# SOLUTIONS OF INITIAL VALUE PROBLEMS OF CAUCHY TYPE IN BANACH SPACES

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# ABSTRACT

## SOLUTIONS OF INITIAL VALUE PROBLEMS OF CAUCHY TYPE IN BANACH SPACES

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This thesis consists of five chapters. The first chapter is devoted to the historical background and introductory concepts.

Partial complex differentiations in the classical sense and in the sense of Sobolev, generalized analytic functions, associated differential operators, associated spaces and interior estimates are introduced in Chapter II. The interior estimate for holomorphic functions in the supremum-norm is also obtained.

In Chapter III, first, the concept of scales of Banach spaces are presented. Then the proof of the abstract Cauchy-Kovalevskaya theorem for the existence and uniqueness of the solutions of initial-value problems of Cauchy type is presented by the method of successive approximations in the scales of Banach spaces.

In Chapter IV, initial value problems defined by Son and Tutschke [25], in the space of functions satisfying the Cauchy-Riemann system, for a system of linear first order partial differential equations for two unknown real-valued functions in the plane is considered. After rewriting the initial value problem in complex form, the solution of the corresponding problem is obtained by applying the abstract Cauchy-Kovalevskaya theorem in the space of holomorphic functions.

In the last chapter, an initial value problem for a first order evolution equation defined by N. Q. Hung [13] in the space of generalized regular functions in Quaternionic Analysis is discussed. Hung has proven only sufficient conditions for the related differential operators to be associated. We have obtained not only sufficient but also necessary conditions for the underlined differential operators to be associated [1]. Further we have corrected a mistake made in the calculation of the interior estimate in that paper.

Keywords: Abstract Cauchy-Kovalevskaya theorem, initial value problems of Cauchy type, associated differential operators, method of associated spaces, generalized analytic functions, quaternionic analysis.

# ÖZ


## CAUCHY TİPİ BAŞLANGIÇ DEĞER PROBLEMLERİNİN BANACH UZAYLARINDA ÇÖZÜMÜ

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Beş bölümden oluşan bu tezde ilk bölüm giriş için ayrıldı.

İkinci bölüm klasik anlamda ve Sobolev anlamında kompleks kısmi türevlere, genelleştirilmiş analitik fonksiyonlara ve iç kestirimlere ayrıldı. Ayrıca holomorf fonksiyonlar için bir iç kestirim supremum normunda elde edildi.

Üçüncü bölümde önce Banach uzayları skalaları tanıtıldı. Sonra Cauchy tipindeki başlangıç değer problemlerinin çözümlerinin varlık ve tekliği için soyut Cauchy-Kovalevskaya teoremi Banach uzayları skalalarında ardışık yaklaşıklıklar metodu yardımıyla kanıtlandı.

Dördüncü bölümde Son ve Tutschke [25] tarafından Cauchy-Riemann sistemini sağlayan bilinmeyen iki tane reel-değerli fonksiyon için tanımlanan, birinci basamaktan iki lineer kısmi türevli denklemin oluşturduğu sisteme ilişkin başlangıç değer problemleri ele alındı. Bu problemler önce kompleks formda yazıldı. Daha sonra karşılık gelen problemin çözümü soyut Cauchy-Kovalevskaya teoremi yardımıyla holomorf fonksiyonlar uzayında elde edildi.

Son bölümde N.Q. Hung [13] tarafından quaterniyon analizinde genelleştirilmiş regüler fonksiyonlar için tanımlanan birinci basamaktan bir evrim denklemine ilişkin başlangıç değer problemi incelendi. Hung bu probleme ilişkin diferansiyel operatörlerin eş olabilmesi için sadece yeter olan koşulları kanıtladı. Biz söz konusu operatörlerin eş olması için sadece yeter olan değil aynı zamanda gerek olan koşulları da elde ettik [1]. Bundan başka söz konusu makalede iç kestirim hesaplanırken yapılan bir hatayı da düzelttik.

Anahtar Kelimeler: Soyut Cauchy-Kovalevskaya teoremi, Cauchy tipinde başlangıç değer problemleri, eş diferansiyel operatörler, eş uzaylar metodu, genelleştirilmiş analitik fonksiyonlar, quaterniyon analizi.

*To my wife, Aziza, my daughter Khadija and my sons Muhammad and Yaseer*

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# TABLE OF CONTENTS

# LIST OF SYMBOLS

| | | |
|---|---|---|
| $\Omega$ | : | A domain in Euclidean space |
| m$\Omega$ | : | Measure of the domain $\Omega$ |
| $C^k(\Omega)$ | : | Space of $k$-times continuously differentiable functions defined in $\Omega$ real quaternion algebra |
| $C^k(\Omega, \mathbb{H})$ | : | Space of functions in $C^k(\Omega)$ taking values in real quaternion algebra $\mathbb{H}$ |
| $C^\alpha(M)$ | : | Space of Hölder-continuous functions with index $\alpha$ defined in $M$ |
| $\mathbb{C}$ | : | Set of complex numbers |
| $\Re(z)$ and $\Im(z)$ | : | Real and imaginary parts of $z$, respectively |
| $\mathcal{D}$ | : | Dirac operator of quaternionic analysis |
| $\Delta$ | : | Laplace operator |
| $\mathcal{L}$ | : | A general linear second order elliptic differential operator of divergence type |
| $\mathcal{L}^*$ | : | Adjoint of $\mathcal{L}$ |
| $\partial_j$ | : | Differential operator with respect to $x_j$ |
| $E(x, \xi)$ | : | A fundamental solution of a linear elliptic partial differential equation with singularity at $\xi$ |
| $\omega_n$ and $\tau_n$ | : | Surface measure and volume of the unit ball in $\mathbb{R}^n$, respectively |
| $\phi$ | : | A test function |
| $\mathcal{F}$ | : | A differential operator on the right hand side of an evolution equation |

# CHAPTER 1

# INTRODUCTION

The development of Complex Analysis was from the very beginning closely connected with the theory of partial differential equations. One of the highlights of these interactions was the development of the theory of generalized analytic functions [41], because it tied complex-analytic and functional-analytic ideas. Whereas the theory of generalized analytic functions is mainly aimed at solving boundary value problems, initial value problems can also be solved by using complex methods. The classical Cauchy-Kovalevskaya theorem is the first fundamental result in this direction. The functional-analytic approach to that theorem, started by M. Nagumo's [19] ideas, led to an abstract version of the Cauchy-Kovalevskaya theorem.

The purpose of the present thesis is to solve initial value problems of type

$$\partial_t u = \mathcal{F}(t, x, u, \partial_{x_i} u) \qquad (1.1)$$

$$u(0, x) = u_0(x) \qquad (1.2)$$

where $\mathcal{F}$ is a first order differential operator acting with respect to the spacelike variable $x$, and $t$ is the time. On the one hand, in view of the famous Lewy example [16] there are even linear differential equations of the form (1.1) with infinitely differentiable coefficients such that no solution exists (see also F. John's book [15]). This implies, especially, that no initial value problem is solvable for Lewy's equation. On the other hand, in view of the classical Cauchy-Kovalevskaya theorem the initial value problem (1.1), (1.2) is uniquely solvable by a power-series provided the right-hand side of the equation (1.1) and the initial function $u_0$ possess power-series representations. Since the power series can be interpreted as holomorphic functions in the framework of the complex analysis initial value problems of type (1.1), (1.2) can be solved within the framework of the complex analysis. This makes it possible to solve the initial value problem not only for holomorphic initial functions but also for more general initial functions such as generalized analytic functions [29].

Clifford-Analysis is the transfer of the methods of complex analysis onto higher dimensions. Holomorphic functions of the plane correspond to the so-called monogenic functions of the Clifford-Analysis, and one can similarly solve initial value problems with monogenic initial functions (see [4, 43, 44, 45], for instance).

The problem (1.1), (1.2) is equivalent to the integro-differential equation

$$u(t,x) = u_0(x) + \int_0^t \mathcal{F}(\tau, x, u(\tau, x), \partial_{x_i} u(\tau, x)) d\tau \tag{1.3}$$

(see [19]). Therefore the solutions of (1.1), (1.2) can be constructed as fixed points of the operator

$$U(t,x) = u_0(x) + \int_0^t \mathcal{F}(\tau, x, u(\tau, x), \partial_{x_i} u(\tau, x)) d\tau. \tag{1.4}$$

It is well-known that the classical Cauchy-Kovaleskaya theorem gives a unique solution to the problem (1.1), (1.2) considered in the case of complex analysis

$$\partial_t w = \mathcal{F}(t, z, w, \partial_z w) \tag{1.5}$$

$$w(0,z) = w_0(z) \tag{1.6}$$

provided $\mathcal{F}(t, z, w, \partial_z w)$ and $w_0(z)$ are holomorphic functions in its variables. The complex-valued solution $w(t,z)$ of (1.5), (1.6) is holomorphic in $z$ and uniquely determined. Lewy [15, 16] proved that this problem has no solution if $\mathcal{F}$ is not holomorphic. He constructed infinitely many differentiable functions $f(t, x, y)$ such that the equation

$$2i(x+iy)\,\partial_t w = \partial_x w + i\partial_y w + f(t,x,y)$$

has no solution. Therefore he showed that the integro-differential equation (1.3) has not always a solution even if $\mathcal{F}(t, x, u, \partial_{x_i} u)$ and $u_0(x)$ are infinitely many differentiable functions.

However the concept of associated spaces [29, 30] leads to conditions under which equation (1.3) has a solution. This concept is originated from complex analysis: In the holomorphic case (1.5), (1.6) the associated space is the space of holomorphic functions and the right-hand side $\mathcal{F}(t, z, w, \partial_z w)$ transforms the space of holomorphic functions into itself. In order to apply a fixed-point theorem, the operator (1.4) has to be estimated in a suitable function space whose elements depends only on the space-like variable $x$. This can be done by the so-called interior estimate for the associated space. Such an estimate describes the behaviour of the derivatives near the boundary. In the case of holomorphic functions, such estimates can be obtained by the Cauchy integral formula. Similar estimates can be shown in the framework of Clifford analysis. This makes it possible to solve initial value problems with monogenic initial functions [13, 43, 44].

Regard the differential equation

$$\mathcal{G}u = 0,$$

where $\mathcal{G}$ is a given differential operator. A second differential operator $\mathcal{F}$ is called associated to $\mathcal{G}$ if $\mathcal{F}$ transforms the set of all solutions to the differential equation $\mathcal{G}u = 0$ into itself, i.e., if

$$\mathcal{G}u = 0 \Rightarrow \mathcal{G}(\mathcal{F}u) = 0$$

(see [29]). Then the function space containing all solutions to the differential equation $\mathcal{G}u = 0$ is called an *associated space to* $\mathcal{F}$. According to the method of associated spaces, an initial value problem of type (1.1), (1.2) is solvable in case the initial function $u_0$ belongs to an associated space of the evolution operator $\mathcal{F}$ on the right-hand side of (1.1). The only condition on the associated space is that its elements have to satisfy an interior estimate of type

$$\left\|\partial_{x_j} u\right\|_{\Omega'} \leq \frac{\text{const}}{\text{dist}(\Omega', \partial\Omega'')} \|u\|_{\Omega''}$$

where $\text{dist}(\Omega', \partial\Omega'')$ is the distance of the domain $\Omega'$ from the boundary $\partial\Omega''$ of a larger domain $\Omega''$ [30].

# CHAPTER 2

# THE METHOD OF ASSOCIATED SPACES

## 2.1 Introduction

This chapter deals with the important basic tools for this study such as partial complex differentiations in the classical sense and according to Sobolev, generalized analytic functions, associated differential operators, associated spaces and interior estimates (see [10, 29, 30, 11, 32, 34]).

## 2.2 Partial complex derivatives in the classical sense

Let $\Omega$ be a given domain in the $z$-plane, $z = x + iy$. Assume, further, that $w = f(z)$ is a (complex-valued) continuously differentiable function defined in the domain $\Omega$. Take any point $z_0 = x_0 + iy_0$ belonging to $\Omega$. Then the linearization $\widetilde{f}$ of $f$ with respect to $z_0$ is defined by

$$\widetilde{f}(z) = f(z_0) + c_1(x - x_0) + c_2(y - y_0) \tag{2.1}$$

where

$$c_1 = \frac{\partial f}{\partial x}(x_0, y_0) \text{ and } c_2 = \frac{\partial f}{\partial y}(x_0, y_0).$$

Since

$$(z - z_0) = (x - x_0) + i(y - y_0)$$

and

$$\overline{(z - z_0)} = (x - x_0) - i(y - y_0),$$

we have

$$x - x_0 = \frac{1}{2}\left((z - z_0) + \overline{(z - z_0)}\right),$$

and

$$y - y_0 = \frac{i}{2}\left(\overline{(z - z_0)} - (z - z_0)\right).$$

Substituting these expressions into (2.1), we obtain

$$\begin{aligned}
\widetilde{f}(z) &= f(z_0) + c_1\left[\frac{1}{2}\left((z - z_0) + \overline{(z - z_0)}\right)\right] + c_2\left[\frac{i}{2}\left(\overline{(z - z_0)} - (z - z_0)\right)\right] \\
&= f(z_0) + \frac{\partial f}{\partial x}(z_0)\left[\frac{1}{2}\left((z - z_0) + \overline{(z - z_0)}\right)\right] + \frac{\partial f}{\partial y}(z_0)\left[\frac{i}{2}\left(\overline{(z - z_0)} - (z - z_0)\right)\right] \\
&= f(z_0) + \frac{1}{2}\left[\frac{\partial f}{\partial x}(z_0) - i\frac{\partial f}{\partial y}(z_0)\right](z - z_0) + \frac{1}{2}\left[\frac{\partial f}{\partial x}(z_0) + i\frac{\partial f}{\partial y}(z_0)\right]\overline{(z - z_0)}
\end{aligned}$$

or

$$\widetilde{f}(z) = f(z_0) + d_1(z - z_0) + d_2\overline{(z - z_0)}$$

where

$$\begin{aligned}
d_1 &= \frac{1}{2}\left[\frac{\partial f}{\partial x}(z_0) - i\frac{\partial f}{\partial y}(z_0)\right] \text{ and} \\
d_2 &= \frac{1}{2}\left[\frac{\partial f}{\partial x}(z_0) + i\frac{\partial f}{\partial y}(z_0)\right].
\end{aligned}$$

The coefficients $d_1$ and $d_2$ are called the **partial complex derivatives** of $f$ with respect to $z$ and $\bar{z}$ at the point $z_0$ and are denoted by

$$\frac{\partial f}{\partial z}(z_0) \text{ and } \frac{\partial f}{\partial \bar{z}}(z_0),$$

respectively. Thus the partial complex derivatives are defined by

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right] \tag{2.2}$$

and

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right]. \tag{2.3}$$

The last two formulae allow us, further, to express the derivatives

$$\frac{\partial f}{\partial x} \text{ and } \frac{\partial f}{\partial y} \text{ by } \frac{\partial f}{\partial z} \text{ and } \frac{\partial f}{\partial \bar{z}}.$$

The partial complex differentiations $\partial/\partial\bar{z}$ and $\partial/\partial z$ defined above satisfy the well-known rules about the derivative of linear combinations, products and so on, too. For example, the derivative of the product $f_1 f_2$ with respect to $\bar{z}$ is given by

$$\frac{\partial}{\partial \bar{z}}(f_1 f_2) = f_1\frac{\partial f_2}{\partial \bar{z}} + f_2\frac{\partial f_1}{\partial \bar{z}}.$$

The composition $g \circ f$ of two differentiable functions $w = f(z)$ and $W = g(w)$ is differentiable, too, and the chain rule may be written in the form

$$\frac{\partial}{\partial z}(g \circ f)(z) = \frac{\partial g}{\partial w}\frac{\partial w}{\partial z} + \frac{\partial g}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial z},$$

and

$$\frac{\partial}{\partial \overline{z}}(g \circ f)(z) = \frac{\partial g}{\partial w}\frac{\partial w}{\partial \overline{z}} + \frac{\partial g}{\partial \overline{w}}\frac{\partial \overline{w}}{\partial \overline{z}}.$$

Note, finally, that

$$\frac{\partial \overline{w}}{\partial z} = \overline{\left(\frac{\partial w}{\partial \overline{z}}\right)}, \text{ and } \frac{\partial \overline{w}}{\partial \overline{z}} = \overline{\left(\frac{\partial w}{\partial z}\right)}$$

hold.

