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ANALYSES OF SPECTROSCOPIC PARAMETERS OF GROUND STATE AND FIRST ORBITALLY EXCITED STATE HEAVY BARYON SIGMA(Q) WITHIN QUANTUM CHROMODYNAMICS SUM RULES

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Approval of the Graduate School of Natural and Applied Sciences, Atilim University.

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I certify that this thesis satisfies all the requirements as a thesis for the degree of **Master of Science in Applied Physics Department, Atilim University.**

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ABSTRACT

ANALYSES OF SPECTROSCOPIC PARAMETERS OF GROUND STATE AND FIRST ORBITALLY EXCITED STATE HEAVY BARYON SIGMA(Q) WITHIN QUANTUM CHROMODYNAMICS SUM RULES

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In this work, we investigate the spectroscopic properties, such as masses and residues, of the ground state Σ_Q baryons having spin-$\frac{1}{2}$ and their first orbital excitation ($1P$) states. In the calculations we use the QCD sum rule method, which is among the successful nonperturbative approaches. The obtained results are compared with the existing experimental ones and present theoretical predictions, as well. The mass calculations of these baryons provide us with important insights into not only understanding the nature and substructure of these baryons but also gaining better understanding on the strong interaction.

Keywords: QCD sum rules, Charmed baryons, Bottom baryons

ÖZ

TABAN DURUM VE İLK ORBİTAL UYARILMIŞ DURUM SIGMA(Q) BARYON İÇİN SPEKTROSKOPİK PARAMETRELERİN KUANTUM RENK DİNAMİĞİ TOPLAM KURALLARI İLE ANALİZİ

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Bu çalışmada spin-$\frac{1}{2}$ taban durum Σ_Q baryonlar ve bunların orbital ($1P$) uyarılmış durumları için kütle ve rezidü gibi spektroskopik özellikleri incelendi. Hesaplamalarda pertürbatif olmayan başarılı yaklaşımlar arasıda yer alan QCD toplam kuralları metodunu kullandık. Elde edilen sonuçlar var olan deney datalarının yanı sıra mevcut teorik tahminler ile karşlaştırıldı. Bu baryonların kütle hesaplamaları bizlere bunların iç yapıları ve doğalarının anlaşılmasının yanı sıra güçlü etkileşmeyi daha iyi anlamamızda önemli bilgiler sağlamaktadır.

Anahtar Kelimeler: KRD toplam kuralları, Tılsımlı baryonlar, Alt baryonlar

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CHAPTER 1

INTRODUCTION

In particle physics the interactions between the particles are explained by a theory which is called as standard model (SM). This model contains three of the four forces of nature. These three forces are the strong, electromagnetic and weak forces which play important roles in our understanding of the nature. The gravitational force is not included in the SM.

According to their properties, such as electrical charge, energy, momentum and mass, one can categorize the elementary particles. Spin is another property used for this purpose. One can classify the elementary particles into two groups considering the spins of these particles. The particles having integer spins such as $0, 1, 2, \ldots$ are bosons and that having half integral spins such as $1/2, 3/2, ...$ are the fermions. Bosons are the force carriers in the theory. The elementary particles in the SM are the spin one-half quarks and leptons, the spin one gauge bosons and the spin zero Higgs field which are presented in Fig. 2.1. For quarks and leptons there are corresponding anti particles having same mass but with quantum numbers, such as electric charge or spin, having opposite sign. The leptons and quarks are arranged in three families called generations. In each lepton generation there are a charged lepton and a neutrino. In each quark generation there are two different quarks. The first families of leptons and quarks contain the lightest fermions and form the stable matter, that is the atoms are composed of electrons, protons and neutrons. Protons and neutrons consist of the up and down quarks. Quarks and leptons have electric charge, but unlike the leptons, quarks also have color charges. The color charges are named as red, green and blue.

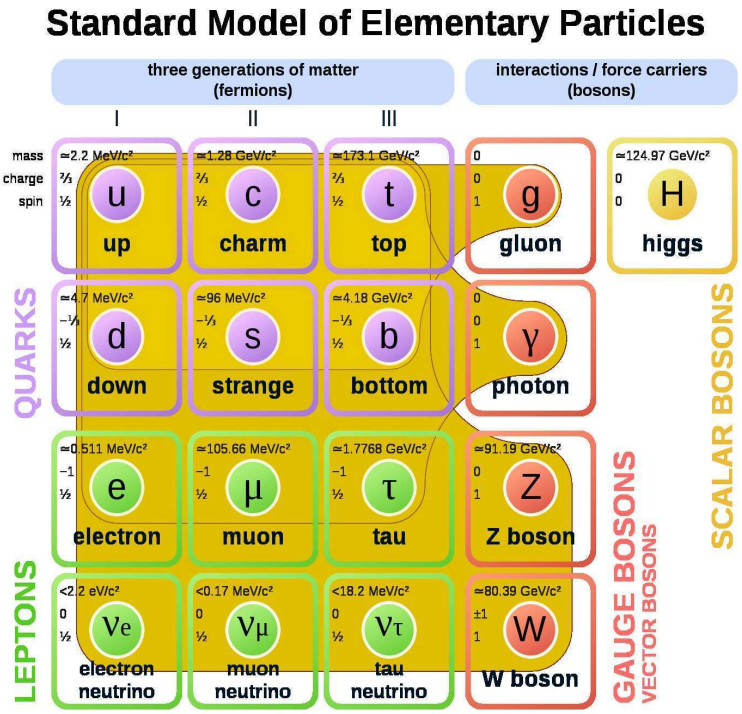


Figure 1.1: The elemetary particles in the SM of particle physics [1].

Although the SM explains lots of experimental processes observed so far, it has also some shortcomings such as that it can not explain the gravitation, the source of the neutrino masses, matter-antimatter asymmetry, dark matter, the reason of difference in the strength of the fundamental forces and mass differences of fundamental particles.

The strong interaction between quarks and gluons is described by the Quantum chromodynamics (QCD), whic is a field theory describing the strong interaction. In nature we can not see quarks as free particles. They are found as bound states of two or three quarks or anti-quarks. Quarks and gluons come together and form the color singled hadrons. Hadrons composed of a quark and an anti-quark are mesons and three quarks are baryons. The mediators of the strong interactions are the gluons which are massless. The gluons carry color charge therefore interact with quarks and themselves, as well. At high energies interaction between quarks and gluons are weak. This property is called as asymptotic freedom. Because of this property the perturbation theory can be applied for high energy regime. However for the low energy region the large val-

ues of the effective coupling constant makes the usage of the perturbative techniques insufficient. In this regime there exist nonperurbative effects which requires the application of the nonperturbative methods. Chiral perturbation theory, quark models, lattice calculations, heavy quark effective theory and the QCD sum rule method are among these nonperturbative approaches.

In this thesis we apply the QCD sum rule method to calculate the spectroscopic parameters of the heavy baryons Σ_Q where Q is used to represent c or b quarks. We obtain the masses of their ground states and first orbitally excited states having opposite parity, as well. Since it is based on the QCD Lagrangian the QCD sum rule method is a powerful method and so far has been applied with success providing predictions consistent with the experimental results. Spectroscopic parameters such as masses and current coupling constants (residue) are among the important parameters of the baryons. These parameters are helpful in the identification of the observed particles. The mass and residue are used as input parameters in the search of the decay widths of these baryons to different final states. In the recent years there are lots of particles observed experimentally. Among these particles there are not only conventional states but also non-conventional ones such as tetraquarks which are composed of four quarks and pentaquarks which are composed of five quarks. If we can explain the properties of the all conventional particles, this may also help us to identify the above mentioned exotic states. Therefore investigations on the spectroscopic parameters of these baryons provide us with important information not only for understanding their internal structures and decay properties but also better identification of other particles. These studies also provide us with deeper understanding of the structure of QCD and its parameters.

Up to the present, almost all the ground state of the baryons having quark content including one c or b quark together with two light ones have been observed in experiments. Among these so called heavy-quark baryons are the Σ_c and Σ_b baryons. In many experiments the properties of the Σ_c were measured (see for instance Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10] and the references therein). The recent mass values of Σ_c baryons can be found in Ref. [11] which are $m_{\Sigma_c(2455)^{++}} = 2453.97 \pm 0.14$ MeV,

$m_{\Sigma_c(2455)^+} = 2452.9 \pm 0.4$ MeV and $\Sigma_c(2455)^0 = 2453.75 \pm 0.14$ MeV. As for the Σ_b, it is firstly observed in 2007 together with Σ_b^* and their masses were reported to be $m_{\Sigma_b^+} = 5807.8^{+2.0}_{-2.2}$ (stat.) $\pm$ 1.7 (syst.) MeV/c^2, and $m_{\Sigma_b^-} = 5815.2 \pm 1.0$ (stat.) $\pm$ 1.7 (syst.) MeV/c^2 for Σ_b^+ and Σ_b^-, respectively [12]. These observations were later confirmed with the usage of larger data sample [13] improving the result of mass measurement which were given as $m_{\Sigma_b^+} = 5811.3^{+0.9}_{-0.8} \pm 1.7$ MeV/c^2 and $m_{\Sigma_b^-} = 5815.5^{+0.6}_{-0.5} \pm$ 1.7 MeV/c^2 providing measurements on their natural widths also. With the improvement in the experimental sector the investigations on the heavy baryons still continue. Besides the ground state heavy baryons the quark model predicts also the existence of their excited states. However the experimental data on them are more limited compared to those on their ground states. Several collaborations reported the observation of the lowest lying orbital excitation of the charmed baryons having negative parity such as $\Lambda_c(2595)$, $\Xi_c(2790)$ which have spin-$\frac{1}{2}$ and $\Lambda_c(2625)$ and $\Xi_c(2815)$ which have spin-$\frac{3}{2}$ [11, 14, 15, 16, 17]. There are other possible P-wave charmed baryon states which were observed by the BaBar and Belle collaborations whose quantum numbers were not determined exactly yet such as $\Sigma_c(2800)$, $\Xi_c(2970)$ and $\Xi_c(3080)$ [18, 19, 20, 21]. In the bottom baryon side there are only few excited states that were observed [22, 23, 24, 25, 26, 27]. Related to such states, one recent observation were reported by the LHCb collaboration for two Σ_b states [27] whose masses and decay widths were given as $m_{\Sigma_b(6097)^-} = 6098.0 \pm 1.7 \pm 0.5$ MeV, $\Gamma(\Sigma_b(6097)^-) = 28.9 \pm 4.2 \pm 0.9$ MeV and $m_{\Sigma_b(6097)^+} = 6095.0 \pm 1.7 \pm 0.4$ MeV, $\Gamma(\Sigma_b(6097)^+) = 31.0 \pm 5.5 \pm 0.7$ MeV whose quantum numbers are not clear also and their determination is required.

These baryons also gain attentions theoretically. In literature one may find many theoretical works on these baryons conducted via different approaches. The masses for the ground state and excited state heavy baryons were investigated using quark model [28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42], lattice QCD [43, 44, 45], relativistic flux tube model [46] and using QCD sum rule method [47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60]. In order to investigate newly observed heavy baryons, beside the mass spectra, their other properties were also considered. In Ref [61] the constituent quark model was used to study the strong and radiative decays of S and P-wave baryons and they indicated the property of the $\Sigma_c(2800)$ resonance being a

$J^P = \frac{3}{2}^-$ or $J^P = \frac{5}{2}^-$ state. Motivated with the recent observations [26, 62, 27] the possible excitation of the heavy bottom baryons were considered in different works. The calculations performed using constituent quark model for the strong decays of P-wave Σ_b, Ξ_b' and Ω_b baryons stated the newly observed $\Sigma_b(6097)$ states are possibly $J^P = \frac{3}{2}^-$ or $J^P = \frac{5}{2}^-$ state [63]. Strong decay analyses for $\Sigma_b(6097)$ resonances presented in Refs. [64, 65] provided predictions on the quantum numbers of the $\Sigma_b(6097)$ indicating the possibility of their being $J^P = \frac{3}{2}^-$ or $J^P = \frac{5}{2}^-$. In Ref. [66] the strong decays of the $\Sigma_b(6097)$ and the mass of these states were investigated using QCD sum rule method taking into account two possibilities for their quantum numbers as $J = \frac{1}{2}$ and $J = \frac{3}{2}$ and their being $1P$ or $2S$ excitation of the ground state Σ_b baryon. Their analysis supported their being $1P$ excitation with quantum numbers $J^P = \frac{3}{2}^-$. Some other analyses for masses and decay properties of the other excited heavy bottom and charmed baryons can be found in Refs. [67, 68, 69, 70] (see also the references therein).

