non-impulsive dynamic equation

$$(t^2 x^{\Delta})^{\Delta} - \Big(1 + \frac{t}{\sigma(t)}\Big) x^{\sigma} = \frac{1}{t}, \quad t \geq 2, \ \mathbb{T} = \mathbb{P}_{1,1} \tag{35}$$

associated with (30) is non-oscillatory.

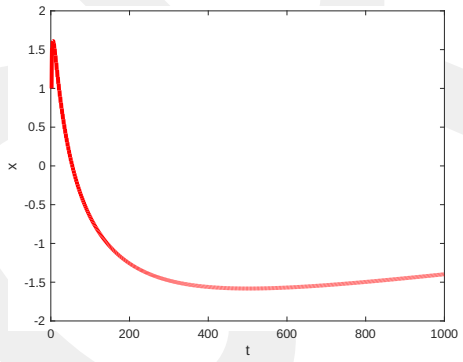

FIGURE 3. Graph of the solution of Eq. (35) for $t \in [2, 1000]_{\mathbb{T}}$.

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Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Code Availability Not applicable.

5. DECLARATIONS

Competing Interests Authors declare not to have financial interests that are directly or indirectly related to the work submitted for publication.

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