Suppose that $f = u + iv$ is holomorphic. Then the real part $u$ and the imaginary part $v$ of $f$ satisfy the well-known Cauchy-Riemann system

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \tag{2.4}$$

Thus holomorphic functions satisfy the relation

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = -i\frac{\partial f}{\partial y} \tag{2.5}$$

In view of (2.2) we get, consequently, the relation

$$\frac{\partial f}{\partial \overline{z}} = 0 \tag{2.6}$$

for any holomorphic function $f$. It may be added that the last equation (2.6) is nothing else than the Cauchy-Riemann system (2.4). Further, in view of (2.3) and (2.5), we conclude for any holomorphic function $f$ that

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{df}{dz} \\ &= \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right] \\ &= \frac{1}{2}\left[-i\frac{\partial f}{\partial y} - i\frac{\partial f}{\partial y}\right] \\ &= -i\frac{\partial f}{\partial y} \end{aligned}$$

or

$$\begin{aligned}\frac{\partial f}{\partial z} &= \frac{df}{dz}\\ &= \frac{1}{2}\left[\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right]\\ &= \frac{1}{2}\left[\frac{\partial f}{\partial x} + \frac{\partial f}{\partial x}\right]\\ &= \frac{\partial f}{\partial x}.\end{aligned}$$

## 2.3 Partial complex derivatives in the sense of Sobolev

If the (real- or complex-valued) function $h$ is continuously differentiable in the closure of a domain $G$ with a smooth boundary $\partial G$ the Ostrogradski-Gauss integral formulae

$$\iint_G \frac{\partial h}{\partial x} dxdy = \int_{\partial G} hdy \tag{2.7}$$

$$\iint_G \frac{\partial h}{\partial y} dxdy = -\int_{\partial G} hdx \tag{2.8}$$

are well-known in the case of the plane. Multiplying (2.8) by $-i$ and adding the resulting expression to (2.7), in view of the definition (2.2) and the relation

$$dy - idx = -i(dx + idy) = -idz$$

one gets

$$\iint_G \frac{\partial h}{\partial \bar{z}} dxdy = \frac{1}{2i} \int_{\partial G} hdz. \tag{2.9}$$

Analogously, we obtain

$$\iint_G \frac{\partial h}{\partial z} dxdy = -\frac{1}{2i} \int_{\partial G} hd\bar{z}, \tag{2.10}$$

by subtracting the equation (2.8) after multiplication with $i$ from the equation (2.7) if one takes into consideration the definition (2.3) and the relation

$$dy + idx = i(dx - idy) = id\bar{z}.$$

Assume that $f$ is a given continuously differentiable function defined in the domain $G$ of the $z$-plane. Assume, further, that $\phi$ is any test function, that is, $\phi$ is a continuously differentiable function in $G$ which is identically equal to zero outside a compact subset of $G$. Therefore $\phi$ and $f\phi$ vanish everywhere in a neighbourhood of the boundary $\partial G$ of $G$. Set $h = f\phi$. Applying the complex form (2.9) of the Ostrogradski-Gauss integral formula to $h$, we get

$$\iint_G \frac{\partial}{\partial \bar{z}} (f\phi)\, dxdy = 0$$

or

$$\iint_G \left( f\frac{\partial \phi}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{z}}\phi \right) dxdy = 0 \qquad (2.11)$$

for each test function $\phi$. Denote the derivative $\partial f/\partial \bar{z}$ by $g$. Hence we have

$$\iint_G \left( f\frac{\partial \phi}{\partial \bar{z}} + g\phi \right) dxdy = 0 \qquad (2.12)$$

for each test function $\phi$ if $g$ is the derivative of $f$ with respect to $\bar{z}$.

**Definition 2.3.1 ([33])** *Let $f$ be any (integrable) function not assumed to be continuously differentiable. If there exists an (integrable) function $g$ such that relation (2.12) is satisfied for each (continuously differentiable) test function $\phi$, then the function $g$ is said to be* ***the derivative of $f$ in Sobolev's sense with respect to*** *$\bar{z}$ and denoted by $\partial f/\partial \bar{z}$ like the classical derivative.*

By comparison of (2.11) and (2.12) we observe immediately that the classical derivative $\partial f/\partial \bar{z}$, in the case of its existence, may be interpreted as derivative in Sobolev's sense, too.

Starting from the Ostrogradski-Gauss integral formula (2.10), one defines **the derivative $g = \partial f/\partial z$ in Sobolev's sense** analogously by the relation

$$\iint_G \left( f\frac{\partial \phi}{\partial z} + g\phi \right) dxdy = 0$$

which has to be satisfied by every test function $\phi$.

These derivatives are also known as **generalized derivatives**, **distributional derivatives** or **derivatives in weak sense**.

Let $f$ be an arbitrary holomorphic function in $G$, i.e. the equation (2.6) holds. In view of (2.12) the function $f$ satisfies the relation

$$\iint_G f \frac{\partial \phi}{\partial \bar{z}} dxdy = 0 \qquad (2.13)$$

for every test function $\phi$.

Conversely assume now that $f$ is any (integrable) function satisfying (2.13) for every test function $\phi$. Then the famous Weyl lemma states that $f$ is necessarily a holomorphic function in the ordinary sense (an elementary proof of Weyl's lemma provided $f$ is continuous is given in [28, 33], for instance).

### 2.4 Generalized analytic functions

Let $\Omega$ be a given domain in the $z$-plane and $a = a(z)$, $b = b(z)$ be two given functions defined and continuous in $\Omega$. Then every solution $w = w(z)$ to the differential equation

$$\frac{\partial w}{\partial \bar{z}} = a(z)w + b(z)\overline{w} \qquad (2.14)$$

is called a **generalized analytic function** [41]. Remark that the derivative $\partial w/\partial \bar{z}$ may be understood in Sobolev's sense. To simplify matters we assume that $w$ as well as $\partial w/\partial \bar{z}$ are continuous and require, consequently, that the differential equation (2.14) is satisfied pointwise at each point of $\Omega$. If the coefficients $a(z)$ and $b(z)$ are equal to zero at each point $z$ of $\Omega$, then the equation (2.14) passes into $\partial w/\partial \bar{z} = 0$, and $w$ turns out to be a holomorphic function.

#### 2.4.1 Pompeiu Integral Operators

**Definition 2.4.1 ([41])** *Suppose $G$ is a domain in $\mathbb{C}$. Also suppose that $F(z, \zeta)$ is a bounded function for each $z$, $\zeta$ in $G$. Then*

$$K(z, \zeta) = \frac{F(z, \zeta)}{(\zeta - z)^{\alpha}}, \quad 0 < \alpha \leq 2$$

*is called a kernel with a **weak** or **strong singularity** depending on whether $\alpha < 2$ or $\alpha = 2$, respectively. The operator*

$$T_G w(z) = \iint_G K(z, \zeta) w(\zeta) d\xi d\eta, \quad \zeta = \xi + i\eta$$

*is called the corresponding singular integral operator.*

**Definition 2.4.2 ([41])** *Let $G$ be a domain in $\mathbb{C}$ and let $f \in L^1(D)$. Then the operator $T_G$ defined by*

$$T_G f(z) := -\frac{1}{\pi} \iint_G f(\zeta) \frac{d\xi d\eta}{\zeta - z}, \quad z \in \mathbb{C},$$

*is called the Pompeiu operator. By $\overline{T}_G$ we denote*

$$\overline{T}_G f(z) := -\frac{1}{\pi} \iint_G f(\zeta) \frac{d\xi d\eta}{\overline{\zeta - z}}, \quad z \in \mathbb{C}.$$

Thus the Pompeiu operator $T_G$ is a weakly singular operator.

**Theorem 2.4.3 ([41])** *If $f \in L^1(D)$ then $T_G f$ has the derivative in Sobolev's sense with respect to $\bar{z}$ equal to $f$, i.e.,*

$$\frac{\partial}{\partial \bar{z}} T_G f = f$$

*holds.*

**Theorem 2.4.4 ([41])** *If $f \in L^p(D)$, $p > 1$, then $T_G f$ has the derivative in Sobolev's sense with respect to $z$ equal to $\Pi_G f$, i.e.,*

$$\frac{\partial}{\partial z} T_G f := \Pi_G f = -\frac{1}{\pi} \iint_G f(\zeta) \frac{d\xi d\eta}{(\zeta - z)^2}, \quad z \in \mathbb{C}$$

*holds.*

So the integral operator $\Pi_G$ is a strongly singular operator.

Notice that, according to the Theorem 2.4.3, the differential operator $\partial/\partial\bar{z}$ is the left-inverse of $T_G$. Let us additionally remark that the operator $\overline{T}_G$ satisfies the differential equation

$$\frac{\partial}{\partial z} \overline{T}_G f = f,$$

analogously.

### 2.4.2 The Hölder space $C^{\alpha}(M)$

**Definition 2.4.5** *A function h is said to be Hölder-continuous with the exponent (or index)* $\alpha$, $0 < \alpha \leq 1$, *if there exists a constant C such that*

$$|h(z_1) - h(z_2)| \leq C\,|z_1 - z_2|^{\alpha} \tag{2.15}$$

*for any two points* $z_1$, $z_2$ *belonging to the set in which h is defined. Notice that the constant C may depend on the choice of the function h. Provided (2.15) is satisfied with* $\alpha = 1$, *the function h is said to be Lipschitz-continuous.*

Let $M$ be a bounded subset of the $z$-plane. The space of all (complex-valued) functions defined in $M$ and Hölder-continuous with respect to the exponent $\alpha$, $0 < \alpha \leq 1$, is denoted by $C^{\alpha}(M)$. This space equipped with the so-called Hölder-norm

$$\|h\|_{C^{\alpha}(M)} = \max\left\{\sup_{M}|h(z)|\,,\,\sup_{z_1\neq z_2}\frac{|h(z_1)-h(z_2)|}{|z_1-z_2|^{\alpha}}\right\}$$

turns out to be a Banach space.

**Theorem 2.4.6 ([41])** *The* $T_G$*-operator is a linear and bounded operator mapping* $C^{\alpha}\left(\overline{G}\right)$ *into itself, where G is a given bounded domain in the z-plane.*

### 2.4.3 The Lebesgue space $L_p(\Omega)$

Let $p$ be a given real number with $p > 1$. The Lebesgue space $L_p(G)$ consists of all $f$ defined in $G$ for which $|f|^p$ is integrable in Lebesgue's sense. The space $L_p(G)$ equipped with so-called $L_p(G)$-norm

$$\|f\|_{L_p(G)} = \left(\int_G |f|^p\right)^{1/p}$$

turns out to be a Banach space (see [2], for instance).

**Theorem 2.4.7 ([41])** *The* $T_G$*- and* $\Pi_G$*-operators are linear and bounded operators mapping* $L_p(G)$ *into itself if* $p > 1$.

## 2.5 Method of Associated Spaces for Solving Initial Value Problems

Regard a differential equation

$$\mathcal{G}w = 0$$

where $\mathcal{G}$ is a given differential operator. Then a second differential operator $\mathcal{F}$ is called associated to $\mathcal{G}$ if $\mathcal{F}$ transforms the set of all solutions to the differential equation $\mathcal{G}w = 0$ into itself, i.e., if

$$\mathcal{G}w = 0 \Rightarrow \mathcal{G}(\mathcal{F}w) = 0.$$

The space **H** of all holomorphic functions defined in a fixed domain may be characterized by the differential equation

$$\mathcal{G}w := \frac{\partial}{\partial \overline{z}} w = 0$$

(cf. (2.6)). Since the derivative $\frac{\partial w}{\partial z}$ of a holomorphic function $w$ is holomorphic again we have

$$\frac{\partial}{\partial \overline{z}}(\frac{\partial w}{\partial z}) = 0.$$

Thus $\mathcal{F} = \partial/\partial z$ is associated to $\mathcal{G} = \partial/\partial \overline{z}$.

Now consider the case of generalized analytic functions and assume that $\mathcal{G}w = 0$ is identical with the differential equation (2.14), i.e., consider

$$\mathcal{G}w := \frac{\partial w}{\partial \overline{z}} - a(z)w - b(z)\overline{w} = 0. \tag{2.16}$$

In the special case that $a(z)$ and $b(z)$ are equal to zero everywhere we know already that $\mathcal{F} = \partial/\partial z$ is associated to $\mathcal{G}$. In the papers [29, 30] W. Tutschke obtains sufficient conditions on the coefficients of the first order operators of the form

$$\mathcal{F}w := C_1\partial_z w + C_2(2)\partial_{\overline{z}} w + A(z)w + B(z)\overline{w},$$

for which the space of generalized analytic functions defined by (2.16) is associated.

More generally, let $\mathcal{F}$ be a differential operator acting with respect to spacelike variables. Then a function space $X$ is said to be an associated space to $\mathcal{F}$ in case $\mathcal{F}$ transforms $X$ into itself. There are two basic problems in the theory of associated spaces: The direct problem is aimed at the construction of associated space $X$ to a given operator $\mathcal{F}$, while the inverse problem determines operators $\mathcal{F}$ to a given space $X$ such that $X$ is associated to $\mathcal{F}$ (see [30]).

Initial value problems of type

$$\begin{aligned} \partial_t u &= \mathcal{F}(t, x, u, \partial_i u) \\ u(0, x) &= u_0(x) \end{aligned}$$

can be solved by the method of associated spaces in the space of functions satisfying a differential equation

$$\mathcal{G}u = 0$$

with coefficients depending only on the spacelike variable $x$ provided the following conditions hold (see [29, 30]):

- $\mathcal{F}$ is associated with $\mathcal{G}$.
- The initial function $u_0$ belongs to the associated space of $\mathcal{F}$ that contains all solutions to the differential equation $\mathcal{G}u = 0$.
- The elements of the associated space satisfy a first order interior estimate of type

$$\left\|\partial_j u\right\|_{\Omega'} \leq \frac{\text{const}}{\text{dist}(\Omega', \partial\Omega'')} \|u\|_{\Omega''}$$

  where $\Omega'$ is a subset of $\Omega''$ having the distance $\text{dist}(\Omega', \partial\Omega'')$ from the boundary $\partial\Omega''$ of $\Omega''$.

## 2.6 Interior estimates

The simplest interior estimate can be obtained from the Cauchy Integral Formula for holomorphic functions. Suppose $\Phi$ is holomorphic in the bounded domain $\Omega$ and continuous in $\overline{\Omega}$. Consider an arbitrary point $z$ in $\Omega$. If $\delta$ is less than the distance $\text{dist}(z, \partial\Omega)$ of $z$ from the boundary $\partial\Omega$ of $\Omega$, then the Cauchy Integral Formula for $\Phi'$ implies

$$\Phi'(z) = \frac{1}{2\pi i} \int_{|\zeta - z| = \delta} \frac{\Phi(\zeta)}{(\zeta - z)^2} d\zeta.$$

Thus carrying out the limiting process $\delta \to \text{dist}(z, \partial\Omega)$ leads to the interior estimate

$$|\Phi'(z)| \leq \frac{1}{\text{dist}(z, \partial\Omega)} \|\Phi\|_\Omega ,$$

where $\|\cdot\|_{\Omega}$ means the supremum norm in $\Omega$. If $\Omega'$ is a subdomain of $\Omega''$, this interior estimate can be written also in the form

$$\|\Phi'(z)\|_{\Omega'} \leq \frac{1}{\operatorname{dist}(\Omega', \partial\Omega'')} \|\Phi\|_{\Omega''} .$$

Note that similar interior estimates for holomorphic functions are also true in other norms (such as the Hölder- and $\mathcal{L}_p$-norms, see [29], for instance).