As it can be understood from the experimental situation, the investigations on these baryons, related to both their ground and especially their excited states, are important not only in experimental sector but also theoretical point of view. By means of these studies we not only understand their nature and internal structures but also gain important information about the nonperturbative nature of the QCD. These type of studies will provide essential information for the experimental investigations, as well. Their investigations may provide essential information about the dynamics of the light quarks present beside the heavy one. In addition their studies are helpful to test the quark model predictions and the results obtained using heavy quark symmetry. With these motivation and the motivation brought by the new experimental results, in this thesis we study the ground state and lowest orbitally excited states of the Σ_c and Σ_b states having spin-$\frac{1}{2}$ and calculate their spectroscopic properties. i.e, masses and residues. To this end, we use the QCD sum rule method proposed firstly by Shifman, Vainshtein and Zakharov [71]. In the calculation we use Ioffe current for the Σ_Q baryons. In the analyses, we first consider only the ground state and obtain the masses and residues for the ground state Σ_c and Σ_b baryons. Then we use the masses of the ground states as input in the further analysis and we extend our calculations to their first orbital excitations and calculate both their masses and residues.

The outline of the thesis is as follow: Chapter 2 summarizes the main properties of the QCD sum rule that is presented mainly following the Refs [72, 73, 74]. Chapter 3 is devoted to the calculations of the QCD sum rules from which the masses and residues of the considered baryons are attained. In the chapter 4 we present the detailed analysis of the results obtained from QCD sum rules calculations. Finally chapter 5 gives the conclusion.

CHAPTER 2

QCD SUM RULES

This chapter is devoted to explain the method used in the calculation which is the QCD sum rule method. This method was firstly proposed by Shifman, Vainshtein and Zakharov [71] to study the mesons and later the method was extended to baryons by Ioffe [79]. The method has been widely applied for the calculation of the hadronic observables such as coupling constants, masses, form factors, etc. giving successful predictions consistent with many experimental observations and theoretical predictions of other methods. The coupling constant which describes the physical property in Quantum Chromodynamics (QCD) has a different behavior compared to that of Quantum Electrodynamics (QED). When the momentum transfer is low the effective coupling constant of QED, represented by α, is around 1/137. α increases slowly by increasing momentum transfer. On the other hand QCD has asymptotic freedom property and the interaction is weak at high momentum transfers and it is strong at low momentum transfer. The Fig. 2.1, obtained from experiments, shows the dependency of the effective coupling constant of the QCD, α_s, on the energy scale. The experimental result is in consistency with QCD, showing that the α_s decreases as the energy increases, whose explicit form is given as

$$\alpha_s(Q^2) = \frac{g_s^2(Q^2)}{4\pi} \simeq \frac{1}{4\pi\beta_0 ln(\frac{Q^2}{\Lambda^2_{QCD}})}. \tag{2.1}$$

In last equation Λ_{QCD} is QCD scale parameter, the β_0 is Gell-Mann-Low function constant. At high momentum region, that is, when $Q^2 \gg \Lambda_{QCD}$ quarks and gluons are nearly free. On the other hand at low momentum region where $Q^2 \sim \Lambda_{QCD}$ strength of the interaction increases and so that the hadronization occurs. In this region because of confinement there exist nonperturbative effects and as a result one can not use

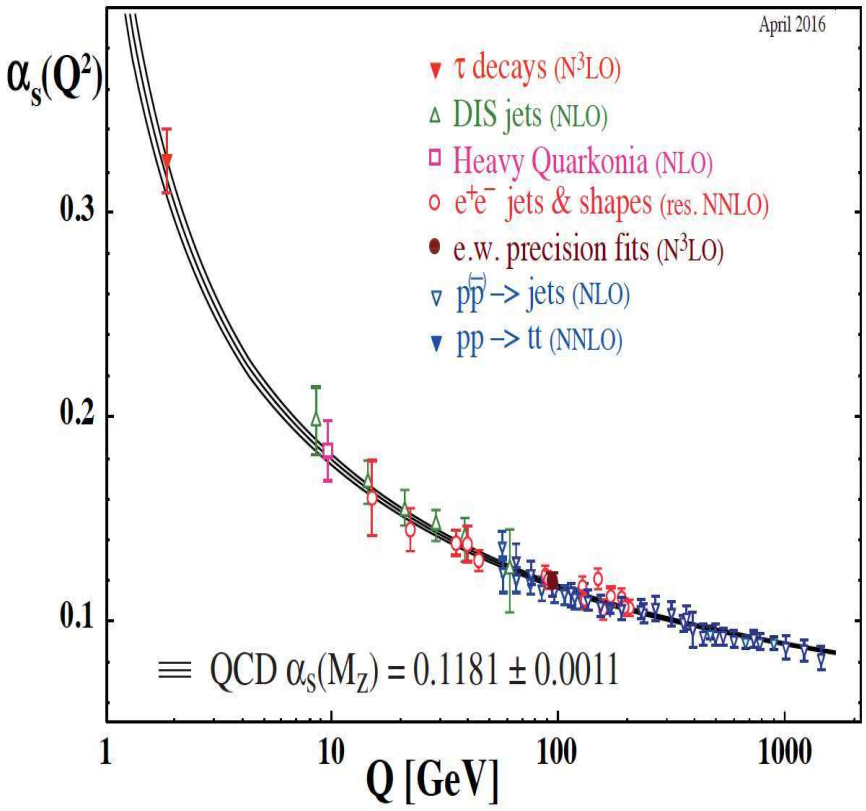


Figure 2.1: The coupling constant, α_s, as a function of the respective energy scale [75].

perturbative calculations. This requires a nonperturbative methods among which is the QCD sum rule.

In this work we use the QCD sum rule and in next section the details of this method will be presented following the Refs. [72, 73, 74, 76].

2.1 QCD Sum Rule

To attain the information on the hadronic observable the method requires starting from short distance where one uses perturbation theory and approaching to long distance where nonperturbative effects show up. Therefore the theory provides a connection

between high and low energy regions. To fulfill this a mathematical expression, the so called correlation function, is used. The correlation function is written in terms of the interpolating currents of the considered hadrons. The interpolating currents carry the quantum numbers of these hadrons and are composed of the quark fields. In the calculation of the correlation function one follows two ways. On one side it is calculated in terms of hadronic degrees of freedom such as masses of the hadrons, current coupling constant. On the other side it is calculated in terms of QCD degrees of freedom including mass of the quarks, quark gluon condensates, etc. The QCD sum rules for the physical quantity under interest is obtained from the match of these two sides via a dispersion relation.

There are two point and three point QCD sum rules used to obtain different physical quantities. While the mass of the hadrons and overlap amplitudes are calculated via two point QCD sum rules, the three point QCD sum rules are used to calculate the physical quantities such as form factors, strong decay constants. In this work we shall calculate the masses and residues of ground states and first orbitally excited states of Σ_c and Σ_b baryons, having spin-$\frac{1}{2}$, through two point QCD sum rules method. Therefore in the next section we shall give details of two point QCD sum rules.

2.2 Two Point Correlation Function

The QCD Lagrangian has the following form

$$L_{QCD} = -\frac{1}{4}G^{\alpha}_{\mu\nu}G^{\mu\nu\alpha} + \sum_{q}\bar{\Psi}_q(i\not{D} - m_q)\Psi_q \tag{2.2}$$

and the QCD sum rule, being based on this Lagrangian, is an advantageous method. In the Lagrangian $G^{\alpha}_{\mu\nu}$ and Ψ_q represent the gluon field tensor and the quark field, respectively, and D represents the covariant derivative. In quark field, the sub-symbol q is used for the quark flavor which are u, d, s, c, b, t. Though in principle the Lagrangian given in Eq. 2.2 describes all the hadronic processes, it has a limited usage in the framework of perturbation theory. For the interactions containing nonperturbative effects other methods are needed. QCD sum rule serves effectively in this respect. The starting point of the two point QCD sum rule is the following two point correlation

function

$$\Pi(q^2) = i \int d^4x e^{iqx} \langle 0|\mathcal{T}\{J(x)\bar{J}(0)\}|0\rangle. \tag{2.3}$$

In this equation $|0\rangle$ is used to represent QCD vacuum state, $\mathcal{T}$ symbolizes the time ordering operator. $J(x)$ is the interpolating current of the hadron under consideration formed considering the quantum numbers such as spin, isospin, parity and as well as valance quark contents of the hadrons. To exemplify the interpolating currents, for the mesons it has the form $J(x) = q_i \Gamma q_j$ and different choice of the Γ gives us the interpolating currents of scalar, pseudoscalar, vector and axialvector mesons as [78] $J^S = \bar{q}_i q_j$, $J^{PS} = i\bar{q}_i \gamma_5 q_j$, $J^V = \bar{q}_i \gamma_\mu q_j$ and $J^{PV} = \bar{q}_i \gamma_\mu \gamma_5 q_j$, respectively. For baryons the Ioffe current [79] of nucleon can be given as an example, which is $J = \epsilon_{abc}\left(u^{aT} C\gamma_\mu u^b\right)\gamma_5 \gamma^\mu d^c$ with superscripts, a, b and c, being the color indices. The chosen current not only annihilate or create the considered states but also the higher states having the same quantum numbers and valance quark contents. For instance, the current of nucleon can also annihilate or create a Roper resonance. Therefore, to get rid of these unwanted contributions, one applies the method called as Borel transformation. The details of this mathematical operation is to be presented in Section 2.6.

As we mentioned before, there are two ways to be followed for the calculation of the correlator. At $q^2 \ll 0$ it can be calculated in terms of QCD degrees of freedom via operator product expansion (OPE). This way of calculation gives the result in terms of QCD degrees of freedom and called as QCD side or theoretical side. The second way is the calculation of the correlator in terms of hadronic degrees of freedom and called as phenomenological or physical side of the calculation.

The QCD sum rule for the physical quantity of interest is attained by the match of the results obtained from these two ways. The QCD sum rule relates the physical parameters such as masses of the interested hadrons to the parameters of the QCD Lagrangian and the condensates. The following sections will give detailed information on the QCD sum rules method.

2.3 Phenomenological (Physical) Side

The behavior of correlation function $\Pi(q^2)$ is determined by the q^2, that is, at large and negative values of the q^2 the contribution to $\Pi(q^2)$ comes from the short distance interactions and, therefore, calculations are performed perturbatively. On the other hand when the q^2 passes to positive (timelike) values the nonperturbative effects start to contribute to the correlator and hadrons start to be produced from the strongly interacting quarks. In the calculation a complete set of hadronic states, which is a unit operator, is inserted into the correlator. This unit operator contains the states having the same quantum numbers with the interpolating current under consideration and given as

$$\begin{aligned} 1 &= |0\rangle\langle 0| + \sum_h \int \frac{d^4 p_h}{(2\pi)^4} 2\pi\theta(p_h^0)\delta(p_h^2 - m_h^2)|h(p_h)\rangle\langle h(p_h)| \\ &+ \text{higher Fock states.} \end{aligned} \tag{2.4}$$

After the insertion of the unit operator and taking the integrals the correlator takes the form

$$\Pi^{Phen}(q^2) = \sum_h \frac{\langle 0|J|h(q)\rangle\langle h(q)|\bar{J}|h(q)\rangle}{q^2 - m_h^2} \tag{2.5}$$

which includes all the resonances. If the considered hadron is a spin-$\frac{1}{2}$ particle as in the subject of this thesis, the interpolating current couples to the states as

$$\langle 0|J|h(q)\rangle = \lambda_h u(q, s) \tag{2.6}$$

where λ_h is the overlap amplitude giving a measure of the coupling strength of the physical state to the interpolating current and $u(p, s)$ is a spinor. All the hadronic states that have the same quantum numbers with the $J(x)$ can be created (annihilated) by the operator $J(x)$. Therefor the $\Pi^{Phen}(q^2)$ includes the information about not only the lowest lying state but also its excited states and it can be written in the narrow state approximation as

$$\begin{aligned} \Pi^{Phen}(q^2) &= -\frac{|\lambda_h|^2(\not{q} \pm m_h)}{q^2 - m_h^2 + i\varepsilon} - \frac{|\lambda_{h_1^*}|^2(\not{q} \pm m_{h_1^*})}{q^2 - m_{h_1^*}^2 + i\varepsilon} \\ &- \frac{|\lambda_{h_2^*}|^2(\not{q} \pm m_{h_2^*})}{q^2 - m_{h_2^*}^2 + i\varepsilon} + \cdots, \end{aligned} \tag{2.7}$$

where m_{h_i} represents the mass of ith resonance, + sign and − sign in front of the m_{h_i} are used for positive and negative parity states, respectively. The coupling strength between the i^{th} resonance and the interpolating current is given by λ_{h_i}. Contribution of higher states are denoted by $\cdots$.

Generally these states form continuous spectrum and, therefore, one can write the correlator as

$$\Pi_j^{Phen}(q^2) = -\int_0^\infty ds \frac{\rho_j^{Phen}(s)}{q^2 - s + i\varepsilon} + \cdots \qquad j = 1, q. \tag{2.8}$$

In the last expression, $\rho_j^{Phen}(s)$ is the spectral density, which is obtained taking the imaginary part of the the correlator, that is, $\rho_j^{Phen}(s) = \frac{1}{\pi} Im[\Pi_j^{Phen}(s)]$. For the Eq. (2.7) the spectral density has the form

$$\rho_q^{Phen}(s) = |\lambda_h|^2 \delta(s - m_h^2) + |\lambda_{h_1^*}|^2 \delta(s - m_{h_1^*}^2) + |\lambda_{h_2^*}|^2 \delta(s - m_{h_2^*}^2) + \cdots, \tag{2.9}$$

and

$$\begin{aligned} \rho_1^{Phen}(s) &= |\lambda_h|^2 m_h \delta(s - m_h^2) \mp |\lambda_{h_1^*}|^2 m_{h_1^*} \delta(s - m_{h_1^*}^2) \\ &\mp |\lambda_{h_2^*}|^2 m_{h_2^*} \delta(s - m_{h_2^*}^2) + \cdots. \end{aligned} \tag{2.10}$$

To be able to make the predictions effectively, one must parameterize the phenomenological spectrum. It mus have dominant contribution from the lowest resonance while the higher ones should be suppressed. Therefore the following representation of the the spectral density, containing a smooth continuum added to a single sharp pole for the lowest resonance, is used:

$$\rho_q^{Phen}(s) = |\lambda_h|^2 \delta(s - m_h^2) + \rho^{Cont}(s), \tag{2.11}$$

and

$$\rho_1^{Phen}(s) = \mp |\lambda_h|^2 m_h \delta(s - m_h^2) + \rho^{Cont}(s). \tag{2.12}$$

Since only the weighted average of the phenomenological spectral density is important in QCD sum rule, the above parameterization suffices for the spectral density, though it is not perfect. To simplify the calculation in QCD sum rules the quark

hadron duality assumption, which is inferred from the asymptotic freedom property of QCD, is applied. This assumption states that the spectral density corresponding to higher states and continuum for phenomenological side is equivalent to that of OPE side above a certain threshold s_0. The expression giving this statement is

$$\rho^{Cont}(s) \cong \rho^{OPE}(s)\theta(s - s_0). \tag{2.13}$$

The s_0 is chosen considering energy of the first (or second) excited resonance and demanding good Borel stability. Considering the Eq. (2.13) being a phenomenological ansatz, it is necessary to check the dependence of any prediction obtained from QCD sum rules on the choice of s_0.

2.4 QCD (Theoretical) Side

The calculation of the correlator in QCD or theoretical side is performed via Operator Product Expansion (OPE) proposed firstly by Wilson [80]. In the OPE, the time ordered product of two local operators, one at point x and the other at point 0, is expanded in terms of a set of local operators. This expansion can be given as

$$\mathcal{T}[J(x)\bar{J}(0)] \overset{x\to 0}{=} \sum_n c_n(x)\hat{O}_n. \tag{2.14}$$

The $c_n(x)$ are the so called Wilson coefficients containing short distance physics and calculated perturbatively. $\hat{O}_n$ are local composite operator which contain the long distance effects. Example of such operators having different dimensions are given below

$$\begin{aligned}
I \text{ (The unit operator)} \qquad & (d = 0),\\
\bar{q}q \qquad & (d = 3),\\
G^a_{\mu\nu}G^{a\mu\nu} \qquad & (d = 4),\\
\bar{q}\sigma_{\mu\nu}G^{a\mu\nu}t^a q \qquad & (d = 5),\\
\bar{q}\Gamma_1 q\bar{q}\Gamma_2 q \qquad & (d = 6),\\
f_{abc}G^{a\nu}_{\mu}G^{b\lambda}_{\nu}G^{c\mu}_{\lambda} \qquad & (d = 6).
\end{aligned} \tag{2.15}$$

Note that the above operators are color neutral, gauge and Lorentz invariant, and there are no color neutral operator having $d = 1, 2$ giving contribution to the vacuum-vacuum correlation function.

When the Eq. (2.14) is placed into Eq. (2.3), the two point correlator turns into

$$\begin{aligned}\Pi(q^2) &= i\int d^4x e^{iqx}\sum_n c_n(x)\langle 0|\hat{O}_n|0\rangle \\ &= \sum_n c_n(q^2)\langle 0|\hat{O}_n|0\rangle. \end{aligned} \tag{2.16}$$

In the calculation the nonperturbative effects are parameterized by means of the vacuum condensates $\langle 0|\hat{O}_n|0\rangle$, which are the vacuum expectation matrix elements of the operators, some of which are presented in Eq. (2.15). In perturbation theory these matrix elements are zero, however in QCD this is not so due to nonperturbative nature of the vacuum. As a result, the free particle propagator gains additional terms related to these nonperturbative effects. When the correlator is calculated using the OPE it gives the spectral density as

$$\rho^{OPE}(q^2) = \frac{1}{\pi} Im[\Pi(q^2)] \tag{2.17}$$

which is used in the dispersion relation as follows

$$\Pi_j^{OPE}(q^2) = -\int_0^\infty ds \frac{\rho_j^{OPE}(s)}{q^2 - s + i\varepsilon} + \cdots \quad j = 1, q \tag{2.18}$$

and the $\cdots$ is used to represent the subtraction terms.

2.5 Analytic Continuation

While the validity region of correlator $\Pi^{Phen}(q^2)$ is $q^2 \sim$ resonance region, that of $\Pi^{QCD}(q^2)$ is $q^2 \to -\infty$. Matching of these two correlators requires the usage of the analytic continuation application to the Π^{QCD}. For this aim, via Cauchy integral formula, from the $\Pi(q^2)$ the dispersion relation is obtained as

$$\begin{aligned}\Pi(q^2) &= \frac{1}{2\pi i}\oint_C dz \frac{\Pi(z)}{z - q^2} \\ &= \frac{1}{2\pi i}\oint_{|z|=R} dz \frac{\Pi(z)}{z - q^2} \\ &+ \frac{1}{2\pi i}\int_{t_{min}}^{R} dz \frac{\Pi(z + i\epsilon) - \Pi(z - i\epsilon)}{z - q^2} \end{aligned} \tag{2.19}$$

for the correlator, Eq. (2.3) using the counter shown in Figure 2.2. In the last equation t_{min} represents the threshold where the real states are created. For the integral, taking

the radius $R \to \infty$ for the circular part of the contour simplifies the calculation, because the integral of circular part on the right hand side goes to zero if the correlation function vanishes fast enough at $q^2 \sim R \to \infty$ leaving the result

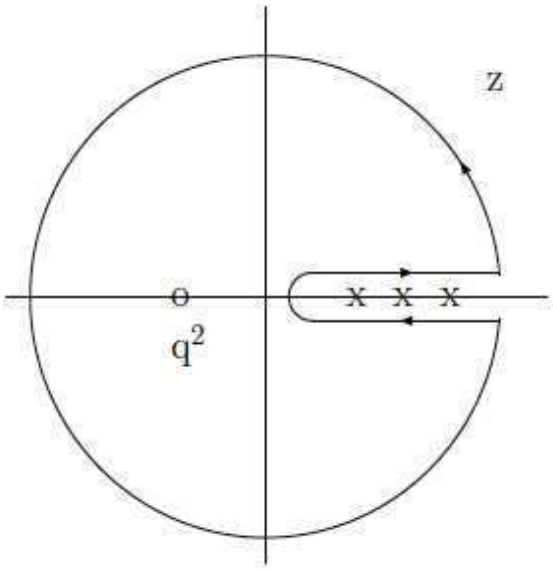


Figure 2.2: The the contour in the complex z-plane used for the calculation of the correlation function at $z = q^2$. The reference point of the QCD calculation is represented by the open point while the crosses are used to represent the hadronic thresholds.

$$\Pi(q^2) = \frac{1}{2\pi i}\int_{t_{min}}^{\infty} dz \frac{\Pi(z+i\epsilon) - \Pi(z-i\epsilon)}{z-q^2} + \text{subtraction terms.} \tag{2.20}$$

For $q^2 < t_{min}$ the $\Pi(q^2)$ is real and for $q^2 > t_{min}$ using the Schwartz reflection principle, for the numerator of the integrand of Eq. (2.20) we can use $\Pi(z+i\epsilon) - \Pi(z-i\epsilon) = 2iIm[\Pi(q^2)]$ which turns the Eq. (2.20) into the following dispersion relation

$$\Pi(q^2) = \frac{1}{\pi}\int_{t_{min}}^{\infty} ds Im \frac{\Pi(s)}{s-q^2} + \text{subtraction terms.} \tag{2.21}$$

The term $\frac{1}{\pi}Im[\Pi(s)] = \rho(s)$ in last equation is called as spectral density and the part of that equation stated as subtraction terms are polynomials in q^2 emerging from the circular integral at infinity.

2.6 Borel Transformation

In the QCD sum rule calculations there are two sides which have been mentioned so far. These two sides are represented by the expressions $\Pi^{Phen}(q^2)$ and $\Pi^{QCD}(q^2)$. $\Pi^{Phen}(q^2)$ is an expression containing physical parameters whereas the $\Pi^{QCD}(q^2)$ contains terms including QCD parameters such as α_s, quark masses, and previously mentioned vacuum expectation values of the operators O_n. Though the match of these two sides provides us with the physical quantity that we search for, this match at this stage is not practical. Before matching the two sides, the contribution coming from the higher dimensional operators should be sufficiently suppressed in the OPE side and lowest pole should dominate the spectrum in the physical side. And also there are the unwanted subtraction terms which should be dealt with. To this end, we employ an operation called as Borel transformation and improve the QCD sum rules. This is not only removes the subtraction terms but also provides suppression on the higher states and continuum contribution.

For a function $f(Q^2)$ the Borel transformation has the following definition

$$f(Q^2) \to \tilde{f}(M^2) = B_{M^2} f(Q^2) = \lim_{Q^2,n\to\infty} \frac{(Q^2)^n}{(n-1)!}\left(\frac{d}{dQ^2}\right)^{n-1} f(Q^2), \tag{2.22}$$

where $Q^2 = -q^2$, $\frac{q^2}{n} = M^2$ and M is the Borel parameter. As a derivative operation this transformation sweeps away polynomial terms in q^2, therefore, this application removes the subtraction terms. Additionally the term $\frac{1}{s-q^2}$ is rendered to $e^{\frac{-s}{M^2}}$ after Borel transformation and this results in an exponential suppression instead of a power suppression for the correlator. The correlator , for instance, for the phenomenological side,

$$\Pi_j^{Phen}(Q^2) = \int_0^\infty ds \frac{\rho_j^{Phen}(s)}{Q^2+s}, \tag{2.23}$$

turns into

$$\Pi_j^{Phen}(M^2) = \int_0^\infty ds e^{\frac{-s}{M^2}} \rho_j^{Phen}(s). \tag{2.24}$$

As for the OPE side, this operation provides also some advantageous for this side improving its convergence. If we consider the nonperturbative part of the OPE for

correlation function that is given as

$$\Pi^{OPE}_{nonpert}(Q^2) = \sum_{nonpert} \frac{c'_n}{(Q^2)^n}\langle O_n\rangle, \tag{2.25}$$

it has a power suppression in Q^2. However after the Borel transformation this turns into a factorial suppression as

$$\Pi^{OPE}_{nonpert}(M^2) = \sum_{nonpert} \frac{c'_n}{(n-1)!(M^2)^n}\langle O_n\rangle, \tag{2.26}$$

together with increasing n values.