Since generalized analytic functions can be represented by holomorphic ones, interior estimates for holomorphic functions lead also to interior estimates for generalized analytic functions. The main tool for getting such estimates are the well-known $T_{\Omega}$- and $\Pi_{\Omega}$-operators (see the Section 2.4.1). Now introduce the auxiliary function

$$\Phi = w - T_{\Omega}(aw + b\overline{w}). \tag{2.17}$$

Since

$$\partial_{\bar{z}}\Phi = \partial_{\bar{z}}w - (aw + b\overline{w}) = 0$$

the Weyl lemma implies that $\Phi$ is a holomorphic function. Differentiating (2.17) with respect to $z$, it follows that

$$\partial_z w = \Phi' + \Pi_{\Omega}(aw + b\overline{w}).$$

Using the boundedness of the $T_{\Omega}$- and $\Pi_{\Omega}$-operators (see Theorems 2.4.6 and 2.4.7), and using an interior estimate (in any norm) for holomorphic functions, the last formula leads to the desired interior estimate for generalized analytic functions. Notice, however, that the $\Pi_{\Omega}$-operator is not a bounded operator in the **supremum-norm**. Therefore no interior estimates for generalized analytic functions is available in the supremum norm, but there are such estimates of type

$$\|\partial_z w\|_{\Omega'} \leq \frac{C}{\operatorname{dist}(\Omega', \partial\Omega'')} \|w\|_{\Omega''} \tag{2.18}$$

in the **Hölder norm** and in the **$\mathcal{L}_p$-norm** as well (see [29, 41]).

Similarly interior estimates (also in the supremum norm) can also be obtained in case the associated space is defined by a higher order differential equation. Suppose, for instance, that the associated space is defined by the Laplace equation $\Delta u = 0$ in $\mathbb{R}^n$. Note that the Poisson Kernel for a ball of radius $\delta$ centred at $x_0$ is given by

$$\frac{1}{\omega_n \delta} . \frac{\delta^2 - |x - x_0|^2}{|\xi - x|^n},$$

where $\omega_n$ means the surface measure of the unit sphere in $\mathbb{R}^n$. This leads to an interior estimate of the form (2.18) with $C = n$ in the supremum-norm (see the Section 5.4, for the proof in case $n = 3$).

## 2.7 Associated spaces in quaternionic analysis

The field $\mathbb{H}$ of quaternions

$$x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3, \quad \text{where } x_0, x_1, x_2, x_3 \in \mathbb{R} \tag{2.19}$$

is a four dimensional non-commutative $\mathbb{R}$-field generated by four basis elements $e_0 = 1$, $e_1 = i$, $e_2 = j$ and $e_3 = k$ with the multiplication rule

$$e_1^2 = e_2^2 = e_3^2 = -1, \quad e_ie_j + e_je_i = -2\delta_{i,j} \quad \text{for } i, j = 1, 2, 3 \tag{2.20}$$

where $\delta_{i,j}$ is the Kronecker symbol.

A function $f$ defined in a bounded domain $\Omega \subset \mathbb{R}^4$ and takes values in $\mathbb{H}$ is a map

$$f : \Omega \to \mathbb{H}$$

and it can be represented in the form

$$f(x) = f_0(x) + \sum_{k=1}^{3} f_k(x)e_k.$$

Its conjugate $\overline{f}$ is defined as

$$\overline{f(x)} = f_0(x) - \sum_{k=1}^{3} f_k(x)e_k,$$

where the components $f_k(x)$ are real-valued functions of $x = (x_0, x_1, x_2, x_3) \in \Omega$.

The Cauchy-Fueter operator $\mathcal{D}$ is defined by

$$\mathcal{D} = \sum_{j=0}^{3} e_j\partial_{x_j}.$$

Clifford analysis replaces the Cauchy-Riemann operator $\partial_{\bar{z}}$ of the complex plane by the Cauchy-Riemann operator

$$\mathcal{D} = \sum_{j=0}^{n} e_j\partial_{x_j}$$

of $\mathbb{R}^{n+1}$ whose coordinates will be denoted by $x_0, x_1, .......x_n$. As usual, one has $e_j^2 = -1$ and $e_ie_j + e_je_i = -2\delta_{i,j}$ (for $i, j = 1, 2, \cdots, n$). While classical complex analysis solves initial value problems with holomorphic initial functions, Clifford analysis constructs solutions of initial value problems whose initial functions are (left-)monogenic, that is, they are solutions of the Cauchy-Riemann equation $\mathcal{D}u = 0$ in $\mathbb{R}^{n+1}$.

It is also possible to solve initial value problems with generalized monogenic initial functions. A generalized monogenic function is a solution of a differential equation of the form $\mathcal{D}u = \mathcal{F}(x, u)$. Explicit interior estimates can be obtained, for instance, for generalized monogenic functions which are solutions of a differential equation with anti-monogenic right-hand sides. This class of generalized monogenic functions is introduced in the paper [39] of U. Yüksel and the W. Tutschke. It generalizes the concept of generalized analytic functions $w(z)$ which satisfy a differential equation with anti-holomorphic right-hand sides, that is

$$\partial_{\bar{z}} w = b(z)\overline{w}.$$

Such functions have the property that $\Delta w$ is a linear combination of $w$ and $\overline{w}$. Analogously, a generalized monogenic function $u$ satisfies a differential equation with anti-monogenic right-hand sides. $\Delta u$ is a linear combination of the real-valued components of $u$. The paper [35] of Nguyen Thanh Van and Wolfgang Tutschke proves interior estimates for such generalized monogenic functions in the supremum norm.

# CHAPTER 3

# ABSTRACT CAUCHY-KOVALEVSKAYA THEOREM

## 3.1 Introduction

Scales of Banach spaces are certain families of Banach spaces depending on a real parameter. Such scales allow the initial value problem

$$\frac{dw}{dt} = \mathcal{F}(t, w(t)) \tag{3.1}$$

$$w(0) = w_0 \tag{3.2}$$

to be solved by the method of successive approximations also in the case of differential equations of type (3.3). In this way one gets an abstract version of the Cauchy-Kovalevskaya theorem (see [10, 29, 11, 27, 21]). The proof of the abstract Cauchy Kovalevskaya theorem is given in the Section 3.5.

Using the Cauchy-Kovalevskaya theorem initial value problems of a more general type

$$\frac{\partial w}{\partial t} = \mathcal{F}(t, w(t,\cdot), Dw(t,\cdot)), \tag{3.3}$$

$$w(0,\cdot) = \Phi, \tag{3.4}$$

can also be solved. Differential operators $D$ involved on the right-hand side $\mathcal{F}$ of (3.3) are not bounded, in general. That is why in a fixed Banach space the initial value problem (3.3), (3.4) cannot be solved by using the method of successive approximations. On the other hand, differential operators may be interpreted as bounded operators if they are regarded not in a fixed Banach space $B$ but in a suitably chosen family $B_s$ of Banach spaces.

An important example for such interpretation is given by holomorphic functions since their derivatives may be estimated by the Cauchy integral formula. Starting with an estimation of the derivatives of holomorphic functions we will elaborate the concept of scales of Banach spaces for solving initial value problems in case of partial differential equations by the method of successive approximations in the Section 3.2. The basic idea of this approach is the following: A partial differential equation can be replaced by an operator equation with bounded operators mapping a scale of Banach spaces into itself.

## 3.2 The behaviour of the derivative of a holomorphic function in compact subsets

Let $\Omega$ be a bounded domain in the $z$-plane. Let, further, $\mathbf{H}$ be the space of all complex-valued functions $w = w(z)$ defined and continuous in $\overline{\Omega}$ and holomorphic in $\Omega$. The space $\mathbf{H}$ equipped with the supremum norm

$$||w|| = \sup_{z\in\Omega} |w(z)| \tag{3.5}$$

turns out to be a normed space. The limit function of a Cauchy sequence of functions belonging to $\mathbf{H}$ is continuous since convergence with respect to the supremum norm means uniform convergence. In view of the Weierstrass' convergence theorem, moreover, the limit function of a uniformly convergent sequence of holomorphic functions is holomorphic, too. Thus the space $\mathbf{H}$ is proved to be complete, i.e., $\mathbf{H}$ is a Banach space. Now let $\Omega'$ be a further domain whose closure $\overline{\Omega'}$ is contained in the given domain $\Omega$. The closure $\overline{\Omega'}$ is, consequently, a compact subset of $\Omega$. Denote by $\delta$ the (positive) distance of $\Omega'$ from the boundary of $\Omega$. Then every closed disk centred at a point $z_0$ of $\Omega'$ belongs entirely to $\Omega$ if its radius $r$ is smaller than $\delta$. Applying the Cauchy integral formula, the value of the derivative $dw/dz$ at the point $z_0$ may be represented by the integral

$$\frac{dw}{dz}(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{w(z)}{(z-z_0)^2} dz.$$

Since the definition of the norm in $\mathbf{H}$ implies that $|w(z)| \leq ||w||$ everywhere in $\Omega$, one obtains the estimate

$$\left|\frac{dw}{dz}(z_0)\right| \leq \frac{1}{2\pi} \cdot \frac{||w||}{r^2} \cdot 2\pi r = \frac{1}{r}\, ||w||$$

This inequality holds for each $r < \delta$. If $r$ tends to $\delta$, one gets for each $z_0$ belonging to $\Omega'$ that

$$\left|\frac{dw}{dz}(z_0)\right| \leq \frac{1}{\delta} \|w\| . \tag{3.6}$$

In the same way as $\mathbf{H}$ corresponds to $\Omega$, the space $\mathbf{H}'$ is defined as the space of all complex-valued functions holomorphic in $\Omega'$ and continuous in $\overline{\Omega'}$. Then the space $\mathbf{H}'$ equipped with the supremum norm

$$\|w\|_{\Omega'} = \sup_{\Omega'} |w(z)| \tag{3.7}$$

turns out to be a Banach space, too. In order to distinguish the norm (3.5) in $\mathbf{H}$ from the norm (3.7) in $\mathbf{H}'$, the norm in $\mathbf{H}$ will be denoted by $\|\cdot\|_\Omega$ instead of $\|\cdot\|$, if necessary. Using these notations, it follows from (3.6) that the estimate

$$\left\|\frac{dw}{dz}\right\|_{\Omega'} \leq \frac{1}{\delta} \|w\|_\Omega$$

holds. In this way we have obtained the following statement:

**Theorem 3.2.1 ([29, 10])** *The complex differentiation is a bounded operator mapping* $\mathbf{H}$ *into* $\mathbf{H}'$ *whose norm can be estimated by*

$$\left\|\frac{d}{dz}\right\| \leq \frac{1}{\delta}. \tag{3.8}$$

## 3.3 Definition of scales of Banach spaces

Regard once more the two Banach spaces $\mathbf{H}$ and $\mathbf{H}'$ introduced in the Section 3.2. Not only the complex differentiation $\frac{d}{dz}$ may be interpreted as bounded linear operator mapping $\mathbf{H}$ into $\mathbf{H}'$ but also the restriction to $\Omega'$ of a function $w = w(z)$ belonging to $\mathbf{H}$ may be interpreted as such an operator. Denote this restriction by $\mathbf{I}$. Then $\mathbf{I}w$ is holomorphic in $\Omega'$ and continuous in $\overline{\Omega'}$ if $w = w(z)$ belongs to $\mathbf{H}$. Equipping both $\mathbf{H}$ and $\mathbf{H}'$ with the supremum norm, we get the estimate

$$\|w\|_{\Omega'} = \sup_{\Omega'} |w(z)| \leq \sup_{\Omega} |w(z)| \leq \|w\|_\Omega ,$$

i.e., $\mathbf{I}$ is proved to be an operator whose norm is not greater than 1, $\|\mathbf{I}\| \leq 1$. We know, further, the following property of a holomorphic function: If a holomorphic function defined in a domain $\Omega$ vanishes identically in a subdomain $\Omega'$, then the given holomorphic function must vanish identically in the whole domain $\Omega$. Therefore, two holomorphic functions must be identical in the whole domain $\Omega$ if they coincide in a subdomain $\Omega'$. The restrictions $\mathbf{I}w_1$ and $\mathbf{I}w_2$ to $\Omega'$ of two holomorphic functions $w_1$ and $w_2$ defined in the whole domain $\Omega$ are identical, consequently, if and only if $w_1 = w_2$ in the whole domain $\Omega$. Thus the equality $\mathbf{I}w_1 = \mathbf{I}w_2$ of the restrictions to $\Omega'$ implies $w_1 = w_2$ in the whole domain $\Omega$. Notice that an operator is said to be injective if two different elements possess images different from each other. Hence we can say that the restriction $\mathbf{I}$ mapping $\mathbf{H}$ into $\mathbf{H}'$ is injective. Summarizing these considerations. we have proved the following [29, 10]:

**Lemma 3.3.1** *The restriction* $\mathbf{I}$ *has the following three properties:*
*a)* $\mathbf{I}$ *is linear.*
*b)* $\mathbf{I}$ *is a bounded operator whose norm is not greater than 1.*
*c)* $\mathbf{I}$ *is injective.*

In the following we will regard not only two domains $\Omega$ and $\Omega'$ but a whole family of domains $\Omega_s$ where $s$ is a real parameter varying in an interval $0 < s < s_0$, where $s_0$ is a given finite number. Strictly speaking, if $\Omega$ is a given bounded domain in the $z$-plane, we choose a family of subdomains $\Omega_s$, $0 < s < s_0$, satisfying the following conditions:

1. the closure $\Omega_{s'}$ is a compact subset of $\Omega_s$ if only $s' < s$,
2. the distance of $\Omega_{s'}$, from the boundary $\partial\Omega_s$ of $\Omega_s$ can be estimated by

$$\operatorname{dist}(\Omega_{s'}, \partial\Omega_s) \geq \text{const} \cdot (s - s') \tag{3.9}$$

   where $s' < s$ and the constant is independent of $s'$ and $s$.

3. every point of $\Omega$ is contained in $\Omega_s$ for sufficiently large $s$.

**Example 3.3.2** *Let* $\Omega$ *be a disk centred at* $z_0$ *with radius* $r$. *Then a family* $\Omega_s$ *of subdomains, where* $s$ *varies in the interval* $0 < s < 1$, *is given by* $\Omega_s = \{z : |z - z_0| < sr\}$. *This family satisfies the three conditions formulated above. The constant entering into (3.9) is equal to* $r$.

Using this definition of an injection, we can say that the Banach spaces $\mathbf{H}_s$, $0 < s < s_0$, form a family, where each $\mathbf{H}_s$ is injected into each space $\mathbf{H}_{s'}$ with $s' < s$. Finally we introduce the concept of a scale of Banach spaces. A family of Banach spaces $B_s$, where $s$ varies in the open interval $0 < s < s_0$ is called a **Scale of Banach Spaces** if every $B_s$ is injected into each space $B_{s'}$ with a smaller $s'$. If a scale of Banach spaces is given, then we have not only a family of Banach spaces $B_s$, $0 < s < s_0$, but at the same time also a family of linear and injective operators $\mathbf{I}_{s,s'}$ with the norms $\left\|\mathbf{I}_{s,s'}\right\| \leq 1$ mapping $B_s$ into $B_{s'}$, $0 < s' < s < s_0$.

The family $\mathbf{H}_s$ is a special scale of Banach spaces.