The three important and frequently used examples of Borel transformation are as follows:

$$B_{M^2}(q^2)^t \;=\; 0, \quad t \geq 0, \tag{2.27}$$

$$B_{M^2}\Big(\frac{1}{(m^2[s]-q^2)^t}\Big) = \frac{1}{(t-1)!}\frac{e^{\frac{m^2[s]}{M^2}}}{(M^2)^{t-1}}, \tag{2.28}$$

and

$$B_{M^2}e^{-\alpha q^2} = \delta(\frac{1}{M^2}-\alpha). \tag{2.29}$$

Actually, the Borel parameter M^2 is not a physical parameter and it is completely arbitrary. Due to this the physical parameters obtained from QCD sum rule should have no dependency on this parameter. However in practice from the analysis a region having moderate dependency can be fixed. In the physical side of the calculation with the $M^2 \to 0$ the dominance by the lowest pole is supplied, on the other hand when $M^2 \to \infty$ the convergence of the OPE is enhanced. So in general a working interval of the Borel parameter is considered to attain dominance for the lowest mass resonance and suppression on the contribution of the higher-dimensional operators.

When the form of the correlation function for OPE side given in Eq. (2.18) considered, after the Borel transformation it becomes

$$\Pi^{OPE}_j(M^2) = \int_0^\infty ds e^{\frac{-s}{M^2}}\rho^{OPE}_j(s). \tag{2.30}$$

Finally the sum rule results in the following form

$$\int_0^\infty ds e^{\frac{-s}{M^2}}\rho^{Phen}_j(s) = \int_0^\infty ds e^{\frac{-s}{M^2}}\rho^{OPE}_j(s). \tag{2.31}$$

And after the application of quark hadron duality assumption given in Eq. (2.13) it becomes

$$\int_0^{s_0} ds e^{\frac{-s}{M^2}} \rho_j^{Phen}(s) = \int_0^{s_0} ds e^{\frac{-s}{M^2}} \rho_j^{OPE}(s), \tag{2.32}$$

where the spectral density of phenomenological side is parameterized as in the Eqs. (2.11) and Eqs. (2.12).

2.7 Vacuum Condensates

The QCD sum rules give the physical parameters in terms of the vacuum condensates. These vacuum condensates are expectation values of the composite operators which do not vanish because of the nonperturbative nature of QCD vacuum. To analyze the obtained QCD sum rule results, their values are needed and for sum of these condensates the standard estimates will be summarized in the present section. Since the vacuum is Lorentz invariant only the spin-0 operators have vacuum expectation value that is not zero. Present section will only include the condensate that are used in the following chapter, which includes details of our calculations.

The lowest dimensional condensate giving nonperturbative contribution to our calculation is the quark condensate which are equivalent for u-quark and d-quark cases because of the isospin symmetry,

$$\langle \bar{q}q \rangle = \langle \bar{u}u \rangle = \langle \bar{d}d \rangle. \tag{2.33}$$

The Gell-Mann-Oakes-Renner relation is used to determine the value of $\langle \bar{q}q \rangle$,

$$(m_u + m_d)\langle \bar{q}q \rangle = -m_\pi^2 f_\pi^2 (1 + O(m_{\pi^2})), \tag{2.34}$$

with m_u and m_d being the current quark masses of u and d quarks, respectively. m_π and f_π are the mass and decay constant of the pion. Considering a particular renormalization scale the value of condensates are determined. At renormalization scale 1 GeV [81]

$$\langle \bar{q}q \rangle (1 \text{ GeV}) = (-240 \pm 10 \text{ MeV})^3 \tag{2.35}$$

The gluon condensate was firstly estimated from the analysis of the leptonic decays of ρ^0 and ϕ^0 mesons [82] and sum rule analysis of charmonium spectrum [71]. Its value is [83]

$$\langle\frac{\alpha_s}{\pi}G^2\rangle = (0.012^{+0.006}_{-0.012})\ \text{GeV}^4. \tag{2.36}$$

The dimension five operators are

$$\langle g_s\bar{q}\sigma.Gq\rangle \equiv \langle g_s\bar{u}\sigma.Gu\rangle \simeq \langle g_s\bar{d}\sigma.Gd\rangle \tag{2.37}$$

due to the isospin symmetry and the values of them can be given in terms of quark condensate as follows

$$\langle g_s\bar{u}\sigma.Gu\rangle = 2\langle\bar{q}D^2q\rangle \equiv 2\lambda_q^2\langle\bar{q}q\rangle. \tag{2.38}$$

To obtain the first equality the following equations of motion are applied

$$\begin{aligned}(i\not{D} - m_f)q_f &= 0,\\ q_f(i\not{D} + m_f) &= 0,\end{aligned} \tag{2.39}$$

together with

$$G^a_{\mu\nu} = \frac{i}{g_s}[D_\mu, D_\nu]^a. \tag{2.40}$$

The value of λ_q^2 can be found in Refs. [84, 85, 86, 87].

Usually using the factorization or vacuum saturation approximation the four quark condensates are approximated in terms of the $\langle\bar{q}q\rangle$ and $\langle\frac{\alpha_s}{\pi}G^2\rangle$. The Ref [88] can be referred for more details of the factorization approximation and finite density estimation of four-quark condensates.

The other condensates having higher dimensions can be effectively determined in terms of the lower-dimensional ones. However in many analysis their contribution becomes negligibly small because of the quick convergence of the Borel transformed OPE and the contributions coming from them can be neglected.

CHAPTER 3

QCD SUM RULES FOR Σ_Q BARYON

This chapter presents the details of our calculations to obtain the masses and the residues of the ground state and first orbital excitation of the baryons Σ_c and Σ_b. To obtain these physical parameters we apply the two point QCD sum rule method whose details are given in Chapter 2.

In the QCD sum rule calculations, to obtain the physical parameters, three steps are followed. In the first part, the correlation function is calculated in terms of hadronic degrees of freedom and, therefore, this part is called as physical side or hadronic side. The second part includes the calculation of the same correlation function in terms of QCD degrees of freedom. This side is called as QCD or theoretical side. The final part requires the match of the results obtained from the previous two sides. To perform this match the coefficients of the same Lorentz structures are chosen from both sides. In what follows we will give the details about first and second steps. We will leave the matching of the obtained results and their numerical analyses to the next section.

3.1 Physical Side

In this section we give the details of the calculations of the physical side to obtain the masses and overlap amplitudes. As is stated in Chapter 2, the calculation of the mass and overlap amplitude starts with the following correlation function:

$$\Pi(q^2) = i \int d^4x e^{iqx} \langle 0|\mathcal{T}\{J(x)\bar{J}(0)\}|0\rangle. \tag{3.1}$$

In Eq. (3.1) the $\mathcal{T}$ represents time ordering operator and $J(x)$ is the interpolating field having the same quantum numbers and valance quark content with the hadron of interest, which are either Σ_c or Σ_b baryons in this work. In this part the interpolating fields are treated as operators creating or annihilating the considered hadrons. To obtain the physical side one inserts a complete set of hadronic states, given as

$$\begin{aligned} 1 &= |0\rangle\langle 0| + \sum_h \int \frac{d^4p_h}{(2\pi)^4} 2\pi\theta(p_h^0)\delta(p_h^2 - m_h^2)|h(p_h)\rangle\langle h(p_h)| \\ &+ \text{higher Fock states}, \end{aligned} \tag{3.2}$$

which has the same quantum numbers with the interpolating field present in the correlation function. After taking the four dimensional integral the correlation function turns into the following form

$$\begin{aligned} \Pi^{\text{Had}}(q^2) &= \frac{\langle 0|J_{\Sigma_Q}|\Sigma_Q(q,s):+\rangle\langle\Sigma_Q(q,s):+|J_{\Sigma_Q}|0\rangle}{m^2-q^2} \\ &+ \frac{\langle 0|J_{\Sigma_Q}|\tilde{\Sigma}_Q(q,s):-\rangle\langle\tilde{\Sigma}_Q(q,s):-|J_{\Sigma_Q}|0\rangle}{\tilde{m}^2-q^2} + \cdots, \end{aligned} \tag{3.3}$$

where $|\Sigma_Q(q,s):+\rangle$ and $|\Sigma_Q(q,s):-\rangle$ represent the ground state Σ_Q having positive parity and its first orbital excitation having negative parity, respectively, and m and $\tilde{m}$ are their corresponding masses. The dots in Eq. (3.3) denote the contributions coming from higher states and continuum. Note that although the current to be used in the calculations has positive parity, they interact with states having both parities. In the last result, in Eq. (3.3), there appear the matrix elements between one particle states and the vacuum and they are parameterized as

$$\begin{aligned} \langle 0|J_{\Sigma_Q}|\Sigma_Q(q,s):+\rangle &= \lambda_{\Sigma_Q} u(q,s) \\ \langle 0|J_{\Sigma_Q}|\Sigma_Q(q,s):-\rangle &= \tilde{\lambda}_{\Sigma_Q}\gamma_5 u(q,s) \end{aligned} \tag{3.4}$$

The λ_{Σ_Q} and $\tilde{\lambda}_{\Sigma_Q}$ are the residues of these satates and $u(q,s)$ denotes the spinor of these spin-$\frac{1}{2}$ states. When we place these matrix elements in the Eq. (3.3) the calculation requires the summation over the spin of the spinors which has the form

$$\sum u(q,s)\bar{u}(q,s) = \not{q} + m. \tag{3.5}$$

With the usage of the Eq. (3.4) and then Eq. (3.5) the final result for the correlation function for the physical side becomes

$$\Pi^{\rm Had}(q^2) = \frac{\lambda_{\Sigma_Q}^2(\not{q}+m)}{m^2-q^2} + \frac{\tilde{\lambda}_{\Sigma_Q}^2(\not{q}-\tilde{m})}{\tilde{m}^2-q^2} + \cdots . \tag{3.6}$$

In the analysis for the physical side we first consider the first term of the Eq. (3.6) and calculate the masses of the ground states with a proper choice of the threshold parameter. Then we consider both terms and using the result obtained for the masses of the ground states as input parameters we obtain the masses and residues for the first orbital excitation of the Σ_Q baryons.

After the Borel transformation the final forms for the physical sides are obtained as

$$\begin{aligned}\frac{1}{\pi}{\rm Im}[\Pi^{\rm Had}(q^2)] &= \lambda_{\Sigma_Q}^2 e^{-\frac{q^2}{M^2}}(\not{q}+m)\delta(q^2-m^2) + \tilde{\lambda}_{\Sigma_Q}^2 e^{-\frac{q^2}{M^2}}(\not{q}-\tilde{m})\delta(q^2-\tilde{m}^2) \\ &+ \cdots ,\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\frac{1}{\pi}{\rm Im}[\Pi_{\not{q}}^{\rm Had}(q^2)] &= \lambda_{\Sigma_Q}^2 e^{-\frac{q^2}{M^2}}\delta(q^2-m^2) + \tilde{\lambda}_{\Sigma_Q}^2 e^{-\frac{q^2}{M^2}}\delta(q^2-\tilde{m}^2) + \cdots , \\ \frac{1}{\pi}{\rm Im}[\Pi_{1}^{\rm Had}(q^2)] &= \lambda_{\Sigma_Q}^2 m e^{-\frac{q^2}{M^2}}\delta(q^2-m^2) - \tilde{\lambda}_{\Sigma_Q}^2 \tilde{m} e^{-\frac{q^2}{M^2}}\delta(q^2-\tilde{m}^2) + \cdots ,\end{aligned} \tag{3.8}$$

where the sub-indices $\not{q}$ and 1 represent that the results given by $\Pi_{\not{q}}$ is the coefficient of the $\not{q}$ structure and Π_1 is that of the 1.

3.2 QCD Side

After the calculation of the physical side, the next step is calculation of the QCD side. For the calculation the same interpolating function given in Eq. (3.1) is used. To calculate this part the explicit form of the interpolating current which is written in terms of quark fields is used. The interpolating currents, $J(x)$ and $\bar{J}(0)$ in Eq. (3.1) are given as

$$J(x) = \epsilon^{ijk}\Big[u_i^T(x)C\gamma^\mu u_j(x)\Big]\gamma_5\gamma_\mu Q_k(x), \tag{3.9}$$

and

$$\begin{aligned}\bar{J}(x) &= J^{\dagger}(0)\gamma^{0} \\ &= -\epsilon^{ijk}\Big[\bar{u}_{j}(0)\gamma^{\mu}C\bar{u}_{i}^{T}(0)\Big]\bar{Q}_{k}(0)\gamma_{5}\gamma_{\mu}. \end{aligned} \tag{3.10}$$

In these interpolating fields the u represents the quark fields for u quark and Q is the quark field for c or b-quark. i, j, k represent the color indices and C is the charge conjugation operator. After their insertion inside the correlation function, one uses the Wick theorem. By means of this theorem the correlator becomes

$$\begin{aligned}\Pi^{QCD}(q^2) &= -i\int d^4x e^{iqx}\epsilon^{ijk}\epsilon^{i'j'k'}\Big[iS_{jj'}^{u,\beta\beta'}(x)(\gamma^{\mu'}C)^{\beta'\alpha'}iS_{ii'}^{u,\alpha'\alpha T}(x)(C\gamma^{\mu})^{\alpha\beta} \\ &\times (\gamma_5\gamma_{\mu})^{\sigma\delta}iS_{kk'}^{Q,\delta\delta'}(x)(\gamma_{\mu'}\gamma_5)^{\delta'\sigma'} \\ &- iS_{ij'}^{u,\alpha\beta'}(x)(\gamma^{\mu'}C)^{\beta'\alpha'}iS_{ji'}^{u,\alpha'\beta T}(x)(C\gamma^{\mu})^{T\beta\alpha} \\ &\times (\gamma_5\gamma_{\mu})^{\sigma\delta}iS_{kk'}^{Q,\delta\delta'}(x)(\gamma_{\mu'}\gamma_5)^{\delta'\sigma'}\Big], \end{aligned} \tag{3.11}$$

where the $iS_{ij}^{u}(x)$ and $iS_{ij}^{Q}(x)$ are the light and heavy quark propagators which include the terms containing perturbative and nonperturbative contributions and are defined as

$$iS_{ij}^{q,\alpha\beta}(x) = \langle 0|\mathcal{T}\{q_i^{\alpha}(x)\bar{q}_j^{\beta}(0)\}|0\rangle \tag{3.12}$$

for light quarks and

$$iS_{ij}^{Q,\alpha\beta}(x) = \langle 0|\mathcal{T}\{Q_i^{\alpha}(x)\bar{Q}_j^{\beta}(0)\}|0\rangle \tag{3.13}$$

for heavy ones. Their explicit forms are given as

$$\begin{aligned}iS_{ij}^{q,\alpha\beta}(x) &= \langle 0|\mathcal{T}\{q_i^{\alpha}(x)\bar{q}_j^{\beta}(0)\}|0\rangle \\ &= +\frac{i}{2\pi^2}\frac{1}{(x^2)^2}\delta_{ij}\not{x}^{\alpha\beta} - \frac{1}{12}\delta_{ij}\delta^{\alpha\beta}\langle\bar{q}q\rangle \\ &- \frac{1}{192}x^2\delta_{ij}\delta^{\alpha\beta}g\langle\bar{q}\sigma.Gq\rangle \end{aligned}$$

$$
\begin{aligned}
&- \frac{i}{32\pi^2}\frac{1}{x^2}g_s G^A_{\mu\nu}t^A_{ij}(\not{x}\sigma^{\mu\nu}+\sigma^{\mu\nu}\not{x})^{\alpha\beta} \\
&- \frac{\pi^2}{3^5 2^7}x^4\delta_{ij}\delta^{\alpha\beta}\langle\bar{q}q\rangle\langle\frac{\alpha_s}{\pi}G^2\rangle - \frac{1}{4\pi^2}\frac{1}{x^2}\delta_{ij}\delta^{\alpha\beta}m_q \\
&+ \frac{i}{48}\delta_{ij}m_q\langle\bar{q}q\rangle\not{x}^{\alpha\beta} + \frac{i}{3^2 2^7}x^2\delta_{ij}m_q g\langle\bar{q}\sigma.Gq\rangle\not{x}^{\alpha\beta} \\
&- \frac{1}{32\pi^2}[ln(-\frac{x^2\Lambda^2}{4})+2\gamma_{EM}]m_q g_s G^A_{\mu\nu}t^A_{ij}(\sigma^{\mu\nu})^{\alpha\beta} \\
&+ \frac{i}{8\pi^2}\frac{1}{x^2}\delta_{ij}m_q^2\not{x}^{\alpha\beta} + \frac{1}{96}x^2\delta_{ij}\delta^{\alpha\beta}m_q^2\langle\bar{q}q\rangle \\
&+ \frac{i}{2^7\pi^2}[ln(-\frac{x^2\Lambda^2}{4})+2\gamma_{EM}]m_q^2 g_s G^A_{\mu\nu}t^A_{ij}(\not{x}\sigma^{\mu\nu}+\sigma^{\mu\nu}\not{x})^{\alpha\beta} \\
&+ \frac{1}{3^2 2^8}x^4\delta_{ij}\delta^{\alpha\beta}m_q^2\langle\bar{q}\sigma.Gq\rangle \\
&- \frac{i}{3^5 2^5}x^2\delta_{ij}g_s^2\langle\bar{q}q\rangle^2\not{x}^{\alpha\beta} - \frac{1}{3^5 2^7}x^4\delta_{ij}\delta^{\alpha\beta}m_q g_s^2\langle\bar{q}q\rangle^2, \qquad (3.14)
\end{aligned}
$$

for the light quark propagator [76] and

$$
\begin{aligned}
iS^{Q,\alpha\beta}_{ij}(x) &= \langle 0|\mathcal{T}\{Q^\alpha_i(x)\bar{Q}^\beta_j(0)\}|0\rangle \\
&= +i\int\frac{d^4k}{(2\pi^4)}e^{-ikx}\Bigg\{\frac{\delta_{ij}(\not{k}+m_Q)^{\alpha\beta}}{k^2-m_Q^2} \\
&- \frac{g_s G^{\mu\nu}_{ij}}{4}\frac{\sigma_{\mu\nu}(\not{k}+m_Q)^{\alpha\beta}+(\not{k}+m_Q)^{\alpha\beta}\sigma_{\mu\nu}}{(k^2-m_Q^2)^2} \\
&+ \frac{g_s^2G^2}{12}\delta_{ij}m_Q\frac{(k^2+m_Q\not{k})^{\alpha\beta}}{(k^2-m_Q^2)^4} + \frac{g_s^3G^3}{48}\delta_{ij}\frac{(\not{k}+m_Q)^{\alpha\beta}}{(k^2-m_Q^2)^6} \\
&\times \Big(\Big[\not{k}(k^2-3m_Q^2)+2m_Q(2k^2-m_Q^2)\Big](\not{k}+m_Q)\Big)^{\alpha\beta}+\cdots\Bigg\}, \qquad (3.15)
\end{aligned}
$$

for the heavy quark propagator [78]. In these propagator equations the α and β represent the spinor indices, i, j are the color indices, $\gamma_{EM} \simeq 0.577$ is the Euler constant and the Λ is a scale parameter that is used to separate the perturbative and nonperturbative regions. As it can be seen from their explicit expressions, these quark propagators contain different terms related to the perturbative part, the quark condensate part, which is only included in the light quark propagator, and the part that contains gluon condensate. Quark condensate part represents the interaction of the quark fields having same flavor, while the gluon condensate part shows the interaction of the gluon fields with the quark fields [72]. Considering these we can shortly represent the propagators as

$$S^{q}(x) = S^{q,\text{Free}}(x) + S^{q,\text{Quark condensate}}(x) + S^{q,\text{Gluon condensate}}(x) \quad (3.16)$$

and

$$S^{Q}(x) = S^{Q,\text{Free}}(x) + S^{Q,\text{Gluon condensate}}(x) \quad (3.17)$$

In the calculation of the OPE side we apply the terms of the propagators one by

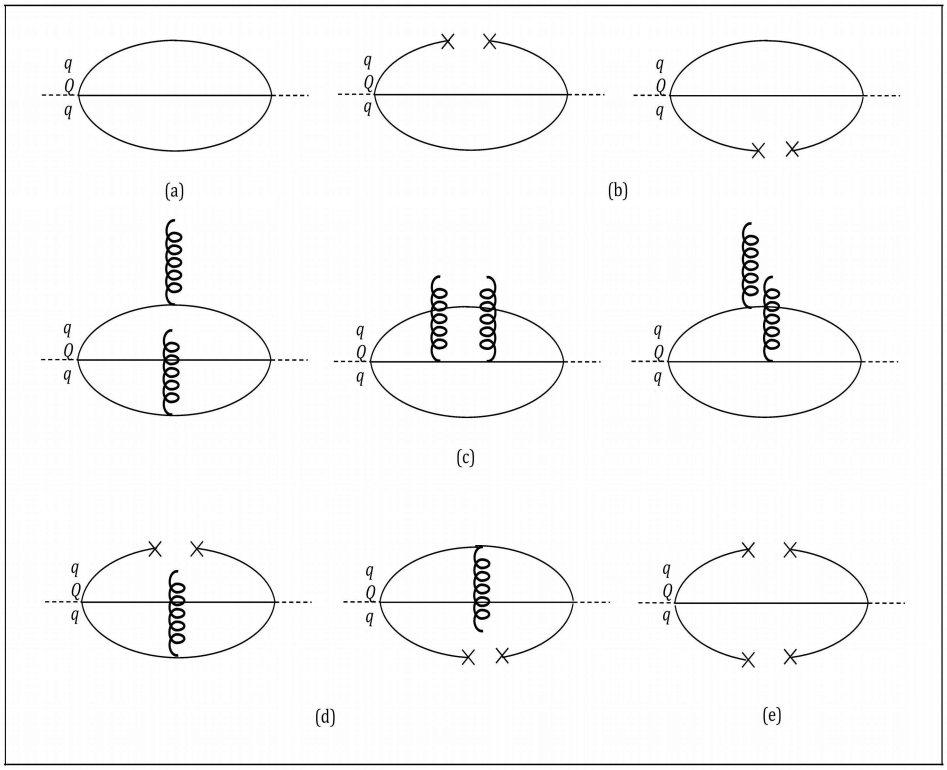


Figure 3.1: Schematic OPE diagrams showing the terms calculated in the QCD sum rule calculations for masses and residues.

one. In Figure 3.1 all these terms are given schematically. In this figure solid lines represent the light and the heavy quark propagators. Therefore (a) schematizes the contribution of the perturbative parts of the propagators. (b) shows the contribution of the terms in which for one of the light quark propagator the term proportional with

$\langle\bar{q}q\rangle$ is considered and the crosses are used to represent the quark condensate. The curly lines are used for the gluons therefore (c) corresponds to diagrams representing the contributions related to $\langle\frac{\alpha_s}{\pi}G^2\rangle$ parts of the propagators. The (d) represents the contributions of the mixed quark-gluon condensate. And finally (e) is related to the contribution of $\langle\bar{q}q\rangle^2$.

In the firs part, we will consider only the contributions coming from the perturbative parts of all the propagators and find the result for the part (a) of Figure 3.1. To do this we insert the terms of each propagator given as

$$iS_{ij}^{q,\alpha\beta}(x) = +\frac{i}{2\pi^2}\frac{1}{(x^2)^2}\delta_{ij}\not{x}^{\alpha\beta} - \frac{m_q}{4\pi^2}\frac{1}{x^2}\delta_{ij}\delta^{\alpha\beta} \tag{3.18}$$

for light u quarks and

$$iS_{ij}^{Q,\alpha\beta}(x) = +i\int\frac{d^4k}{(2\pi^4)}e^{-ikx}\frac{\delta_{ij}(\not{k}+m_Q)^{\alpha\beta}}{k^2-m_Q^2} \tag{3.19}$$

for heavy one. We place them into the following result of the correlation function obtained from Eq. (3.11) using the matrix relation $A^{\alpha\beta}B^{\beta\lambda}E^{\lambda\alpha} = Tr[ABE]$ as

$$\begin{aligned}\Pi^{QCD(a)}(q^2) &= -i\int d^4x e^{iqx}\epsilon^{ijk}\epsilon^{i'j'k'}\Big\{Tr[iS^{u}_{jj'}(x)(\gamma^{\mu'}C)iS^{uT}_{ii'}(x)(C\gamma^{\mu})] \\ &\times (\gamma_5\gamma_\mu)iS^{Q}_{kk'}(x)(\gamma_{\mu'}\gamma_5) - Tr[iS^{u}_{ij'}(x)(\gamma^{\mu'}C)iS^{uT}_{ji'}(x)(C\gamma^{\mu})^T] \\ &\times (\gamma_5\gamma_\mu)iS^{Q}_{kk'}(x)(\gamma_{\mu'}\gamma_5)\Big\}.\end{aligned} \tag{3.20}$$