### 3.4 Differential equations in scales of Banach spaces

In order to explain the concept of differential equations in scales of Banach spaces, we start with a simple but typical example. We look for a complex-valued function $w = w(t, z)$ depending on both a real variable $t$ and a complex variable $z$. Assume that this function depends holomorphically on $z$ for each $t$. Regard the partial differential equation

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial z} \tag{3.10}$$

where the differentiation $\frac{\partial}{\partial z}$ means the ordinary complex differentiation with respect to the variable $z$. Such interpretation is possible because $w = w(t, z)$ is supposed to be a holomorphic function in $z$. Now regard the scale $\mathbf{H}_s$, $0 < s < s_0$, of Banach spaces of holomorphic functions introduced in the Section 3.3. Then the right-hand side of the differential equation may be interpreted as an operator mapping each $\mathbf{H}_s$ into $\mathbf{H}_{s'}$, if $s' < s$. Since we look for solutions holomorphic in $z$ for each $t$ we may interpret the solution as an element of every Banach space $\mathbf{H}_s$ for each fixed $t$. Summarizing these interpretations of both the right-hand side and the solution of the differential equation (3.10), we get the following concept of differential equations in scales of Banach spaces:

Let $B_s$, $0 < s < s_0$, be a given scale of Banach spaces. Let, further, $\mathcal{F}(t, w(t))$ be an operator mapping each $B_s$ into each $B_{s'}$ if $s' < s$ and $t$ is fixed. Then we look for a function $w = w(t)$ belonging to each $B_s$ for every fixed $t$ and satisfying the differential equation (3.1), where the derivative $dw/dt$ is defined in every Banach space $B_s$ of the given scale. If we interpret $w(t)$ as an element of $B_s$, then $\mathcal{F}(t, w(t))$ belongs only to $B_{s'}$. Therefore, also $dw(t)/dt$ must be interpreted as an element of $B_{s'}$. Properly speaking, the left-hand side $dw/dt$ of (3.1) must be replaced by $\mathbf{I}_{s,s'} \frac{dw}{dt}$, where $\mathbf{I}_{s,s'}$ is the injective operator connecting $B_s$ with $B_{s'}$. For short we will use, however, the simplified denotation $dw/dt$.

In the Section 3.5 we shall look for a solution of this differential equation satisfying the additional condition (3.2) where $w_0$ is a given element belonging to every Banach space of the given scales $B_s$.

### 3.5 The Abstract Cauchy-Kovalevskaya theorem

Initial value problems of type (3.1), (3.2) can be solved using an abstract Cauchy-Kovalevskaya theorem. Here one starts from scales of Banach spaces, i.e. one has a family of (abstract) Banach spaces $B_s$ with norms $\|\cdot\|_s$, $0 < s < s_0$, which are embedded into each other. The latter means that $B_s$ is a subspace of $B_{s'}$ if $0 < s' < s < s_0$, where $\|w\|_{s'} \leq \|w\|_s$. In this case embedded means nothing but the restriction of a function in $\Omega_s$ to a smaller domain $\Omega_{s'}$.

The classical Cauchy-Kovalevskaya theorem is based on the fact that the derivative of a holomorphic function is holomorphic, again. Generalizing this property of holomorphic functions to the case of generalized analytic functions, one gets the concept of associated differential operators.

The Cauchy-Kovaleskaya theorem states that the initial value problem (3.1), (3.2) is solvable in an abstract scale of Banach space $B_s$, $0 < s < s_0$, provided the linear operator $\mathcal{F}$ has suitable properties in the given scale if the abstract operator equation comes from a partial differential equation. In many cases the operator $\mathcal{F}$ acts in a so-called associated space defined by a differential equation $\mathcal{G}w = 0$, that is the operator $\mathcal{F}$ is mapping an associated space into itself, where the space $B_s$ can be defined as the set of all solutions of $\mathcal{G}w = 0$, in a family of subdomains $\Omega_s$ exhausting the domain $\Omega$ in which the initial function $w_0$ is given.

In the sequel we will prove that the initial value problem (3.1), (3.2) is solvable by the method of successive approximations in the scale $B_s$, $0 < s < s_0$ (see [10, 29, 27, 21]). To introduce the required symbols, the usual case of a linear operator $\mathcal{F}$ acting in a fixedly chosen scale of Banach spaces $B_s, 0 < s < s_0$, equipped with norm $\|\cdot\|_s$. $\mathcal{F}(t, w(t))$ is a given linear and continuous mapping from $B_s$ into $B_{s'}$ for each $t$ with $0 < t < T$, where $0 < s' < s < s_0$. Suppose further that the conditions

$$\|\mathcal{F}(t,w) - \mathcal{F}(t,v)\|_{s'} \leq \frac{C}{s-s'}\|w-v\|_s \tag{3.11}$$

and

$$\|\mathcal{F}(t,w_0)\|_s \leq \frac{K}{s_0-s} \tag{3.12}$$

are satisfied, where $C$ and $K$ do not depend on $w$ and $v$, and again $s'$ is an arbitrary positive number being less than $s$. Then define the sequence $w^{(k)}(t)$ as the following successive approximations

$$w^{(k+1)}(t) = w_0 + \int_0^t \mathcal{F}(\tau, w^{(k)}(\tau))d\tau, \quad k = 0, 1, \cdots \tag{3.13}$$

where $w^{(0)} = w_0$.

**Theorem 3.5.1 ([10, 29, 27, 21])** *Suppose the conditions (3.11) and (3.12) are satisfied in a scale of Banach spaces $B_s$, $s \in (0, s_0)$, and let $w_0 \in B_s$ for all $s \in (0, s_0)$. Then the initial value problem (3.1), (3.2) is solvable by the successive approximations (3.13), i.e. the sequence $w^{(k+1)}(t)$ is convergent for any given $s \in (0, s_0)$ on the interval $t \in \left[0, \frac{s_0-s}{Ce}\right)$ and limit function $w_*(t) = \lim_{k\to\infty} w^{(k)}(t)$ on this interval represents a solution of the initial value problem (3.1), (3.2).*

**Proof.** Let us commence by proving the inequality

$$\|w^{(k+1)}(t) - w^{(k)}(t)\|_s \leq \frac{K}{Ce}\left(\frac{Cet}{s_0-s}\right)^{k+1}, \forall k \in \mathbb{N}, s \in (0, s_0) \tag{3.14}$$

by induction, where $C$ and $K$ are positive constants in the conditions *(3.11) and (3.12)* above.

For $k = 0$:

$$w^{(1)}(t) = w_0 + \int_0^t \mathcal{F}(\tau, w_0(\tau))d\tau,$$

$$
\begin{aligned}
\|w^{(1)}(t) - w_0\|_s &\leq \int\limits_0^t \|\mathcal{F}(\tau, w_0(\tau))\|_s d\tau \\
&\leq \int\limits_0^t \frac{K}{s_0 - s} d\tau \\
&\leq \frac{K}{s_0 - s} \cdot t
\end{aligned}
$$

So the inequality (3.14) is true for $k = 0$. Now assume as induction hypothesis that (3.14) is true for any $k$. Then we will show that (3.14) is true for $k + 1$, i.e. that

$$
\|w^{(k+2)}(t) - w^{(k+1)}(t)\|_s \leq \frac{K}{Ce} \left( \frac{Cet}{s_0 - s} \right)^{k+2}, \ \forall k \in \mathbb{N},\ s \in (0, s_0) \tag{3.15}
$$

holds.

For $k \to k + 1$:

$$
w^{(k+2)}(t) - w^{(k+1)}(t) = \int_0^t \left[ \mathcal{F}(\tau, w^{(k+1)}(\tau) - \mathcal{F}(\tau, w^{(k)}(\tau)) \right] d\tau,
$$

$$
\begin{aligned}
\|w^{(k+2)}(t) - w^{(k+1)}(t)\|_{s'} &\leq \int_0^t \|\mathcal{F}(\tau, w^{(k+1)}(\tau) - \mathcal{F}(\tau, w^{(k)}(\tau))\|_{s'} d\tau, \\
&\leq \int_0^t \frac{C}{s - s'} \cdot \|w^{(k+1)}(t) - w^{(k)}(t)\|_s d\tau \\
&\leq \frac{C}{s - s'} \int_0^t \|w^{(k+1)}(t) - w^{(k)}(t)\|_s d\tau \\
&\leq \frac{C}{s - s'} \cdot \frac{K}{Ce} \int_0^t \left( \frac{Ce\tau}{s_0 - s} \right)^{k+1} d\tau \\
&\leq \frac{C}{s - s'} \cdot \frac{K}{Ce} \cdot \left( \frac{Ce}{s_0 - s} \right)^{k+1} \int_0^t \tau^{k+1} d\tau \\
&\leq \frac{C}{s - s'} \cdot \frac{K}{Ce} \left( \frac{Ce}{s_0 - s} \right)^{k+1} \cdot \frac{t^{k+2}}{k + 2},
\end{aligned} \tag{3.16}
$$

where $0 < s' < s < s_0$. Suppose, for $0 < s' < s_0$, that

$$
s = s' + \frac{s_0 - s'}{k + 2},\ k = 0, 1, \cdots.
$$

Then we have

$$
\frac{1}{s_0 - s} = \frac{1 + \frac{1}{k+1}}{s_0 - s'} \tag{3.17}
$$

and

$$
\frac{1}{s - s'} = \frac{k + 2}{s_0 - s'}. \tag{3.18}
$$

Substituting (3.17) and (3.18) into (3.16) one gets

$$\begin{aligned}
\|w^{(k+2)}(t) - w^{(k+1)}(t)\|_{s'} &\leq C \cdot \frac{k+2}{s_0 - s'} \cdot \frac{K}{Ce} \left( Ce \frac{1 + \frac{1}{k+1}}{s_0 - s'} \right)^{k+1} \cdot \frac{t^{k+2}}{k+2} \\
&< C \left( \frac{1}{s_0 - s'} \right)^{k+2} \cdot \frac{K}{Ce} (Ce)^{k+1} \cdot e \cdot t^{k+2}
\end{aligned}$$

which implies, in view of

$$\frac{1}{s_0 - s'} \leq \frac{1}{s_0 - s},$$

that

$$\|w^{(k+2)}(t) - w^{(k+1)}(t)\|_{s'} \leq \frac{K}{Ce} \left( \frac{Cet}{s_0 - s} \right)^{k+2}. \tag{3.19}$$

Carrying out the limiting process $s' \to s$ in (3.19) leads to the inequality (3.15). Thus we have proved by induction that the inequality (3.14) is true.

Provided $t$ is restricted to the interval $0 \leq t \leq q\frac{s_0 - s}{Ce}$, where $q$ is any number with $0 < q < 1$, the inequality (3.14) leads to

$$\|w^{(k+1)}(t) - w^{(k)}(t)\|_s \leq \frac{K}{Ce} q^{k+1}, \ \forall k \in \mathbb{N},\ s \in (0, s_0).$$

Let $w^{(-1)}(t) \equiv 0$. Then

$$\begin{aligned}
\sum_{k=0}^{\infty} \left[ w^{(k)}(t) - w^{(k-1)}(t) \right] &= \lim_{n \to \infty} \sum_{k=0}^{n} \left[ w^{(k)}(t) - w^{(k-1)}(t) \right] \\
&= \lim_{n \to \infty} w^{(n)}(t) \\
&= w_*(t).
\end{aligned} \tag{3.20}$$

In view of the successive approximations (3.13) we have

$$\begin{aligned}
w_*(t) &= \lim_{n \to \infty} w^{(n)}(t) \\
&= \lim_{n \to \infty} \left[ w_0 + \int_0^t \mathcal{F}(\tau, w^{(n-1)}(\tau)) d\tau \right]
\end{aligned}$$

or

$$w_*(t) = w_0 + \int_0^t \mathcal{F}(\tau, w_*(\tau)) d\tau,$$

which leads to the initial value problem

$$\begin{aligned}
\frac{dw_*}{dt} &= \mathcal{F}(t, w_*) \\
w_*(0) &= w_0
\end{aligned}$$

by the help of the Leibnitz rule. So the limit function is a solution to the initial value problem *(3.1), (3.2)*.

On the other hand, it follows from (3.20) that

$$w_*(t) = \sum_{k=0}^{\infty} \left[ w^{(k)}(t) - w^{(k-1)}(t) \right]$$

or that

$$\begin{aligned} \|w_*(t)\|_s &\leq \sum_{k=0}^{\infty} \|w^{(k)}(t) - w^{(k-1)}(t)\|_s \\ &\leq \lim_{n\to\infty} \sum_{k=0}^{n} \|w^{(k)}(t) - w^{(k-1)}(t)\|_s \\ &\leq \frac{K}{Ce} \cdot \lim_{n\to\infty} \sum_{k=0}^{\infty} q^k \\ &= \frac{K}{Ce} \cdot \frac{1}{1-q} < \infty. \end{aligned}$$

This shows that $w_*(t) \in B_s$. This completes the proof of the theorem. □

Under the same restriction made for $t$ in the proof of the Theorem *3.5.1,* we will prove now that the error made at the $k^{th}$ approximation can be estimated by

$$\|w_*(t) - w^{(k)}(t)\|_s \leq \frac{K}{Ce} q^{k+1} \cdot \frac{1}{1-q}.$$

For this end consider first

$$\begin{aligned} \sum_{m=k}^{\infty} \left[ w^{(m+1)}(t) - w^{(m)}(t) \right] &= \lim_{n\to\infty} \sum_{m=k}^{n} \left[ w^{(m+1)}(t) - w^{(m)}(t) \right] \\ &= \lim_{n\to\infty} \left[ w^{(n+1)}(t) - w^{(k)}(t) \right] \\ &= w_*(t) - w^{(k)}(t). \end{aligned}$$

Then apply the norm in $B_s$ to the both sides of the last equality to obtain

$$\begin{aligned} \|w_*(t) - w^{(k)}(t)\|_s &\leq \lim_{n\to\infty} \sum_{m=k}^{n} \|w^{(m+1)}(t) - w^{(m)}(t)\|_s \\ &\leq \frac{K}{Ce} \cdot \lim_{n\to\infty} \sum_{m=k}^{n} q^{m+1} \\ &= \frac{K}{Ce} q^{k+1} \cdot \lim_{n\to\infty} \sum_{m=0}^{n} q^m \\ &= \frac{K}{Ce} q^{k+1} \cdot \frac{1}{1-q}. \end{aligned}$$

# CHAPTER 4

# FIRST ORDER DIFFERENTIAL OPERATORS ASSOCIATED TO THE CAUCHY-RIEMANN OPERATOR IN THE PLANE

## 4.1 Introduction

This chapter deals with initial value problems of type

$$\begin{aligned} \frac{\partial w}{\partial t} &= \mathcal{F} w \\ w(0, \cdot) &= \Phi \end{aligned}$$

where $t$ is the time and $\mathcal{F}$ is a linear first order differential operator acting in the $z$-plane. In case of the classical Cauchy-Kovalevkaya theorem, $\mathcal{F} w$ has the form

$$\mathcal{F} w := A(t, z) \frac{\partial w}{\partial z} + E(t, z) w + G(t, z) \tag{4.1}$$

and the initial value problem is solvable provided the coefficients $A$, $E$, and $G$ and the initial function $\Phi$ are holomorphic. On the other hand, the Lewy example [16] shows that there are equations of the above form with infinitely differentiable coefficients not having any solutions. We will construct, conversely, all linear operators $\mathcal{F}$ for which the initial value problem with an arbitrary holomorphic initial function is always solvable. In particular, we will see that there are equations of that type whose coefficients are only continuous (see [25]).

Consider the linear first order system

$$\partial_t u = a_{11}\partial_x u + a_{12}\partial_y u + a_{21}\partial_x v + a_{22}\partial_y v + c_1 u + c_2 v + c_3 \tag{4.2}$$

$$\partial_t v = b_{11}\partial_x u + b_{12}\partial_y u + b_{21}\partial_x v + b_{22}\partial_y v + d_1 u + d_2 v + d_3 \tag{4.3}$$

for two desired real-valued functions $u(t,x,y)$ and $v(t,x,y)$, where $t$ means the time and $(x,y)$ runs in a (bounded) domain in the $x,y$-plane. The coefficients are supposed to depend at least continuously on $t$, $x$ and $y$. On the one hand, in view of the classical Cauchy–Kovalevskaya Theorem the initial value problems

$$u(0,x,y) = \Phi(x,y) \tag{4.4}$$

$$v(0,x,y) = \Psi(x,y) \tag{4.5}$$

are solvable provided the coefficients of (4.2), (4.3), and the initial functions $\Phi$, $\Psi$ possess power-series representations (in $x$ and $y$). The present chapter formulates sufficient conditions on the coefficients of (4.2) and (4.3) under which each initial value problem (4.4) and (4.5) is solvable provided the initial functions $\Phi$, $\Psi$ satisfy the Cauchy–Riemann system. We will see that four coefficients of (4.2) and (4.3) can be chosen as arbitrary continuous functions. This result will be reached by the technique of associated differential operators applied to a complex rewriting of the initial value problems (4.2)-(4.5) (see [25]).