After insertion of the quark propagators the next step requires the calculation of the colour factors existing inside these terms which has the following form

$$\epsilon^{ijk}\epsilon^{i'j'k'}\delta^{ii'}\delta^{jj'}\delta^{kk'} = \epsilon^{ijk}\epsilon^{ijk} = \delta^{jj}\delta^{kk} - \delta^{jk}\delta^{kj} = 9-3 = 6 \tag{3.21}$$

in the first term of the last result and

$$\epsilon^{ijk}\epsilon^{i'j'k'}\delta^{ij'}\delta^{ji'}\delta^{kk'} = \epsilon^{ijk}\epsilon^{jjk} = \delta^{ij}\delta^{ji} - \delta^{ii}\delta^{jj} = 3-6 = -6 \tag{3.22}$$

in the second term of the last result. Besides the color factors we also use the relation

$$(C\gamma^{\mu})^{T} = (C\gamma^{\mu}). \tag{3.23}$$

Application of these colour terms and the last relation turns the correlation function, Eq. (3.20), into

$$\begin{aligned}\Pi^{QCD(a)}(q^2) &= -12i\int d^4x e^{iqx}\Big\{Tr[iS^{u}(x)(\gamma^{\mu'}C)iS^{uT}(x)(C\gamma^{\mu})] \\ &\times (\gamma_5\gamma_{\mu})iS^{Q}(x)(\gamma_{\mu'}\gamma_5)\Big\}. \end{aligned}\tag{3.24}$$

And finally we get

$$\begin{aligned}\Pi^{QCD(a)}(q^2) &= -12(i)^4\int d^4x e^{iqx}\frac{1}{(2\pi^2)^2}\frac{1}{(x^2)^4}\Big\{Tr[\not{x}(\gamma^{\mu'}C)\ \not{x}^{T}(C\gamma^{\mu})] \\ &\times (\gamma_5\gamma_{\mu})\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{\not{k}+m_Q}{k^2-m_Q^2}(\gamma_{\mu'}\gamma_5)\Big\}. \end{aligned}\tag{3.25}$$

Using

$$C\ \not{x}^{T}C = \not{x} \tag{3.26}$$

we obtain the trace part of the above relation as

$$\begin{aligned}Tr[\not{x}(\gamma^{\mu'}C)\ \not{x}^{T}(C\gamma^{\mu})] &= Tr[\not{x}\gamma^{\mu'}\ \not{x}\gamma^{\mu})] \\ &= x_{\alpha}x_{\beta}Tr[\gamma^{\alpha}\gamma^{\mu'}\gamma^{\beta}\gamma^{\mu})] \\ &= x_{\alpha}x_{\beta}4(g^{\alpha\mu'}g^{\beta\mu} - g_{\alpha\beta}g^{\mu'\mu} + g^{\alpha\mu}g^{\mu'\beta}) \\ &= 8x^{\mu}x^{\mu'} - 4x^2g^{\mu'\mu}. \end{aligned}\tag{3.27}$$

with this result the correlation function for the perturbative terms becomes

$$\begin{aligned}\Pi^{QCD(a)}(q^2) &= -12(i)^4\int d^4x e^{iqx}\frac{1}{(2\pi^2)^2}\frac{1}{(x^2)^4}\Big\{(8x^{\mu}x^{\mu'} - 4x^2g^{\mu'\mu}) \\ &\times (\gamma_5\gamma_{\mu})\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{\not{k}+m_Q}{k^2-m_Q^2}(\gamma_{\mu'}\gamma_5)\Big\}. \end{aligned}\tag{3.28}$$

After this step the four integrals over x and k are performed using the relations given in App. A and finally the perturbative part results in

$$\Pi^{QCD(a)}(q^2) = \frac{-1}{2^4\pi^4}\int_0^1 dz\frac{1}{(1-z)}\Big[3z[-L(z)]^2 - 2q^2z^2(1-z)[-L(z)]\Big] \not{q} ln[-L(z)] - \frac{3m_Q}{2^5\pi^4}\int_0^1 dz\frac{1}{(1-z)^2}[-L(z)]ln[-L(z)] \quad (3.29)$$

whose imaginary part gives us the spectral density as

$$\frac{1}{\pi}\mathrm{Im}[\Pi^{QCD(a)}(q^2)] = \frac{1}{2^4\pi^4}\int_0^{1-\frac{m_Q^2}{q^2}} dz\frac{1}{(1-z)}\Big[3z[-L(z)]^2 - 2q^2z^2(1-z)[-L(z)]\Big] \not{q} + \frac{3m_Q}{2^5\pi^4}\int_0^{1-\frac{m_Q^2}{q^2}} dz\frac{1}{(1-z)^2}[-L(z)]^2. \quad (3.30)$$

The $L(z)$ in these equation is

$$L(z) = q^2z(1-z) - m_Q^2z. \quad (3.31)$$

For the calculation of the terms containing quark condensate we insert the perturbative terms for the heavy and one of the light quark propagators and for the remaining light quark propagator we use its part which is given as

$$iS_{ij}^{q,\alpha\beta}(x) = (-\frac{1}{12}\delta^{\alpha\beta} - \frac{I}{48}m_q \not{x}^{\alpha\beta})\delta_{ij}\langle\bar{q}q\rangle. \quad (3.32)$$

As it can be seen from the colour factor of this propagator, the colour factor calculation for this term will not change and we use the results of previous calculations. With the new terms for one of the light quark propagator and using the first term of the perturbative part of the second light quark propagator, the correlation function becomes

$$\Pi^{QCD(b)}(q^2) = -12(i)^3\int d^4x e^{iqx}\frac{1}{(2\pi^2)}\frac{1}{(x^2)^2}\frac{-1}{12}\langle\bar{q}q\rangle\Big\{\Big[Tr[(\gamma^{\mu'}C)\ \not{x}^T(C\gamma^{\mu})] + Tr[\not{x}(\gamma^{\mu'}C)(C\gamma^{\mu})]\Big](\gamma_5\gamma_{\mu})\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{\not{k}+m_Q}{k^2-m_Q^2}(\gamma_{\mu'}\gamma_5)\Big\}. \quad (3.33)$$

The traces in this equation are as follows

$$Tr[(\gamma^{\mu'}C)\ \not{k}^T(C\gamma^{\mu})] = Tr[\gamma^{\mu'}\ \not{k}\gamma^{\mu}] = 0,$$
$$Tr[\not{k}(\gamma^{\mu'}C)(C\gamma^{\mu})] = -Tr[\not{k}\gamma^{\mu'}\gamma^{\mu}] = 0. \quad (3.34)$$

Therefore our results do not have a contribution coming from the term $\langle\bar{q}q\rangle$ when for just one light quark propagator we take its quark condensate term. If we use the second term of the Eq. (3.32) and the whole terms in perturbative part and we follow the similar steps as in the calculations of the perturbative term, the imaginary part of the correlation function for the results that are proportional to $\langle\bar{q}q\rangle$ takes the form

$$\frac{1}{\pi}\mathrm{Im}[\Pi^{QCD(b)}(q^2)] = \frac{m_u\langle\bar{u}u\rangle}{\pi^2}\int_0^{1-\frac{m_Q^2}{q^2}} dz\Big[(-1+(4-3z)z)\Big]\not{q}$$
$$- \frac{3m_Q m_u\langle\bar{u}u\rangle}{2\pi^2}\int_0^{1-\frac{m_Q^2}{q^2}} dz. \quad (3.35)$$

As for the terms proportional to gluon condensate, $\langle\frac{\alpha_s}{\pi}G^2\rangle$, there are different parts. In the first case we take the part of the heavy quark propagator given below keeping the perturbative terms for the light quark propagators. For this purpose we take the part of the heavy quark propagator that is given as

$$iS_{ij}^{Q,\alpha\beta}(x) = +i\int\frac{d^4k}{(2\pi^4)}e^{-ikx}\frac{g_s^2}{12}G^2\frac{\delta_{ij}(k^2+m_Q\ \not{k})^{\alpha\beta}}{(k^2-m_Q^2)^4} \quad (3.36)$$

and the correlation function becomes

$$\Pi^{QCD(c1)}(q^2) = -12(i)^4\frac{1}{12}\langle g_s^2G^2\rangle m_Q\int d^4x e^{iqx}\frac{1}{(2\pi^2)^2}\frac{1}{(x^2)^4}\Big\{Tr[\not{k}(\gamma^{\mu'}C)\ \not{k}^T(C\gamma^{\mu})]$$
$$\times\ (\gamma_5\gamma_{\mu})\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{(k^2+m_Q\ \not{k})}{(k^2-m_Q^2)^4}(\gamma_{\mu'}\gamma_5)\Big\}. \quad (3.37)$$

After the usage of the trace result given in Eq.(3.27), it becomes

$$\Pi^{QCD(c1)}(q^2) = -(i)^4\langle g_s^2G^2\rangle m_Q\int d^4x e^{iqx}\frac{1}{(2\pi^2)^2}\frac{1}{(x^2)^4}\Big\{(8x^{\mu}x^{\mu'}-4x^2g^{\mu'\mu})$$
$$\times\ \gamma_{\mu}\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{(k^2-m_Q\ \not{k})}{(k^2-m_Q^2)^4}\gamma_{\mu'}\Big\}. \quad (3.38)$$

The last result is obtained as

$$\Pi^{QCD(c1)}(q^2) = -(i)^4\langle g_s^2G^2\rangle m_Q\int d^4x e^{iqx}\frac{1}{(2\pi^2)^2}\frac{1}{(x^2)^4} \times 4\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{1}{(k^2-m_Q^2)^4}(-2x^2k^2-4\not{k}.xm_Q), \quad (3.39)$$

after making the contraction for Dirac matrices. Performing the four dimensional integrals over x and k leads us to the result for this part which is given as follows

$$\Pi^{QCD(c1)}(q^2) = -\frac{im_Q}{3.2^7\pi^3}\langle g_s^2G^2\rangle(1-\frac{m_Q^2}{q^2})^2\Theta[1-\frac{m_Q^2}{q^2}] - \frac{m_Q}{3.2^6\pi^4}\langle g_s^2G^2\rangle\int_0^1 dz\frac{z^3}{(1-z)^2}ln[-L(z)] \quad (3.40)$$

where $\Theta[\cdots]$ is the step function and $L(z)$ is given in Eq.(3.31). Taking the imaginary part of the above equation we get the spectral density for this side as

$$\frac{1}{\pi}\text{Im}[\Pi^{QCD(c1)}(q^2)] = -\frac{m_Q}{3.2^5\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle(1-\frac{m_Q^2}{q^2})^2\Theta[1-\frac{m_Q}{q^2}] + \frac{m_Q}{3.2^4\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\int_0^{1-\frac{m_Q^2}{q^2}}dz\frac{z^3}{(1-z)^2}, \quad (3.41)$$

where

$$\frac{g_s^2}{4\pi^2} = \frac{\alpha_s}{\pi}. \quad (3.42)$$

Two gluon can also be emitted from the different light quark lines, and to consider this contribution, we will study with the term of the light quark propagator given as

$$iS_{ij}^{q,\alpha\beta}(x) = -\frac{i}{32\pi^2}\frac{1}{x^2}g_sG^A_{\mu\nu}t^A_{ij}(\not{x}\sigma^{\mu\nu}+\sigma^{\mu\nu}\not{x})^{\alpha\beta}. \quad (3.43)$$

When we place this term for the light quark propagators and keep the perturbative term in heavy quark propagator the correlation function becomes

$$\Pi^{QCD(c2)}(q^2) = -i\epsilon^{ijk}\epsilon^{i'j'k'}(\frac{-i}{32\pi^2})^2\langle g_s^2G^A_{\alpha\beta}G^B_{\nu\rho}\rangle t^A_{jj'}t^B_{ii'}\int d^4x e^{iqx}\frac{1}{(x^2)^2} \times Tr[(\not{x}\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not{x})(\gamma^{\mu'}C)(\not{x}\sigma^{\nu\rho}+\sigma^{\nu\rho}\not{x})^T(C\gamma^{\mu})]$$

$$
\begin{aligned}
&\times \quad i\int\frac{d^4k}{(2\pi)^4}\delta_{kk'}e^{-ikx}\frac{(\not k+m_Q)}{k^2-m_Q^2}\\
&+ \quad i\epsilon^{ijk}\epsilon^{i'j'k'}(\frac{-i}{32\pi^2})^2\langle g_s^2G^A_{\alpha\beta}G^B_{\nu\rho}\rangle t^A_{ij'}t^B_{ji'}\int d^4xe^{iqx}\frac{1}{(x^2)^2}\\
&\times \quad Tr[(\not k\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not k)(\gamma^{\mu'}C)(\not k\sigma^{\nu\rho}+\sigma^{\nu\rho}\not k)^T(C\gamma^{\mu})^T]\\
&\times \quad i\int\frac{d^4k}{(2\pi)^4}\delta_{kk'}e^{-ikx}\frac{(\not k+m_Q)}{k^2-m_Q^2}
\end{aligned}
\tag{3.44}
$$