## 4.2 The Complex Rewriting of the Given System

Let us consider the complex variable $z = x + iy$ and $w = u + iv$. Then we have

$$\begin{aligned}
\partial_z w &= \frac{1}{2}\left[\left(\partial_x u + \partial_y v\right) + i\left(\partial_x v - \partial_y u\right)\right], \\
\overline{\partial_z w} &= \frac{1}{2}\left[\left(\partial_x u + \partial_y v\right) - i\left(\partial_x v - \partial_y u\right)\right], \\
\partial_{\bar z} w &= \frac{1}{2}\left[\left(\partial_x u - \partial_y v\right) + i\left(\partial_x v + \partial_y u\right)\right], \text{ and} \\
\overline{\partial_{\bar z} w} &= \frac{1}{2}\left[\left(\partial_x u - \partial_y v\right) - i\left(\partial_x v + \partial_y u\right)\right].
\end{aligned}$$

Thus the derivatives of $u$ and $v$ with respect to the real variables $x$ and $y$ can be expressed by the partial complex derivatives of $w$ with respect to $z$ and $\bar z$ as follows

$$\begin{aligned}
\partial_x u &= \frac{1}{2}\left(\partial_z w + \overline{\partial_z w} + \partial_{\bar z} w + \overline{\partial_{\bar z} w}\right), \\
\partial_y u &= \frac{i}{2}\left(\partial_z w - \overline{\partial_z w} - \partial_{\bar z} w + \overline{\partial_{\bar z} w}\right), \\
\partial_x v &= \frac{i}{2}\left(-\partial_z w + \overline{\partial_z w} - \partial_{\bar z} w + \overline{\partial_{\bar z} w}\right), \text{ and} \\
\partial_x v &= \frac{1}{2}\left(\partial_z w + \overline{\partial_z w} - \partial_{\bar z} w - \overline{\partial_{\bar z} w}\right).
\end{aligned} \tag{4.6}$$

Multiplying (4.3) by $i$ and adding the resulting expression to (4.2), in view of (4.6), and $\partial_t w = \partial_t u + i\partial_t v$ one can express the system (4.2), (4.3) in complex form as

$$\partial_t w = \mathcal{F} w \tag{4.7}$$

where

$$\mathcal{F} w := A\partial_z w + B\overline{\partial_z w} + C\partial_{\bar{z}} w + D\overline{\partial_{\bar{z}} w} + Ew + F\overline{w} + G, \tag{4.8}$$

and

$$\begin{aligned}
2A &= (a_{11} - b_{12} + b_{21} + a_{22}) + i(b_{11} + a_{12} - a_{21} + b_{22}) && (4.9)\\
2B &= (a_{11} + b_{12} - b_{21} + a_{22}) + i(b_{11} - a_{12} + a_{21} + b_{22}) && (4.10)\\
2C &= (a_{11} + b_{12} + b_{21} - a_{22}) + i(b_{11} - a_{12} - a_{21} - b_{22}) && (4.11)\\
2D &= (a_{11} - b_{12} - b_{21} - a_{22}) + i(b_{11} + a_{12} + a_{21} - b_{22}) && (4.12)\\
2E &= (c_1 + d_2) + i(d_1 - c_2) && (4.13)\\
2F &= (c_1 - d_2) + i(d_1 + c_2) && (4.14)\\
G &= (c_3 + id_3). && (4.15)
\end{aligned}$$

Since

$$\begin{aligned}
u(0, x, y) + iv(0, x, y) &= \Phi(x, y) + i\Psi(x, y) \text{ or}\\
u(0, z) + iv(0, z) &= \Phi(z) + i\Psi(z),
\end{aligned}$$

the complex rewriting of the initial conditions is

$$w(0, z) = w_0(z)$$

with

$$w_0(z) = \Phi(z) + i\Psi(z).$$

Therefore the initial value problems (4.2)-(4.5) can be written in complex form as

$$\begin{aligned}
\partial_t w &= \mathcal{F} w,\\
w(0, z) &= w_0(z).
\end{aligned}$$

## 4.3 Necessary and Sufficient Conditions for associated pairs

We formulate necessary and sufficient conditions on the coefficients of $\mathcal{F}$ defined by (4.8) under which $\mathcal{F}$ is associated with the Cauchy-Riemann operator $\mathcal{G} := \partial_{\bar{z}}$, i.e.

$$\mathcal{G}w = 0 \Rightarrow \mathcal{G}(\mathcal{F}w) = 0.$$

Assume now $\mathcal{G}w = 0$, that is $w_{\bar{z}} = 0$. Note that for any continuously differentiable function $w$ the properties

$$\begin{aligned} w_{\bar{z}} &= \overline{(\overline{w})_z} \text{ and} \\ w_z &= \overline{(\overline{w})_{\bar{z}}} \end{aligned}$$

hold. Applying $\mathcal{G}$ to $\mathcal{F}w$, as a consequence of the product rule, we get

$$(\mathcal{F}w)_{\bar{z}} = A_{\bar{z}}w_z + A(w_z)_{\bar{z}} + B_{\bar{z}}\overline{w_z} + B(\overline{w_z})_{\bar{z}} + E_{\bar{z}}w + Ew_{\bar{z}} + F_{\bar{z}}\overline{w} + F(\overline{w})_{\bar{z}} + G_{\bar{z}}$$

or

$$(\mathcal{F}w)_{\bar{z}} = A_{\bar{z}}w_z + B_{\bar{z}}\overline{w_z} + B\overline{w_{zz}} + E_{\bar{z}}w + F_{\bar{z}}\overline{w} + F\overline{w_z} + G_{\bar{z}}. \tag{4.16}$$

Thus $\mathcal{G}(\mathcal{F}w)$ will vanish if all the coefficients in (4.16) are equal to zero, i.e.

$$\begin{aligned} A_{\bar{z}} &= 0 \\ B_{\bar{z}} &= 0 \\ B &= 0 \\ E_{\bar{z}} &= 0 \\ F_{\bar{z}} &= 0 \\ F &= 0 \\ G_{\bar{z}} &= 0. \end{aligned}$$

Therefore we obtain the following sufficient conditions for $\mathcal{F}$ and $\mathcal{G}$ to be associated:

1. $B$ and $F$ are identically equal to zero.
2. $A$, $E$ and $G$ are holomorphic functions.

In the sequel we will prove that these conditions are also necessary. For this end, assume that $\mathcal{F}w$ is holomorphic if only $w$ is so. In order to obtain the conditions on the coefficients of the operator $\mathcal{F}$, we will choose special functions from the associated space, in this case special holomorphic functions, and obtain the conditions implying that $\mathcal{F}w$ is holomorphic for those functions. So choosing the holomorphic function $w \equiv 0$ in (4.8) we have

$$\mathcal{F}w = G.$$

Since $\mathcal{F}w$ must be holomorphic as an image of the holomorphic function $w \equiv 0$, $G$ has to be holomorphic, and term $G_{\bar{z}}$ can be omitted in (4.16).

Choose now the holomorphic function $w \equiv 1$. Then, by means of (4.8) we obtain

$$\mathcal{F}w = E \cdot 1 + F \cdot \overline{1}$$

and, thus

$$(\mathcal{F}w)_{\bar{z}} = E_{\bar{z}} + F_{\bar{z}} = 0 \tag{4.17}$$

must hold. Choose, similarly, the holomorphic function $w \equiv i$. So

$$\begin{aligned} \mathcal{F}w &= E \cdot i + F \cdot \bar{i} \\ &= (E - F) \cdot i \end{aligned}$$

and, thus,

$$(\mathcal{F}w)_{\bar{z}} = (E_{\bar{z}} - F_{\bar{z}})i = 0 \tag{4.18}$$

has to be satisfied. Equations (4.17) and (4.18) imply that $E$ and $F$ are holomorphic functions necessarily.

Similarly for $w \equiv z$, it follows from (4.16)

$$(\mathcal{F}w)_{\bar{z}} = A_{\bar{z}}(1) + B_{\bar{z}}\overline{(1)} + B(0) + F\overline{(1)} = 0$$

or

$$A_{\bar{z}} + B_{\bar{z}} + F = 0. \tag{4.19}$$

For the choice $w \equiv iz$, we get

$$(\mathcal{F}w)_{\bar{z}} = A_{\bar{z}}(i) + B_{\bar{z}}(-i) + B(0) + F(-i) = 0$$

or

$$A_{\bar{z}} - B_{\bar{z}} - F = 0. \tag{4.20}$$

Adding (4.19) to (4.20) yields $A_{\bar{z}} \equiv 0$, showing that $A$ is holomorphic. Subtracting (4.20) from (4.19) implies

$$B_{\bar{z}} + F = 0. \tag{4.21}$$

So far we know that $A$, $E$, $F$ and $G$ are holomorphic. It remains to choose $w \equiv z^2$ in (4.16) to get

$$(\mathcal{F}w)_{\bar{z}} = B \cdot \overline{w_{zz}} = B \cdot \overline{2} = 0$$

or

$$B \equiv 0.$$

Taking into account $B \equiv 0$ in (4.21) yields

$$F \equiv 0.$$

Summarizing the above considerations we have proved the following lemma:

**Lemma 4.3.1 ([25])** *Suppose $A$, $B$, $E$, $F$ and $G$ are continuously differentiable with respect to $z$ and $\bar{z}$. Then the operator (4.8) is associated to the Cauchy–Riemann operator if and only if the following conditions are satisfied:*

1. $B$ and $F$ are identically equal to zero.

2. $A$, $E$ and $G$ are holomorphic functions.

## 4.4 Permissible Coefficients

The above Lemma determines all complex equations (4.7) to which the Cauchy–Riemann system is associated. In the sequel we characterize the corresponding real systems (4.2) and (4.3). In view of (4.13), (4.14) and (4.15), the coefficients $c_j$ and $d_j$, $j = 1, 2$, are uniquely determined by the real and imaginary parts of two arbitrary holomorphic functions. Let $A_1 = 2\Re(A)$, $A_2 = 2\Im(A)$, $B_1 = 2\Re(B)$, $B_2 = 2\Im(B)$, $C_1 = 2\Re(C)$, $C_2 = 2\Im(C)$, $D_1 = 2\Re(D)$, and $D_2 = 2\Im(D)$. Then splitting up the equations (4.9)-(4.12) into real and imaginary parts, one gets the following 8 linear equations for the 8 coefficients $a_{ij}$, $b_{ij}$ where $i, j = 1, 2$:

$$
\begin{aligned}
a_{11} - b_{12} + b_{21} + a_{22} &= A_1 \\
b_{11} + a_{12} - a_{21} + b_{22} &= A_2 \\
a_{11} + b_{12} - b_{21} + a_{22} &= B_1 \\
b_{11} - a_{12} + a_{21} + b_{22} &= B_2 \\
a_{11} + b_{12} + b_{21} - a_{22} &= C_1 \\
b_{11} - a_{12} - a_{21} - b_{22} &= C_2 \\
a_{11} - b_{12} - b_{21} - a_{22} &= D_1 \\
b_{11} + a_{12} + a_{21} - b_{22} &= D_2.
\end{aligned}
$$

This system can be written in matrix form as follows:

$$
\begin{bmatrix}
1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 & 1 \\
1 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 \\
1 & 1 & 1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 \\
1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
a_{11} \\ b_{12} \\ b_{21} \\ a_{22} \\ b_{11} \\ a_{12} \\ a_{21} \\ b_{22}
\end{bmatrix}
=
\begin{bmatrix}
A_1 \\ A_2 \\ B_1 \\ B_2 \\ C_1 \\ C_2 \\ D_1 \\ D_2
\end{bmatrix}.
$$

The determinant of the coefficient matrix of this linear system is 256, i.e. nonzero, and therefore the $a_{ij}$ and $b_{ij}$ are uniquely determined in case the coefficients $A$, $B$, $C$ and $D$ are given. Observe, however, that $C$ and $D$ are completely free for systems to which the Cauchy–Riemann system is associated. In other words, for the desired associated operators 4 of the 8 coefficients can be chosen arbitrarily.

Notice that $A$ must be holomorphic and $B$ must be identically equal to zero in view of Lemma 4.3.1. Thus

$$
\begin{aligned}
B_1 &= a_{11} + b_{12} - b_{21} + a_{22} \equiv 0, \text{ and} \\
B_2 &= b_{11} - a_{12} + a_{21} + b_{22} \equiv 0
\end{aligned}
$$

which implies

$$a_{22} = -b_{12} + b_{21} - a_{11}, \text{ and}$$
$$a_{21} = -b_{11} - b_{22} + a_{12}. \tag{4.22}$$

So, taking (4.22) into account, we have

$$\begin{aligned} 2A &= (a_{11} - b_{12} + b_{21} + a_{22}) + i\,(b_{11} + a_{12} - a_{21} + b_{22}) \\ &= [a_{11} - b_{12} + b_{21} + (-b_{12} + b_{21} - a_{11})] + i\,[b_{11} + a_{12} - (-b_{11} - b_{22} + a_{12}) + b_{22}] \\ &= 2\,(-b_{12} + b_{21}) + 2i\,(b_{11} + b_{22}), \end{aligned}$$

showing that

$$A_1 = 2\Re(A) = 2\,(-b_{12} + b_{21}) \text{ and}$$
$$A_2 = 2\Im(A) = 2\,(b_{11} + b_{22}). \tag{4.23}$$

Considering (4.23) together with (4.22) we obtain, finally

$$\begin{aligned} a_{21} &= -A_2^* + a_{12} \\ a_{22} &= A_1^* - a_{11} \\ b_{21} &= A_1^* + b_{12} \\ b_{22} &= A_2^* - b_{11} \end{aligned} \tag{4.24}$$

where $A_1^* = A_1/2$ and $A_2^* = A_2/2$. To sum up we have proved the following statement:

**Lemma 4.4.1 ([25])** *Suppose $a_{11}$, $a_{12}$, $b_{11}$ and $b_{12}$ are arbitrarily chosen. Then permissible coefficients are given by (4.24) where $A_1^* + iA_2^*$ is an arbitrary holomorphic function.*

Note that 4 of the 14 coefficients of the systems (4.2) and (4.3) are completely free, whereas the remaining 10 coefficients depend on the choice of 3 arbitrary holomorphic functions. The 4 free ones need only to be continuous.

## 4.5 Solution of the initial value problem

In order to solve the initial value problems (4.2)-(4.5), consider an exhaustion of the (bounded) domain $\Omega$ by a family of subdomains $\Omega_s$, $B_s$ $0 < s < s_0$, satisfying the usual condition $\mathrm{dist}(\Omega_{s_1}, \partial\Omega_{s_2}) \geq const \cdot (s_2 - s_1)$ if and only $0 < s_1 < s_2 < s_0$. Let

$\mathbf{H}_s$ be the Banach space of functions holomorphic in $\Omega_s$ and continuous in $\overline{\Omega}_s$. Then the $\mathbf{H}_s$, $0 < s < s_0$ form a scale of Banach spaces. Rewrite the complex version of the given initial value problem as abstract operator equation in the scale $\mathbf{H}_s$. The necessary interior estimate for the complex derivative of a holomorphic function is obtained by means of Cauchy's Integral Formula (see Section 2.6). Therefore the abstract Cauchy–Kovalevskaya Theorem (see Chapter 3, see also [21, 27]) is applicable. Consequently, the following theorem has been proved:

**Theorem 4.5.1 ([25])** *Suppose the coefficients of the systems (4.2) and (4.3) are given in accordance with Section (4.4). Suppose, further, that the initial functions* $\Phi$ *and* $\Psi$ *in (4.4) and (4.5) satisfy the Cauchy–Riemann system. Then the initial value problems (4.2)-(4.5) is solvable. The solution exists at least in the time-interval* $0 \leq t < a_*(s_0 - s)$ *if* $a_*$ *is sufficiently small and* $(x, y)$ *belongs to* $\Omega_s$ *(where the subdomains* $\Omega_s$ *form an exhaustion of* $\Omega$*).*

## CHAPTER 5

# NECESSARY AND SUFFICIENT CONDITIONS FOR FIRST ORDER DIFFERENTIAL OPERATORS TO BE ASSOCIATED WITH A DISTURBED DIRAC OPERATOR IN QUATERNIONIC ANALYSIS

### 5.1 Introduction

In the paper [13] N. Q. Hung has solved the initial value problem

$$\frac{\partial u}{\partial t} = \mathcal{L}u := \sum_{i=1}^{3} A^{(i)}(t,x)\frac{\partial u}{\partial x_i} + B(t,x)u + C(t,x) \qquad (5.1)$$

$$u(0,x) = u_0(x) \qquad (5.2)$$

using the method of *associated spaces* in the space of generalized regular functions in the sense of quaternionic analysis satisfying the equation $\mathcal{D}_\alpha u = 0$, where

$$\mathcal{D}_\alpha u := \mathcal{D}u + \alpha u, \quad \alpha \in \mathbb{R}$$

and

$$\mathcal{D} = \sum_{j=1}^{3} e_j \frac{\partial}{\partial x_j}$$

is the Dirac operator, and $t$ is the time variable. The author has proven only sufficient conditions on the coefficients of the operator $\mathcal{L}$ under which $\mathcal{L}$ is associated with the operator $\mathcal{D}_\alpha$, i.e. $\mathcal{L}$ transforms the set of all solutions of the differential equation $\mathcal{D}_\alpha u = 0$ into solutions of the same equation for fixedly chosen $t$. We will prove necessary and sufficient conditions for the underlined operators to be associated. This criterion makes it possible to construct all linear operators $\mathcal{L}$ for which the initial value problem with an arbitrary initial generalized regular function is always solvable (see [1]).

## 5.2 Preliminaries and notations

Let $\mathbb{H}$ be a real quaternion algebra with the units $e_0 = 1$, $e_1 = i$, $e_2 = j$ and $e_3 = k$ of (real) quaternion algebra $\mathbb{H}$. Suppose that $\Omega$ is a bounded and simply connected domain of the Euclidean space $\mathbb{R}^3$. A function $f$ defined in the bounded domain $\Omega \subset \mathbb{R}^3$ and takes values in $\mathbb{H}$ is a map

$$f : \Omega \to \mathbb{H}$$

and it can be represented in the form

$$f(x) = f_0(x) + \sum_{k=1}^{3} f_k(x)e_k.$$

Its conjugate $\overline{f}$ is defined as

$$\overline{f(x)} = f_0(x) - \sum_{k=1}^{3} f_k(x)e_k,$$

where the components $f_k(x)$ are real-valued functions of $x = (x_1, x_2, x_3) \in \Omega$.

The Dirac operator $\mathcal{D}$ is defined by

$$\mathcal{D} = \sum_{j=1}^{3} e_j \partial_j,$$

where $\partial_j = \partial/\partial x_j$, and acts on $f$ (on the left-hand side) as follows

$$\mathcal{D}f = \sum_{j=1}^{3} \sum_{k=0}^{3} \left(\partial_j f_k\right) e_j e_k.$$

The conjugate of $\mathcal{D}$ is

$$\overline{\mathcal{D}} = -\sum_{j=1}^{3} e_j \partial_j.$$

We denote the space of continuous or $k$-times continuously differentiable functions defined in $\Omega$ and taking values in $\mathbb{H}$ by $C(\Omega, \mathbb{H})$, $C^k(\Omega, \mathbb{H})$, respectively.

**Lemma 5.2.1 ([5, 8])** *If* $u, v \in C^1(\Omega, \mathbb{H})$ *then*

$$\mathcal{D}(u \cdot v) = (\mathcal{D}u) \cdot v + \bar{u} \cdot (\mathcal{D}v) - 2\sum_{j=1}^{3} u_j \cdot \partial_j v. \tag{5.3}$$

**Definition 5.2.2 ([5, 8])** *A function* $f \in C^1(\Omega, \mathbb{H})$ *is said to be (left) regular if it satisfies* $\mathcal{D}f = 0$ *in* $\Omega$.

Note that if $u$ and $v$ are regular functions then

$$\mathcal{D}(u \cdot v) = -2\sum_{j=1}^{3} u_j \cdot \partial_j v$$

holds.

For further definitions concerning quaternions and regular functions we refer the reader to [5, 8].

Now we introduce the generalized regular function:

**Definition 5.2.3 ([13])** *The disturbed Dirac operator is the operator which is denoted by* $\mathcal{D}_\alpha$ *and defined as*

$$\mathcal{D}_\alpha u := \mathcal{D}u + \alpha u, \tag{5.4}$$

*where* $\alpha$ *is a real number.*

**Definition 5.2.4 ([13])** *A function* $u \in C^1(\Omega, \mathbb{H})$ *is called a generalized regular function if it satisfies* $\mathcal{D}_\alpha u = 0$ *in* $\Omega$.

Next we will prove three lemmas that will be used in the next section.

**Lemma 5.2.5 ([1])** *If* $u, v \in C^1(\Omega, \mathbb{H})$ *then*

$$\partial_i(u \cdot v) = \partial_i u \cdot v + u \cdot \partial_i v, \ \forall i \in \{1, 2, 3\}.$$

**Proof.** Let $u = \sum\limits_{k=0}^{3} u_k e_k,\, v = \sum\limits_{n=0}^{3} v_n e_n \in C^1(\Omega, \mathbb{H})$ and $i \in \{1,2,3\}$. Then

$$\begin{aligned}
\partial_i (u \cdot v) &= \partial_i \left( \sum_{k=0}^{3} u_k e_k \cdot \sum_{n=0}^{3} v_n e_n \right) \\
&= \sum_{k,n=0}^{3} \partial_i (u_k v_n) e_k e_n \\
&= \sum_{k,n=0}^{3} (\partial_i u_k \cdot v_n + u_k \cdot \partial_i v_n) e_k e_n \\
&= \partial_i \left( \sum_{k=0}^{3} u_k e_k \right) \cdot \sum_{n=0}^{3} v_n e_n + \sum_{k=0}^{3} u_k e_k \cdot \partial_i \left( \sum_{n=0}^{3} v_n e_n \right) \\
&= \partial_i u \cdot v + u \cdot \partial_i v.
\end{aligned}$$

□

**Lemma 5.2.6 ([1])** *If $u, v \in C^1(\Omega, \mathbb{H})$ then*

$$\mathcal{D}_\alpha (u \cdot v) = (\mathcal{D}_\alpha u) \cdot v + \bar{u} \cdot (\mathcal{D}_\alpha v) - 2 \sum_{j=1}^{3} u_j \cdot \partial_j v - \alpha (\bar{u} \cdot v). \tag{5.5}$$

**Proof.** Taking (5.4) and Lemma 5.2.5 into account we obtain

$$\begin{aligned}
\mathcal{D}_\alpha (u \cdot v) &= \mathcal{D}(u \cdot v) + \alpha (u \cdot v) \\
&= \sum_{i=1}^{3} e_i \partial_i (u \cdot v) + \alpha (u \cdot v) \\
&= \sum_{i=1}^{3} e_i (\partial_i u \cdot v + u \cdot \partial_i v) + \alpha (u \cdot v) \\
&= \left( \sum_{i=1}^{3} e_i \partial_i u \right) \cdot v + \sum_{i=1}^{3} e_i (u \cdot \partial_i v) + \alpha (u \cdot v) \\
&= \mathcal{D} u \cdot v + \alpha u \cdot v + \sum_{i=1}^{3} e_i (u \cdot \partial_i v) \\
&= (\mathcal{D}_\alpha u) \cdot v + \sum_{i=1}^{3} e_i (u \cdot \partial_i v).
\end{aligned}$$

After an easy calculation one can show that the summation in the last equality can be represented as

$$\sum_{i=1}^{3} e_i (u \cdot \partial_i v) = \bar{u} \cdot (\mathcal{D}_\alpha v) - 2 \sum_{j=1}^{3} u_j \cdot \partial_j v - \alpha (\bar{u} \cdot v).$$

Thus the proof is complete. □

Note that if $u$ and $v$ are generalized regular functions then

$$\mathcal{D}_\alpha (u \cdot v) = -2 \sum_{j=1}^{3} u_j \cdot \partial_j v - \alpha (\bar{u} \cdot v) .$$

Observe, further, that in case $\alpha = 0$, $\mathcal{D}_\alpha = \mathcal{D}$ and the formula (5.5) is reduced to the formula (5.3).

**Lemma 5.2.7 ([1])** *If $u \in C^2(\Omega, \mathbb{H})$ is generalized regular then any first order derivative of $u$ is also generalized regular.*

**Proof.** $\mathcal{D}_\alpha (\partial_i u) = \partial_i (\mathcal{D}_\alpha u) = 0, \ \forall i \in \{1, 2, 3\}$. □

## 5.3 Necessary and sufficient conditions for the associated differential operators

**Definition 5.3.1 ([29, 30])** *Let $\mathcal{F}$ be a first order differential operator depending on $t$, $x$, $u$ and on the first order partial derivatives $\partial_i u$, while $\mathcal{G}$ is a differential operator with respect to the space variables $x_i$ with coefficients not depending on time $t$. Then $\mathcal{F}$ is said to be 'associated' with $\mathcal{G}$ if $\mathcal{F}$ transforms the set of all solutions to the differential equation $\mathcal{G}u = 0$ into solutions of the same equation for fixedly chosen $t$, i.e.*

$$\mathcal{G}u = 0 \Rightarrow \mathcal{G}(\mathcal{F}u) = 0.$$

*The function space containing all solutions to the differential equation $\mathcal{G}u = 0$ is called an associated space of $\mathcal{F}$ (see also Section 2.5).*

Next we consider the operator $\mathcal{L}$ defined by

$$\mathcal{L}u := \sum_{i=1}^{3} A^{(i)}(t, x)\partial_i u + B(t, x)u + C(t, x), \tag{5.6}$$

having the quaternion-valued coefficients with real-valued components

$$\begin{aligned} A^{(i)}(t, x) &= \sum_{k=0}^{3} a_k^{(i)}(t, x)e_k, \\ B(t, x) &= \sum_{k=0}^{3} b_k(t, x)e_k, \end{aligned}$$

and $C(t,x)$, and determine the conditions over these coefficients guaranteeing that $\mathcal{L}$ is associated with $\mathcal{D}_\alpha$, i.e.

$$\mathcal{D}_\alpha u = 0 \Rightarrow \mathcal{D}_\alpha (\mathcal{L}u) = 0.$$

The equation

$$\mathcal{D}_\alpha u = 0 \tag{5.7}$$

implies

$$\partial_1 u = \sum_{j=2}^{3} e_1 e_j \partial_j u + \alpha e_1 u, \tag{5.8}$$

which shows that the derivatives of $u$ with respect to $x_1$ can be expressed in terms of the derivatives of $u$ with respect to $x_j$ $(j = 2, 3)$ and $u$ itself. This formula leads to the equality

$$\partial_k \partial_1 u = \sum_{j=2}^{3} e_1 e_j \partial_k \partial_j u + \alpha e_1 \partial_k u, \ \forall k \in \{1, 2, 3\}. \tag{5.9}$$

Applying $\mathcal{D}_\alpha$ to $\mathcal{L}u$ and taking Lemma 5.2.6, Lemma 5.2.7, (5.7), (5.8) and (5.9) into account it follows that $\mathcal{D}_\alpha (\mathcal{L}u)$ can be expressed as

$$\mathcal{D}_\alpha (\mathcal{L}u) = (-2P) \cdot \partial_2 \partial_3 u + \sum_{j=2}^{3} \left(-2Q_j\right) \cdot \partial_j^2 u + \sum_{i=2}^{3} R_i \cdot \partial_i u + S \cdot u + \mathcal{D}_\alpha C,$$

where

$$\begin{aligned}
P &:= \left[\left(a_2^{(1)} + a_1^{(2)}\right) e_1 e_3 + \left(a_3^{(1)} + a_1^{(3)}\right) e_1 e_2 + \left(a_3^{(2)} + a_2^{(3)}\right) e_0\right], \\
Q_j &:= \left[\left(a_j^{(j)} - a_1^{(1)}\right) e_0 + \left(a_j^{(1)} + a_1^{(j)}\right) e_1 e_j\right], \\
R_i &:= \left[\left(\mathcal{D}_\alpha A^{(1)} - \alpha \overline{A^{(1)}} - 2b_1\right) e_1 e_i + \right. \\
&\qquad \left. \mathcal{D}_\alpha A^{(i)} - \alpha \overline{A^{(i)}} - 2\alpha \left(a_i^{(1)} + a_1^{(i)}\right) e_1 - 2b_i\right], \\
S &:= \left[\left(\mathcal{D}_\alpha A^{(1)} - \alpha \overline{A^{(1)}}\right) \alpha e_1 + \mathcal{D}_\alpha B - \alpha \overline{B} + 2\alpha^2 a_1^{(1)} e_0 - 2\alpha b_1 e_1\right].
\end{aligned} \tag{5.10}$$

After an easy calculation one can show that $\mathcal{D}_\alpha (\mathcal{L}u)$ is equivalent to the following linear combination

$$
\begin{aligned}
&(-2)\left[\left(a_2^{(1)}+a_1^{(2)}\right)(-e_2)+\left(a_3^{(1)}+a_1^{(3)}\right)e_3+\left(a_3^{(2)}+a_2^{(3)}\right)e_0\right]\partial_2\partial_3 u+\\
&(-2)\left[\left(a_2^{(2)}-a_1^{(1)}\right)e_0+\left(a_2^{(1)}+a_1^{(2)}\right)e_3\right]\partial_2^2 u+\\
&(-2)\left[\left(a_3^{(3)}-a_1^{(1)}\right)e_0-\left(a_3^{(1)}+a_1^{(3)}\right)e_2\right]\partial_3^2 u+\\
&\left[\mathcal{D}\left(A^{(2)}+A^{(1)}e_3\right)+2\alpha\left(a_2^{(2)}-a_1^{(1)}\right)e_2+\right.\\
&\quad\left.2\left(\alpha a_3^{(2)}-b_1\right)e_3-2\left(\alpha a_3^{(1)}+b_2\right)e_0\right]\partial_2 u+\\
&\left[\mathcal{D}\left(A^{(3)}-A^{(1)}e_2\right)+2\alpha\left(a_3^{(3)}-a_1^{(1)}\right)e_3+\right.\\
&\quad\left.2\left(\alpha a_2^{(3)}+b_1\right)e_2+2\left(\alpha a_2^{(1)}-b_3\right)e_0\right]\partial_3 u+\\
&\left[\mathcal{D}\left(B+\alpha A^{(1)}e_1\right)+2\alpha\left(\alpha a_3^{(1)}+b_2\right)e_2-2\alpha\left(\alpha a_2^{(1)}-b_3\right)e_3\right]u+\\
&D_\alpha C.
\end{aligned}
\tag{5.11}
$$

Therefore the operator $\mathcal{L}$ is associated with the operator $\mathcal{D}_\alpha$ if the coefficients of this linear combination vanish identically, i.e. if the conditions

$$
\left.
\begin{aligned}
a_1^{(1)}=a_2^{(2)}&=a_3^{(3)}\\
a_2^{(1)}&=-a_1^{(2)}\\
a_3^{(1)}&=-a_1^{(3)}\\
a_3^{(2)}&=-a_2^{(3)}
\end{aligned}
\right\}
\tag{5.12}
$$

$$
\left.
\begin{aligned}
&\mathcal{D}\left(A^{(2)}+A^{(1)}e_3\right)+2\alpha\left(a_2^{(2)}-a_1^{(1)}\right)e_2+\\
&\qquad 2\left(\alpha a_3^{(2)}-b_1\right)e_3-2\left(\alpha a_3^{(1)}+b_2\right)e_0=0\\
&\mathcal{D}\left(A^{(3)}-A^{(1)}e_2\right)+2\alpha\left(a_3^{(3)}-a_1^{(1)}\right)e_3+\\
&\qquad 2\left(\alpha a_2^{(3)}+b_1\right)e_2+2\left(\alpha a_2^{(1)}-b_3\right)e_0=0
\end{aligned}
\right\}
\tag{5.13}
$$

$$
\mathcal{D}\left(B+\alpha A^{(1)}e_1\right)+2\alpha\left(\alpha a_3^{(1)}+b_2\right)e_2-2\alpha\left(\alpha a_2^{(1)}-b_3\right)e_3=0
\tag{5.14}
$$

$$
\mathcal{D}_\alpha C=0
\tag{5.15}
$$

hold.