To calculate this term we use the following relation:

$$
\langle g_s^2G^A_{\alpha\beta}G^B_{\nu\rho}\rangle=\frac{\delta^{AB}}{96}\langle g_s^2G^2\rangle(g_{\alpha\nu}g_{\beta\rho}-g_{\alpha\rho}g_{\beta\nu}).
\tag{3.45}
$$

Besides the Eq. (3.45) one needs the values of the color factors and the trace present in this equation. The colour factors are calculated as

$$
\begin{aligned}
\epsilon^{ijk}\epsilon^{i'j'k'}t^A_{jj'}t^B_{ii'}\delta^{AB}\delta_{kk'} &= \epsilon^{ijk}\epsilon^{i'j'k}t^A_{jj'}t^A_{ii'}=(\delta_{ii'}\delta_{jj'}-\delta_{ij'}\delta_{ji'})t^A_{jj'}t^A_{ii'}\\
&= t^A_{jj}t^A_{ii}-t^A_{ji}t^A_{ij}=-4,
\end{aligned}
\tag{3.46}
$$

and

$$
\begin{aligned}
\epsilon^{ijk}\epsilon^{i'j'k'}t^A_{ij'}t^B_{ji'}\delta^{AB}\delta_{kk'} &= \epsilon^{ijk}\epsilon^{i'j'k}t^A_{ij'}t^A_{ji'}=(\delta_{ii'}\delta_{jj'}-\delta_{ij'}\delta_{ji'})t^A_{ij'}t^A_{ji'}\\
&= t^A_{ji}t^A_{ij}-t^A_{ii}t^A_{jj}=4.
\end{aligned}
\tag{3.47}
$$

When we use the color factors and the Eq. (3.45) in Eq. (3.44) we obtain

$$
\begin{aligned}
\Pi^{QCD(c2)}(q^2) &= \frac{8(i)^2}{96}(\frac{-i}{32\pi^2})^2\langle g_s^2G^2\rangle\int d^4xe^{iqx}\frac{1}{(x^2)^2}\\
&\times \quad Tr[(\not k\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not k)(\gamma^{\mu'}C)(\not k\sigma_{\alpha\beta}+\sigma_{\alpha\beta}\not k)^T(C\gamma^{\mu})]\\
&\times \quad \gamma_5\gamma_\mu(\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{(\not k+m_Q)}{k^2-m_Q^2})\gamma_{\mu'}\gamma_5\\
&+ \quad \frac{8(i)^2}{96}(\frac{-i}{32\pi^2})^2\langle g_s^2G^2\rangle\int d^4xe^{iqx}\frac{1}{(x^2)^2}\\
&\times \quad Tr[(\not k\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not k)(\gamma^{\mu'}C)(\not k\sigma_{\alpha\beta}+\sigma_{\alpha\beta}\not k)^T(C\gamma^{\mu})^T]\\
&\times \quad \gamma_5\gamma_\mu\int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{(\not k+m_Q)}{k^2-m_Q^2}\gamma_{\mu'}\gamma_5
\end{aligned}
\tag{3.48}
$$

As for the traces, they are obtained as:

$$
\begin{aligned}
& Tr[(\not{x}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\not{x})(\gamma^{\mu'}C)(\not{x}\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\not{x})^T(C\gamma^{\mu})] \\
= & -Tr[(\not{x}\sigma^{\alpha\beta} + \sigma^{\alpha\beta}\not{x})\gamma^{\mu'}(\not{x}\sigma_{\alpha\beta} + \sigma_{\alpha\beta}\not{x})\gamma^{\mu}] \\
= & -32(x^2 g^{\mu\mu'} + 2x^{\mu}x^{\mu'}). \qquad (3.49)
\end{aligned}
$$

Therefore the result of this part becomes

$$
\begin{aligned}
\Pi^{QCD(c2)}(q^2) &= \frac{1}{48\pi^2}\langle\frac{g_s^2}{4\pi^2}G^2\rangle\int d^4x e^{iqx}\frac{1}{(x^2)^2}(x^2 g^{\mu\mu'} + 2x^{\mu}x^{\mu'}) \\
&\times \gamma_5\gamma_{\mu}\int\frac{d^4k}{(2\pi)^4}\delta_{kk'}e^{-ikx}\frac{(\not{k}+m_Q)}{k^2-m_Q^2}\gamma_{\mu'}\gamma_5. \qquad (3.50)
\end{aligned}
$$

Finally the integrals are performed in the Eq. (3.50) and its result is obtained as

$$
\begin{aligned}
\Pi^{QCD(c2)}(q^2) &= \frac{1}{3.2^4\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\int_0^1 dz(1-z)(1-3z)ln[-L(z)]\not{q} \\
&+ \frac{m_Q}{2^5\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\int_0^1 dz(1-z)ln[-L(z)]. \qquad (3.51)
\end{aligned}
$$

The spectral density is attained from the last result as

$$
\begin{aligned}
\frac{1}{\pi}\mathrm{Im}[\Pi^{QCD(c2)}(q^2)] &= -\frac{1}{3.2^4\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\int_0^{1-\frac{m_Q^2}{q^2}} dz(1-z)(1-3z)\not{q} \\
&- \frac{m_Q}{2^5\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\int_0^{1-\frac{m_Q^2}{q^2}} dz(1-z). \qquad (3.52)
\end{aligned}
$$

There is one more term in the calculation proportional to $\langle\frac{\alpha_s}{\pi}G^2\rangle$. In this case one gluon is emitted from one of the light quark line and the other is emitted from the heavy quark line. In these calculation the two light quark line is taken into account one by one. For one of each light quark line we consider the Eq. (3.43), for the heavy quark one we use the term

$$
iS_{ij}^{Q,\alpha\beta}(x) = -i\int\frac{d^4k}{(2\pi^4)}e^{-ikx}\frac{g_s G_{ij}^{\mu\nu}}{4}\frac{\sigma_{\mu\nu}(\not{k}+m_Q)^{\alpha\beta} + (\not{k}+m_Q)^{\alpha\beta}\sigma_{\mu\nu}}{(k^2-m_Q^2)^2}, \quad (3.53)
$$

and for the remaining light quark line we use perturbative term of its propagator which results in

$$
\begin{aligned}
\Pi^{QCD(c3)}(q^2) &= -i\epsilon^{ijk}\epsilon^{i'j'k'}(\frac{i}{2\pi^2})\frac{-i}{32\pi^2}\frac{-i}{4}\langle g_s^2 G^A_{\alpha\beta}G^B_{\nu\rho}\rangle \int d^4x e^{iqx}\frac{1}{(x^2)^2}\Big[t^A_{jj'}t^B_{kk'}\delta_{ii'} \\
&\times Tr[\not x(\gamma^{\mu'}C)(\not x\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not x)^T(C\gamma^\mu)] + t^A_{ii'}t^B_{kk'}\delta_{jj'} \\
&\times Tr[(\not x\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not x)(\gamma^{\mu'}C)(\not x)^T(C\gamma^\mu)] - t^A_{ij'}t^B_{kk'}\delta_{ji'} \\
&\times Tr[\not x(\gamma^{\mu'}C)(\not x\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not x)^T(C\gamma^\mu)] - t^A_{ji'}t^B_{kk'}\delta_{ij'} \\
&\times Tr[(\not x\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not x)(\gamma^{\mu'}C)(\not x)^T(C\gamma^\mu)]\Big] \\
&\times \int \frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{\sigma^{\nu\rho}(\not k+m_Q)+(\not k+m_Q)\sigma^{\nu\rho}}{(k^2-m_Q^2)^2}. \qquad (3.54)
\end{aligned}
$$

Again we apply the relation given in Eq. (3.45) and using

$$
\begin{aligned}
\epsilon^{ijk}\epsilon^{i'j'k'}t^A_{jj'}t^B_{kk'}\delta_{ii'}\delta_{AB} &= \epsilon^{ijk}\epsilon^{ij'k'}t^A_{jj'}t^A_{kk'} = (\delta_{jj'}\delta_{kk'}-\delta_{jk'}\delta_{kj'})t^A_{jj'}t^A_{kk'} = -4, \\
\epsilon^{ijk}\epsilon^{i'j'k'}t^A_{ii'}t^B_{kk'}\delta_{jj'}\delta_{AB} &= \epsilon^{ijk}\epsilon^{i'jk'}t^A_{ii'}t^A_{kk'} = (\delta_{ii'}\delta_{kk'}-\delta_{ik'}\delta_{ki'})t^A_{ii'}t^A_{kk'} = -4, \\
\epsilon^{ijk}\epsilon^{i'j'k'}t^A_{ij'}t^B_{kk'}\delta_{ji'}\delta_{AB} &= -\epsilon^{ijk}\epsilon^{j'jk'}t^A_{ij'}t^A_{kk'} = -(\delta_{ij'}\delta_{kk'}-\delta_{ik'}\delta_{kj'})t^A_{ij'}t^A_{kk'} = 4, \\
\epsilon^{ijk}\epsilon^{i'j'k'}t^A_{ji'}t^B_{kk'}\delta_{ij'}\delta_{AB} &= -\epsilon^{ijk}\epsilon^{ii'k'}t^A_{ji'}t^A_{kk'} = -(\delta_{ji'}\delta_{kk'}-\delta_{jk'}\delta_{ki'})t^A_{ji'}t^A_{kk'} = 4, \qquad (3.55)
\end{aligned}
$$

and the results of the traces having the form

$$
\begin{aligned}
Tr[\not x(\gamma^{\mu'}C)(\not x\sigma^{\alpha\beta}+\sigma^{\alpha\beta}\not x)^T(C\gamma^\mu)] &= 8i\Big[x^2g^{\alpha\mu'}g^{\beta\mu}-x^2g^{\alpha\mu}g^{\beta\mu'}-x^\alpha x^{\mu'}g^{\beta\mu} \\
&+ x^{\mu'}x^\beta g^{\alpha\mu}+x^\mu x^\alpha g^{\beta\mu'}-x^\mu x^\beta g^{\alpha\mu'}\Big], \qquad (3.56)
\end{aligned}
$$

we get

$$
\begin{aligned}
\Pi^{QCD(c3)}(q^2) &= \frac{(i)^4}{3\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\int d^4x e^{iqx}\frac{1}{(x^2)^3} \\
&\times \int\frac{d^4k}{(2\pi)^4}e^{-ikx}\frac{1}{(k^2-m_Q^2)^2}(3m_Qx^2+x^2\not k+2\not k k.x). \qquad (3.57)
\end{aligned}
$$

The result of this part, after performing the integrals, gives the spectral density as

$$
\begin{aligned}
\frac{1}{\pi}\mathrm{Im}[\Pi^{QCD(c3)}(q^2)] &= \frac{1}{3.2^4\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\int_0^{1-\frac{m_Q^2}{q^2}}dz(z+3z^2)\not q \\
&+ \frac{m_Q}{2^4\pi^2}\langle\frac{\alpha_s}{\pi}G^2\rangle\int_0^{1-\frac{m_Q^2}{q^2}}dzm_Q\frac{z}{(1-z)}. \qquad (3.58)
\end{aligned}
$$

The next term requires the usage of the following part of the light quark propagator for one of the light quark propagator, whereas the other propagators are free parts of the corresponding propagators

$$iS_{ij}^{q,\alpha\beta}(x) = -\frac{x^2}{192}m_0^2\langle\bar{u}u\rangle\delta_{ij}(\delta^{\alpha\beta} - \frac{i}{6}m_q \not{x}^{\alpha\beta}). \tag{3.59}$$

The similar steps given for the other terms are followed and the calculations give the imaginary part of the correalator as

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}[\Pi^{QCD(d)}(q^2)] &= \frac{1}{2^6\pi^6}m_0^2 m_u\langle\bar{u}u\rangle\delta(q^2 - m_Q^2)\ \not{q} \\ &+ \frac{5}{3.2^6\pi^6}m_Q m_0^2 m_u\langle\bar{u}u\rangle\delta(q^2 - m_Q^2).\end{aligned} \tag{3.60}$$

Finally we calculate the last term taking both of the light quark propagators as given in Eq. (3.32) and from the heavy quark propagator taking free term. Following similar steps as before we get the imaginary part of this calculation as

$$\begin{aligned}\frac{1}{\pi}\mathrm{Im}[\Pi^{QCD(e)}(q^2)] &= \frac{2}{3}\langle\bar{u}u\rangle^2\delta(q^2 - m_Q^2)\ \not{q} \\ &+ \frac{4}{3}m_Q\langle\bar{u}u\rangle^2\delta(q^2 - m_Q^2).\end{aligned} \tag{3.61}$$

Having calculated the spectral densities for both perturbative and nonperturbative terms, we now use them in the dispersion relation together with the result of physical side and get the result for the masses and residues of the hadronic states under question. The analyses for the obtained results are given in the following section.