Now we will prove the following statement:

**Theorem 5.3.2 ([1])** *Assume that $A^{(i)}(t,x)\in C^2(\Omega,\mathbb{H})$ (for $i=1,2,3$), and $B(t,x)$, $C(t,x)\in C^1(\Omega,\mathbb{H})$ for each $t\in[0,T]$. Then the operator $\mathcal{L}$ is associated with $\mathcal{D}_\alpha$ if and only if the conditions (5.12)-(5.15) are satisfied.*

**Proof.** Assume that $u$ is generalized regular. Then it is obvious in view of (5.11) that $\mathcal{L}u$ is also generalized regular in case the conditions of the theorem are satisfied.

Now assume, conversely, that $\mathcal{L}u$ is always generalized regular if only $u$ is so. In order to obtain the conditions on the coefficients of $\mathcal{L}$, we will choose special functions from the associated space, in this case special generalized regular functions. Note that

$$u = v \cdot \exp(\alpha x_1 e_1)$$

is generalized regular provided $v$ is regular. Choose, especially, $u \equiv 0$. Then (5.11) passes into $\mathcal{D}_\alpha(\mathcal{L}u) = \mathcal{D}_\alpha C$. Since $\mathcal{L}u$ is generalized regular as image of the generalized regular function $u \equiv 0$, we conclude that (5.15) has to be satisfied. Thus the term $\mathcal{D}_\alpha C$ can be omitted in (5.11). Next we choose $u \equiv 1 \cdot \exp(\alpha x_1 e_1)$. Then $\mathcal{D}_\alpha(\mathcal{L}u)$ becomes

$$\left[\mathcal{D}\left(B + \alpha A^{(1)} e_1\right) + 2\alpha\left(\alpha a_3^{(1)} + b_2\right) e_2 - 2\alpha\left(\alpha a_2^{(1)} - b_3\right) e_3\right] \exp(\alpha x_1 e_1)$$

and, thus, the condition (5.14) must hold. For the choices

$$u \equiv (-x_1 e_1 + x_2 e_2) \cdot \exp(\alpha x_1 e_1),$$

and,

$$u \equiv (-x_1 e_1 + x_3 e_3) \cdot \exp(\alpha x_1 e_1),$$

we conclude, in view of (5.11), that the two conditions (5.13) have to be satisfied, respectively. Finally choosing

$$\begin{aligned}
u &\equiv \left(\frac{1}{2}(x_2^2 - x_1^2)e_1 + x_1 x_2 e_2\right) \cdot \exp(\alpha x_1 e_1), \\
u &\equiv \left(\frac{1}{2}(x_3^2 - x_1^2)e_1 + x_1 x_3 e_3\right) \cdot \exp(\alpha x_1 e_1), \text{ and} \\
u &\equiv \left((x_2 x_3 + \frac{1}{2}x_3^2 - \frac{1}{2}x_1^2)e_1 + x_1 x_3 e_2 + (x_1 x_2 + x_1 x_3)e_3\right) \cdot \exp(\alpha x_1 e_1)
\end{aligned}$$

in (5.11), we obtain the conditions (5.12). This completes the proof of the theorem. □

**Example 5.3.3** *Suppose that $\alpha$ is any real number, $C(t,x) \in C^1(\Omega, \mathbb{H})$ is any generalized regular function, and $A^{(1)}(t,x) \in C^2(\Omega, \mathbb{H})$ is any quaternion-valued function for each $t \in [0,T]$. Suppose, further, that $A^{(2)}(t,x) = -A^{(1)}(t,x)e_3$, $A^{(3)}(t,x) = A^{(1)}(t,x)e_2$, and $B(t,x) = -\alpha A^{(1)}(t,x)e_1$. Then all the conditions (5.12)-(5.15) are satisfied, and $\mathcal{L}$ is associated with $\mathcal{D}_\alpha$.*

**Remark 5.3.4** *Note that the coefficients in Example 5.3.3 does not satisfy the (sufficient) conditions formulated in N. Q. Hung's paper (see [13], Theorem 2), and therefore, the criterion in that paper gives no information whether $\mathcal{L}$ is associated with $\mathcal{D}_\alpha$ or not, in this case.*

**Example 5.3.5** *Consider the following example given in [13]: Suppose that $\alpha = 1$, $A^{(i)}(t,x) = f(t,x)e_i$ (for $i = 1,2,3$), $B = f(t,x)e_0$, and $C(t,x) \equiv 0$, where real-valued function $f(t,x)$ is in $C^2(\Omega)$ for each $t \in [0,T]$. Then all the conditions (5.12)-(5.15) are satisfied, and $\mathcal{L}$ is associated with $\mathcal{D}_\alpha$.*

**Note 5.3.6** *Setting $Z := \alpha a_2^{(3)} + b_1$, $T := \alpha a_2^{(1)} - b_3$, and $W := \alpha a_3^{(1)} + b_2$, in view of (5.12), the conditions (5.13) and (5.14) turn out to be*

$$\left.\begin{aligned} \mathcal{D}\left(A^{(2)} + A^{(1)}e_3\right) - 2\left(Ze_3 + We_0\right) = 0 \\ \mathcal{D}\left(A^{(3)} - A^{(1)}e_2\right) + 2\left(Ze_2 + Te_0\right) = 0 \end{aligned}\right\}$$

*and*

$$\mathcal{D}\left(B + \alpha A^{(1)}e_1\right) + 2\alpha\left(We_2 - Te_3\right) = 0$$

*respectively.*

### 5.3.1 Special case of $\alpha = 0$

In case $\alpha = 0$ the operator $\mathcal{D}_\alpha$ coincides with the operator $\mathcal{D}$ and the conditions (5.13)-(5.15) are reduced to

$$\left.\begin{aligned} \mathcal{D}\left(A^{(2)} + A^{(1)}e_3\right) - 2\left(b_1e_3 + b_2e_0\right) = 0 \\ \mathcal{D}\left(A^{(3)} - A^{(1)}e_2\right) + 2\left(b_1e_2 - b_3e_0\right) = 0 \end{aligned}\right\} \tag{5.16}$$

$$\mathcal{D}B = 0 \tag{5.17}$$

$$\mathcal{D}C = 0 \tag{5.18}$$

respectively. Thus we obtain the following statement as a corollary of Theorem 5.3.2:

**Theorem 5.3.7 ([1])** *Assume that $A^{(i)}(t,x) \in C^2(\Omega, \mathbb{H})$ (for $i = 1,2,3$), and $B(t,x)$, $C(t,x) \in C^1(\Omega, \mathbb{H})$ for each $t \in [0,T]$. Then the operator $\mathcal{L}$ is associated with the Dirac operator $\mathcal{D}$ if and only if the conditions (5.12), (5.16)-(5.18) are satisfied.*

**Remark 5.3.8** *In the paper [17] the conditions (5.17), (5.18), and*

$$\left(\mathcal{D}A^{(k)}\right)e_k - \left(\mathcal{D}A^{(l)}\right)e_l = 2\left(b_k e_k - b_l e_l\right) = 0, \quad k, l = 1, 2, 3 \tag{5.19}$$

*are given as necessary conditions for the operators $\mathcal{L}$ and $\mathcal{D}$ to be associated. Note that (5.19) becomes an identity if $k = l$. One can easily show that the remaining six conditions in (5.19) are either equivalent to (5.16) or linear combinations of (5.16). Note, further, that the conditions (5.12), (5.16)-(5.18) are necessary and sufficient for associated operators $\mathcal{L}$ and $\mathcal{D}$.*

## 5.4 The interior estimate for the generalized regular functions

The functions considered in [20] and [17] are regular functions and their components are harmonic functions. Therefore, the interior estimate of regular functions follows from the Poisson integral formula as we will obtain in this section as an auxiliary interior estimate (see (5.22), see also [22, 30], for instance). To solve the initial value problem (5.1), (5.2), we need the interior estimate for the generalized regular functions. Let $u$ be a generalized regular function. Since $0 = D_\alpha u = Du + \alpha u$, we have $Du = -\alpha u$. Applying $D$ to both sides of this equation leads to $D^2 u = -\alpha Du = -\alpha(-\alpha u) = \alpha^2 u$. Using the fact that $D^2 = -\Delta$ we get that $-\Delta u = \alpha^2 u$, or that a generalized regular function $u$ satisfies also the Helmholtz equation

$$(\Delta + \alpha^2)u = 0.$$

Therefore the components $u_k$ of a generalized regular function $u$ are solutions to the Helmholtz equations $\left(\Delta + \alpha^2\right)u_k = 0$. Hence, we have to use another method to prove the interior estimate for generalized regular functions. For the proof of the interior estimate, consider an exhaustion $\Omega_s$, $0 < s < s_0$, of a given (bounded) domain $\Omega$ in $\mathbb{R}^3$. Let $B_s$ be the space of all generalized regular functions $f(x)$ defined in $\Omega_s$ which are continuous in $\overline{\Omega_s}$, i.e.

$$B_s := \left\{f \in \mathbb{H} \mid D_\alpha f = 0 \text{ in } \Omega_s \text{ and } f \in C\left(\overline{\Omega_s}\right)\right\}.$$

Define the norm

$$\|f\|_s := \max_{k\in\{0,1,2,3\}}\left(\sup_{x\in\Omega_s}|f_k(x)|\right). \tag{5.20}$$

Let $B_s$ be the space equipped with the norm $\| \, . \, \|_s$ defined by (5.20). The limit function of a Cauchy sequence of functions belonging to $B_s$ is continuous because convergence with respect to the supremum norm means uniform convergence. In view of Weierstrass convergence theorem, further, the limit function of a uniformly convergent sequence of generalized regular functions is generalized regular, too. Thus the space $B_s$ is proved to be complete, i.e. $B_s$ is a Banach space.

Now we will give a definition and a lemma which will be helpful for obtaining an interior estimate for a generalized regular function in sup-norm:

**Definition 5.4.1 ([31])** *Suppose $\mathcal{L}^*$ is adjoint to the linear k-th order differential operator $\mathcal{L}$ of divergence type, that is,*

$$\mathcal{L}u = \sum_{i,j} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_i b_i \frac{\partial u}{\partial x_i} + cu,$$

*and u is an integrable function satisfying the relation*

$$\int_\Omega \left(\phi h + (-1)^{k+1}\, u\mathcal{L}^*\phi\right) dx = 0$$

*where $\phi$ is any test function, that is, $\phi$ is a continuously differentiable function in $\Omega$ which is identically equal to zero outside a compact subset of $\Omega$. Then u is called a **weak solution** in the **distributional sense** (or **distributional solution)** of the differential equation $\mathcal{L}u = h$. Additionally, a weak solution in distributional sense is necessarily a solution in the classical sense provided that u is k-times continuously differentiable. Weak solution u of the homogeneous equation $\mathcal{L}u = 0$, consequently, are characterized by the relation*

$$\int_\Omega u\mathcal{L}^*\phi dx = 0.$$

**Lemma 5.4.2 ([31])** *Suppose $E(x,\xi)$ is a fundamental solution of $\mathcal{L}u = 0$ with singularity at $\xi$, where $\mathcal{L}$ is k-th order differential operator of divergence type. Then the function defined by*

$$u(x) = \int_\Omega E(x,\xi)h(\xi)d\xi$$

*turns out to be a **distributional solution** of the inhomogeneous differential equation $\mathcal{L}u = h$.*

**Proof.** Denote $\Omega$ as domain of the $x$-space and the $\xi$-space by $\Omega_x$ and $\Omega_\xi$, respectively. If $\phi$ is $k$-times continuously differentiable test function then one has

$$\begin{aligned}\int_{\Omega_x} u\mathcal{L}^*\phi dx &= \int_{\Omega_x}\left(\int_{\Omega_\xi} E(x,\xi)h(\xi)d\xi\right)\mathcal{L}^*\phi dx \\ &= \int_{\Omega_\xi} h(\xi)\left(\int_{\Omega_x} E(x,\xi)\mathcal{L}^*\phi dx\right)d\xi \quad \text{by Fubini theorem} \\ &= (-1)^k\int_{\Omega_\xi} h(\xi)\phi(\xi)d\xi\end{aligned}$$

which completes the proof. □

We refer the reader to [31, 18] for the concept of fundamental solutions.

In the following we will obtain an interior estimate for a generalized regular function in sup-norm.

Consider now two subdomains $\Omega_{s'}$ and $\Omega_{s''}$ with $\overline{\Omega_{s'}} \subset \Omega_{s''} \subset \Omega$, where $s' < s'' < s_0$, and define an auxiliary function

$$u^{(0)}(x) = u(x) + \int_{\Omega_{s'}} E(x,\xi)\alpha^2 u(\xi)d\xi, \tag{5.21}$$

where $u(x)$ is a generalized regular function in $\Omega$, and

$$E(x,\xi) = \frac{-1}{4\pi \mid x-\xi \mid}$$

is the fundamental solution of the Laplace equation in $\mathbb{R}^3$. It follows from Lemma 5.4.2 that

$$\Delta\left(\int_{\Omega_{s'}} E(x,\xi)\alpha^2 u(\xi)d\xi\right) = \alpha^2 u(x).$$

Thus applying $\Delta$ to both sides of (5.21) one gets

$$\Delta u^{(0)}(x) = \Delta u(x) + \alpha^2 u(x) = \left(\Delta + \alpha^2\right)u(x) = 0.$$

So the function $u^{(0)}$ is harmonic in $\Omega$. On the other hand, a ball $\mathbf{B}_\delta(x_0)$ centred at a point $x_0 \in \Omega_{s'}$ having the radius $\delta$ with $\delta < \operatorname{dist}(\Omega_{s'}, \partial\Omega_{s''})$ is contained in $\Omega_{s''}$. Thus the Poisson integral formula for harmonic functions applied to $\mathbf{B}_\delta(x_0)$ yields

$$u^{(0)}(x) = \frac{1}{4\pi\delta}\int_{|\xi-x_0|=\delta} u^{(0)}(\xi)P(x,\xi)d\sigma(\xi),$$

where $P(x, \xi)$ is the Poisson Kernel

$$P(x, \xi) = \frac{\delta^2 - |x - x_0|^2}{|\xi - x|^3}.$$

Now we will prove the interior estimate

$$\left\| \frac{\partial u^{(0)}}{\partial x_i} \right\|_{s'} \leq \frac{3}{\operatorname{dist}(\Omega_{s'}, \Omega_{s''})} \left\| u^{(0)} \right\|_{s''} \tag{5.22}$$

for the harmonic function $u^{(0)}$. For this end let

$$r := |\xi - x| = \left[ \sum_{j=1}^{3} (\xi_j - x_j)^2 \right]^{\frac{1}{2}}.$$

Note that $|x - x_0|^2 = \sum_{j=1}^{3} (x_j - x_{0_j})^2$. Taking the derivative of $P(x, \xi)$ with respect to $x_i$ we get

$$\frac{\partial P(x, \xi)}{\partial x_i} = \frac{-2(x_i - x_{0,i}) \cdot r^3 + 3r(\xi_i - x_i) \cdot \left[ \delta^2 - |x - x_0|^2 \right]}{|\xi - x|^6}. \tag{5.23}$$

Now take the derivative of $u^{(0)}(x)$ given by (5.21), take (5.23) into account in the resulting expression to obtain

$$\frac{\partial u^{(0)}(x)}{\partial x_i} = \frac{1}{4\pi\delta} \int_{|\xi - x_0| = \delta} u^{(0)}(\xi) \cdot \frac{-2(x_i - x_{0i}) \cdot r^3 + 3r(\xi_i - x_i) \cdot \left[ \delta^2 - |x - x_0|^2 \right]}{|\xi - x|^6} d\sigma(\xi).$$

Let $x_0$ be any element of $\Omega_{s'}$ and take $x = x_0$. Then $r = |\xi - x_0| = \delta$. Thus, since $|x - x_0| = 0$ and $x_i - x_{0i} = 0$, the last equality yields

$$\frac{\partial u^{(0)}(x_0)}{\partial x_i} = \frac{1}{4\pi\delta} \int_{|\xi - x_0| = \delta} u^{(0)}(\xi) \cdot \frac{3(\xi_i - x_{0i})}{\delta^3} d\sigma(\xi).$$

Using $|\xi_i - x_{0i}| \leq |\xi - x_0| = \delta$ and taking the absolute value of the last equality we obtain that

$$\begin{aligned} \left| \frac{\partial u^{(0)}(x_0)}{\partial x_i} \right| &\leq \frac{3}{\delta} \cdot \sup_{|\xi - x_0| = \delta} \left| u^{(0)}(\xi) \right| \cdot \frac{1}{4\pi\delta^2} \cdot \int_{|\xi - x_0| = \delta} d\sigma(\xi) \\ &\leq \frac{3}{\delta} \cdot \sup_{|\xi - x_0| = \delta} \left| u^{(0)}(\xi) \right| \cdot \frac{1}{4\pi\delta^2} \cdot 4\pi\delta^2 \end{aligned}$$

or

$$\left| \frac{\partial u^{(0)}(x_0)}{\partial x_i} \right| \leq \frac{3}{\delta} \cdot \sup_{|\xi - x_0| = \delta} \left| u^{(0)}(\xi) \right|, \quad \forall x_0 \in \Omega_{s'}.$$

By the maximum principle for harmonic functions we have

$$\sup_{|\xi-x_0|=\delta} \left|u^{(0)}(\xi)\right| = \sup_{|\xi-x_0|\le\delta} \left|u^{(0)}(\xi)\right|$$

and consequently the previous inequality can be expressed also as

$$\left|\frac{\partial u^{(0)}(x_0)}{\partial x_i}\right| \le \frac{3}{\delta} \cdot \sup_{|\xi-x_0|\le\delta} \left|u^{(0)}(\xi)\right|, \quad \forall x_0 \in \Omega_{s'}.$$

This implies, in view of the definition of the norm in $B_s$, that

$$\left\|\frac{\partial u^{(0)}}{\partial x_i}\right\|_{s'} \le \frac{3}{\delta} \left\|u^{(0)}\right\|_{s''}.$$

Carrying out the limiting process $\delta \to \mathrm{dist}(\Omega_{s'}, \Omega_{s''})$, we get (5.22). So we have proved the following statement:

**Proposition 5.4.3** *If $u^{(0)}$ is a harmonic function in $\Omega$ then the interior estimate (5.22) holds.*

The proof of the following lemma concerning an upper bound of a weakly singular integral can be found in [34], for instance:

**Lemma 5.4.4 (Schmidt's Inequality)** *Suppose $\Omega$ is a domain in $\mathbb{R}^n$, with finite measure $m\Omega$ not necessarily bounded. Denote the volume of a unit ball in $\mathbb{R}^n$ by $\tau_n$, while the measure of surface of unit ball is $\omega_n$. Then for $0 \le \alpha < n$*

$$\int_\Omega \frac{1}{|\xi - x|^\alpha} d\xi \le \frac{\omega_n}{n-\alpha} \left(\frac{m\Omega}{\tau_n}\right)^{1-\frac{\alpha}{n}}$$

*for each $x \in \mathbb{R}^n$.*

In the sequel we will obtain the interior estimate for generalized regular functions in sup-norm making use of the interior estimate (5.22) for harmonic functions. By the way the interior estimate can also be obtained in $L_p$-norm applying the Theorem 3 of [38] to the fundamental solution of the Laplace equation in $\mathbb{R}^3$.

From (5.21) we have

$$u_k^{(0)}(x) = u_k(x) + \int\limits_{\Omega_{s'}} E(x,\xi)\alpha^2 u_k(\xi) d\xi, \tag{5.24}$$

where $x \in \Omega_{s'}$, $u_k^{(0)}(x)$ and $u_k(x)$ are the $k^{\text{th}}$ components of functions $u^{(0)}(x)$ and $u(x)$ respectively, i.e.

$$u^{(0)}(x) = \sum_{k=0}^{3} u_k^{(0)}(x) e_k$$

and

$$u(x) = \sum_{k=0}^{3} u_k(x) e_k.$$

On the other hand, the Schmidt's inequality (see Lemma 5.4.4) for $n = 3$ and $\alpha = 1$ ($\omega_3 = 4\pi$, $\tau_3 = \frac{4}{3}\pi$) states that

$$\int_{\Omega} \frac{1}{|\xi - x|} d\xi \leq \frac{4\pi}{2} \left( \frac{3\text{m}\Omega}{4\pi} \right)^{\frac{2}{3}}.$$

Recalling that $E(x, \xi) = -1/\left(4\pi |x - \xi|\right)$ and applying the Schmidt's inequality to the absolute value of (5.24) we obtain

$$\begin{aligned}
\left| u_k^{(0)}(x) \right| &\leq |u_k(x)| + \int_{\Omega_{s'}} \alpha^2 \left| E(x, \xi) \right| \cdot |u_k(\xi)| \, d\xi \\
&\leq \sup_{x \in \Omega_{s'}} |u_k(x)| + \frac{\alpha^2}{4\pi} \sup_{\xi \in \Omega_{s'}} |u_k(\xi)| \cdot \int_{\Omega_{s'}} \frac{1}{|x - \xi|} d\xi \qquad (5.25) \\
&\leq \left[ 1 + \frac{\alpha^2}{4\pi} \cdot \frac{4\pi}{2} \left( \frac{3\text{m}\Omega}{4\pi} \right)^{\frac{2}{3}} \right] \cdot \sup_{x \in \Omega_{s'}} |u_k(x)| .
\end{aligned}$$

By means of the definition of sup-norm (5.20) it follows from the last ineqaulity in (5.25) that

$$\left\| u^{(0)} \right\|_{s'} \leq \left[ 1 + \frac{\alpha^2}{2} . \left( \frac{3\text{m}\Omega}{4\pi} \right)^{\frac{2}{3}} \right] \cdot \|u\|_{s'} \qquad (5.26)$$

The equality (5.21) implies that

$$u_k(x) = u_k^{(0)}(x) - \int_{\Omega_{s'}} E(x, \xi) \alpha^2 u_k(\xi) d\xi, \quad x \in \Omega_{s'}. \qquad (5.27)$$

Taking the derivatives of the both sides of (5.27) with respect to $x_i$ (for $i = 1, 2, 3$) it follows that

$$\begin{aligned}
\frac{\partial u_k(x)}{\partial x_i} &= \frac{\partial u_k^{(0)}}{\partial x_i} - \int_{\Omega_{s'}} \frac{\partial E(x, \xi)}{\partial x_i} \cdot \alpha^2 u_k(\xi) d\xi \\
&= \frac{\partial u_k^{(0)}}{\partial x_i} + \frac{\alpha^2}{4\pi} \int_{\Omega_{s'}} \frac{\partial}{\partial x_i} \left[ |x - \xi|^{-1} \right] \cdot u_k(\xi) d\xi. \qquad (5.28)
\end{aligned}$$

Taking the absolute value of the both sides of (5.28) we get

$$\left|\frac{\partial u_k(x)}{\partial x_i}\right| \leq \left|\frac{\partial u_k^{(0)}}{\partial x_i}\right| + \frac{\alpha^2}{4\pi}\sup_{\xi\in\Omega_{s'}}|u_k(\xi)|\cdot\int\limits_{\Omega_{s'}}\left|\frac{\partial}{\partial x_i}\left[|x-\xi|^{-1}\right]\right|d\xi.$$

Note that

$$\frac{\partial}{\partial x_i}\left[|x-\xi|^{-1}\right] = -\frac{x_i-\xi_i}{|x-\xi|^3},$$

which implies

$$\left|\frac{\partial}{\partial x_i}\left[|x-\xi|^{-1}\right]\right| = \frac{|x_i-\xi_i|}{|x-\xi|^3} \leq \frac{1}{|x-\xi|^2}.$$

Therefore applying the Schmitd's inequality once more for $\alpha = 2$ and $n = 3$ we obtain

$$\begin{aligned}\int\limits_{\Omega_{s'}}\left|\frac{\partial}{\partial x_i}\left[|x-\xi|^{-1}\right]\right|d\xi &\leq \int\limits_{\Omega_{s'}}\frac{1}{|x-\xi|^2}d\xi \\ &\leq 4\pi\left(\frac{3\mathrm{m}\Omega_{s'}}{4\pi}\right)^{\frac{1}{3}}\end{aligned}$$

and so

$$\left|\frac{\partial u_k(x)}{\partial x_i}\right| \leq \left|\frac{\partial u_k^{(0)}}{\partial x_i}\right| + \frac{\alpha^2}{4\pi}\sup_{\xi\in\Omega_{s'}}|u_k(\xi)|\cdot 4\pi\left(\frac{3\mathrm{m}\Omega_{s'}}{4\pi}\right)^{\frac{1}{3}}.$$

Using the fact that $\|\cdot\|_{s'} \leq \|\cdot\|_{s''}$ and the definition of sup-norm the last inequality implies that

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{s'} \leq \left\|\frac{\partial u^{(0)}}{\partial x_i}\right\|_{s'} + \alpha^2\left(\frac{3\mathrm{m}\Omega}{4\pi}\right)^{\frac{1}{3}}\cdot\|u\|_{s''}. \tag{5.29}$$

Let d := dist$(\Omega_{s'}, \Omega_{s''})$. Employing (5.22) and (5.26), we get from (5.29) that

$$\begin{aligned}\left\|\frac{\partial u}{\partial x_i}\right\|_{s'} &\leq \frac{3}{\mathrm{d}}\cdot\left\|u^{(0)}\right\|_{s''} + \alpha^2\left(\frac{3\mathrm{m}\Omega}{4\pi}\right)^{\frac{1}{3}}\cdot\|u\|_{s''} \\ &\leq \frac{3}{\mathrm{d}}\cdot\left[1+\frac{\alpha^2}{2}.\left(\frac{3\mathrm{m}\Omega}{4\pi}\right)^{\frac{2}{3}}\right]\cdot\|u\|_{s''} + \alpha^2\left(\frac{3\mathrm{m}\Omega}{4\pi}\right)^{\frac{1}{3}}\cdot\|u\|_{s''} \\ &\leq \frac{3}{\mathrm{d}}\cdot\|u\|_{s''} + \frac{3\alpha^2}{2\mathrm{d}}.\left(\frac{3\mathrm{m}\Omega}{4\pi}\right)^{\frac{2}{3}}\cdot\|u\|_{s''} + \alpha^2\left(\frac{3\mathrm{m}\Omega}{4\pi}\right)^{\frac{1}{3}}\cdot\|u\|_{s''} \\ &\leq \left\{\frac{3}{\mathrm{d}} + \alpha^2\left(\frac{3\mathrm{m}\Omega}{4\pi}\right)^{\frac{1}{3}}\left[1+\frac{3}{2\mathrm{d}}\left(\frac{3\mathrm{m}\Omega}{4\pi}\right)^{\frac{1}{3}}\right]\right\}\|u\|_{s''}.\end{aligned} \tag{5.30}$$

Let $d_0 = \mathrm{diam}\,(\Omega)$, then the dist$(\Omega_{s'}, \Omega_{s''}) \leq d_0$. Thus $1 \leq \frac{d_0}{\mathrm{dist}(\Omega_{s'},\Omega_{s''})}$. Since $0 < s' < s''$, (5.30) leads to the following interior estimate for the first order partial derivatives of a generalized regular function

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{s'} \leq \frac{3+\alpha^2\left(\frac{3\mathrm{m}\Omega}{4\pi}\right)^{\frac{1}{3}}\left[\frac{3}{2}\left(\frac{3\mathrm{m}\Omega}{4\pi}\right)^{\frac{1}{3}}+d_0\right]}{\mathrm{dist}(\Omega_{s'},\Omega_{s''})}\|u\|_{s''}. \tag{5.31}$$

**Remark 5.4.5** *In the paper [13] the author obtains the interior estimate in the form*

$$\left\|\frac{\partial u}{\partial x_i}\right\|_{s'} \leq \frac{\alpha^2\left(\frac{3m\Omega}{4\pi}\right)^{\frac{1}{3}}\left[3+\frac{1}{2}\left(\frac{3m\Omega}{4\pi}\right)^{\frac{1}{3}}\right]}{dist(\Omega_{s'},\Omega_{s''})}\|u\|_{s''}\,. \tag{5.32}$$

*But (5.32) is not correct. The interior estimate (5.31) is the corrected one.*

## 5.5 Initial value problems with a Helmholtz-Type Dirac Operator

Applying the abstract Cauchy-Kovalevskaya theorem (see Section 3.5) the following statement has been proved:

**Theorem 5.5.1 ([13])** *Suppose the operator $\mathcal{L}$ is associated with the disturbed Dirac operator $\mathcal{D}_\alpha$ of the quaternionic analysis. Then the initial value problem (5.1), (5.2) is solvable in case $u_0$ is an arbitrary generalized regular initial function. The solution $u(t,x)$ is also generalized regular for each $t$.*

## 5.6 Concluding Remarks

Paper [45] deals with the initial value problem

$$\frac{\partial u}{\partial t} = \mathcal{L}u := \sum_{i=0}^{3} A^{(i)}(t,x)\frac{\partial u}{\partial x_i} + B(t,x)u + C(t,x) \tag{5.33}$$

$$u\,(0,x) = u_0(x) \tag{5.34}$$

in the space of generalized regular functions satisfying the differential equation $\mathcal{D}_\alpha u = 0$, where $\mathcal{D}_\alpha u := \mathcal{D}u + \alpha u$, and

$$\mathcal{D} = \partial_0 + \sum_{j=1}^{3} e_j \partial_j$$

is the Cauchy-Fueter operator of the quaternionic analysis. Necessary and sufficient conditions are given on the coefficients of the operator $\mathcal{L}$ under which $\mathcal{L}$ is associated with the operator $\mathcal{D}_\alpha$. For such operators $\mathcal{L}$ the initial value problem (5.33), (5.34) is solved uniquely.

It can be expected, therefore, that similar constructions for the solution of a more general initial value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \mathcal{L}u := \sum_{i=0}^{n} A^{(i)}(t,x)\frac{\partial u}{\partial x_i} + B(t,x)u + C(t,x) \\ u(0,x) &= u_0(x) \end{aligned}$$

can also be carried out, by replacing the Cauchy-Fueter operator of the quaternionic analysis with the Cauchy-Riemann operator

$$\mathcal{D} = \partial_0 + \sum_{j=1}^{n} e_j \partial_j$$

of the Clifford analysis in Clifford algebras depending on parameters (see [36]).

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