CHAPTER 4

NUMERICAL ANALYSES

4.1 QCD Sum Rule for Ground State Baryons

In this section we present the numerical analyses of our results that we obtain in the Chapter 3. The QCD sum rule for the ground state Σ_Q baryon is obtained, from the match of the physical and the QCD sides, as follows

$$|\lambda_{\Sigma_Q}|^2 e^{-\frac{m^2_{\Sigma_Q}}{M^2}} = \int_{m_Q^2}^{s_0} ds e^{-\frac{s}{M^2}} \frac{1}{\pi} Im[\Pi_i^{QCD}(s)] \tag{4.1}$$

where sub index is $i = \not{q}$ or 1. From this result the mass of the ground state Σ_Q baryon is obtained by taking the derivative of the Eq. (4.1) with respect to $-\frac{1}{M^2}$ and dividing the obtained result by Eq. (4.1), which results in

$$m^2_{\Sigma_Q} = \frac{\int_{m_Q^2}^{s_0} ds e^{-\frac{s}{M^2}}\ s\ \frac{1}{\pi} Im[\Pi_i^{OPE}(s)]}{\int_{m_Q^2}^{s_0} ds e^{-\frac{s}{M^2}}\ \frac{1}{\pi} Im[\Pi_i^{OPE}(s)]}. \tag{4.2}$$

The current coupling strength is obtained from the Eq. (4.1) in terms of the mass of the ground state baryon as

$$\lambda^2_{\Sigma_Q} = e^{\frac{m^2_{\Sigma_Q}}{M^2}} \int_{m_Q^2}^{s_0} ds \frac{1}{\pi} Im[\Pi_i^{QCD}(s)] \tag{4.3}$$

4.2 QCD Sum Rule for $1P$ Excited State Baryons

For the calculation of the $1P$ excitations we consider the match of the physical and theoretical sides containing not only the ground state but also the first orbitally excited

(1P) state and we obtain

$$\begin{aligned}
\lambda_{\Sigma_Q}^2 e^{-\frac{m_{\Sigma_Q}^2}{M^2}} + \tilde{\lambda}_{\Sigma_Q}^2 e^{-\frac{\tilde{m}_{\Sigma_Q}^2}{M^2}} &= \int_{m_Q^2}^{s_0} ds e^{-\frac{s}{M^2}} \frac{1}{\pi} Im[\Pi_{q'}^{QCD}(s)] = \tilde{\Pi}_{q'}^{QCD} \\
m_{\Sigma_Q}\lambda_{\Sigma_Q}^2 e^{-\frac{m_{\Sigma_Q}^2}{M^2}} - \tilde{m}_{\Sigma_Q}\tilde{\lambda}_{\Sigma_Q}^2 e^{-\frac{\tilde{m}_{\Sigma_Q}^2}{M^2}} &= \int_{m_Q^2}^{s_0} ds e^{-\frac{s}{M^2}} \frac{1}{\pi} Im[\Pi_1^{QCD}(s)] = \tilde{\Pi}_1^{QCD}. \quad (4.4)
\end{aligned}$$

In the last equations we have three unknown physical quantities which are the mass and the residue of the orbitally excited Σ_Q state and the residue of the ground state Σ_Q. Note that, in these equations we will use the ground state mass as input parameter. Therefore, we need a third equation which is obtained from the first equation of Eq. (4.4) by taking its derivative with respect to $-\frac{1}{M^2}$. Then the result is used together with Eq. (4.4) and, solving them together, we get the mass and residue of the first orbitally excited Σ_Q as

$$\tilde{m}_{\Sigma_Q}^2 = \frac{\frac{d}{d(-\frac{1}{M^2})}(\tilde{\Pi}_1^{QCD} - m_{\Sigma_Q}\tilde{\Pi}_{q'}^{QCD})}{\tilde{\Pi}_1^{QCD} - m_{\Sigma_Q}\tilde{\Pi}_{q'}^{QCD}}, \quad (4.5)$$

and

$$\tilde{\lambda}_{\Sigma_Q}^2 = \frac{\tilde{\Pi}_1^{QCD} - m_{\Sigma_Q}\tilde{\Pi}_{q'}^{QCD}}{m_{\Sigma_Q} + \tilde{m}_{\Sigma_Q}} e^{\frac{\tilde{m}_{\Sigma_Q}}{M^2}}. \quad (4.6)$$

4.3 Numerical Analysis for Σ_c States

After obtaining the necessary relations to calculate the masses of the ground state and orbitally excited state Σ_Q in their analysis we use some input parameters which are given in Table 4.1

Table 4.1: Some input parameters used in the calculations.

Parameters	Values
m_c	$1.275^{+0.025}_{-0.035}$ GeV [75]
m_b	$4.18^{+0.04}_{-0.03}$ GeV [75]
m_u	$2.2^{+0.05}_{-0.04}$ MeV [75]
$\langle\bar{u}u\rangle$	$(-0.24 \pm 0.01)^3$ GeV^3 [89]
m_0^2	(0.8 ± 0.1) GeV^2 [89]
$\langle g_s^2 G^2\rangle$	$4\pi^2(0.012 \pm 0.004)$ GeV^4[90]

However, these are not the only input parameters that are needed in the calculation. In addition to these we need two auxiliary parameters which are threshold parameter s_0 and the Borel parameter M^2. The threshold parameter is not completely arbitrary and have a relation with the lowest excited state of the considered baryon. In the calculation we consider the ground state + $1P$ + continuum approximation and firs determine the mass of the ground state baryon with suitable choices of the working intervals for the threshold parameter, s_0, and Borel parameter, M^2.

For continuum threshold, the $\sqrt{s_0}$ is related to the beginning of continuum states and therefore its working interval is chosen considering this. Its value is fixed, related to the next excited state, as about 500 $\sim$ 700 MeV above the mass value of the hadron under question. With this consideration we chose this working interval as

$$8.7\ \text{GeV}^2 \leq s_0 \leq 10.0\ \text{GeV}^2, \tag{4.7}$$

for the analysis on the Σ_c baryon.

To fix the working interval for the Borel parameter, we consider the convergence of the OPE calculation. Since it is an asymptotic expansion, the contribution of the perturbative terms must be dominant compared to the nonperturbative ones. Besides, the results should have weak dependence on the Borel parameter since it is not a physical parameter. These requirements lead us to the lower limit of the Borel parameter. To obtain the lower limit we plot the Figure 4.1, which shows the results of the OPE side for perturbative and the total non-perturbative contributions for central value of the threshold parameter s_0. We also consider the mass result that we obtain for Σ_c and considering the region where it has least variation we determine the lower limit for Borel parameter (see Figure. 4.3). To decide its upper limit we consider the pole dominance over the continuum. To determine the corresponding working interval of Borel parameters, we consider the analyses for the ratio of the pole contribution to total result as well as continuum contribution to total which are given as

$$R(M^2) = \frac{\int_{m_Q^2}^{s_0} ds e^{-\frac{s}{M^2}} \rho^{OPE}(s)}{\int_{m_Q^2}^{\infty} ds e^{-\frac{s}{M^2}} \rho^{OPE}(s)}, \tag{4.8}$$

and

$$C(M^2) = \frac{\int_{s_0}^{\infty} ds e^{-\frac{s}{M^2}} \rho(s)^{OPE}}{\int_{m_Q^2}^{\infty} ds e^{-\frac{s}{M^2}} \rho^{OPE}(s)}, \tag{4.9}$$

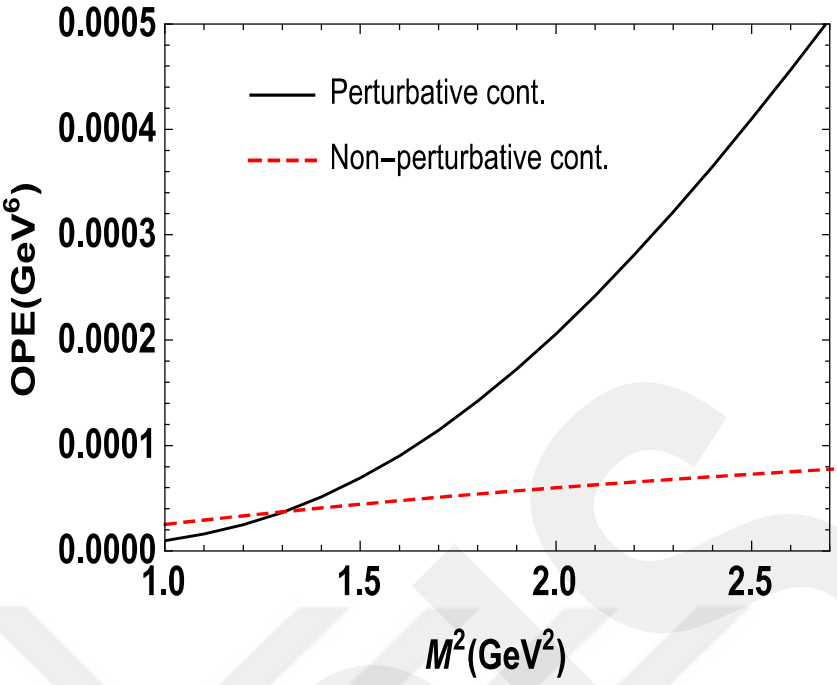


Figure 4.1: The perturbative and non-perturbative contributions obtained from Σ_c calculations for central value of s_0.

respectively. Using the above equations we plot Figure 4.2. To determine the upper limit of the Borel parameter we demand that the contribution of the pole is equal at least % 50 of the total. By these analyses the we set the upper limit of the interval for the Borel parameter of the Σ_c calculations. We also require that the pole contribution be between % 75 and % 50 of the total so that we also use this in fixing the lower limit for M^2. From all these analyses the working window of the Borel parameter is obtained as follows

$$1.9\ \text{GeV}^2 \leq M^2 \leq 2.6\ \text{GeV}^2. \tag{4.10}$$

Using the auxiliary parameters we obtain the final value for the mass of the ground state Σ_c baryon, from the QCD sum rule calculation obtained using the $\not{q}$ structure, as

$$m_{\Sigma_c} = 2.44^{+0.17}_{-0.15}\ \text{GeV}, \tag{4.11}$$

where the + and − parts in the mass result represent the errors which are due to variations of the auxiliary parameters and the input parameters, as well. The Figures 4.4 and 4.5 show the final mass result as functions of Borel parameter M^2 and threshold parameter s_0, respectively.

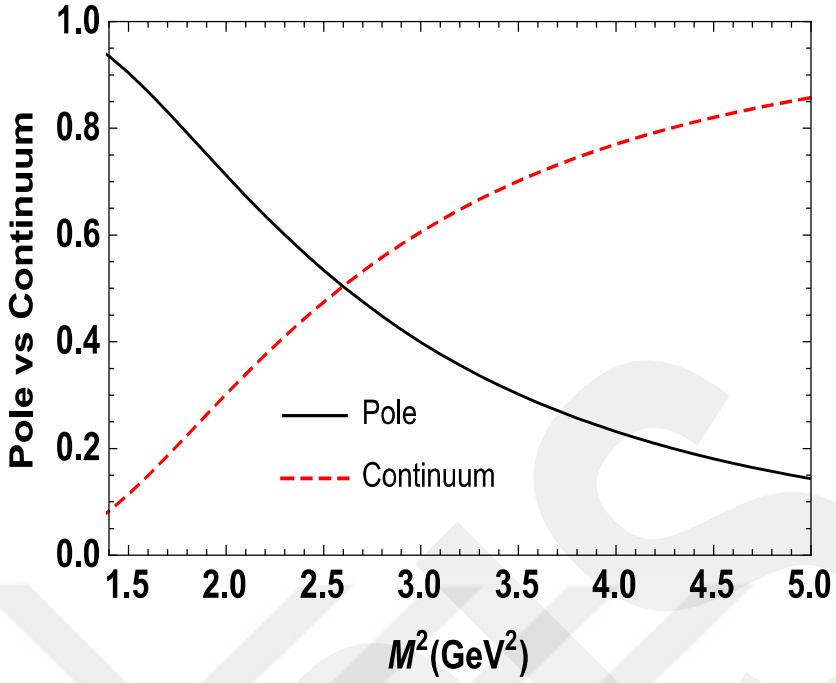


Figure 4.2: The pole and continuum contributions obtained from Σ_c calculations for $s_0 = 10$ GeV^2.

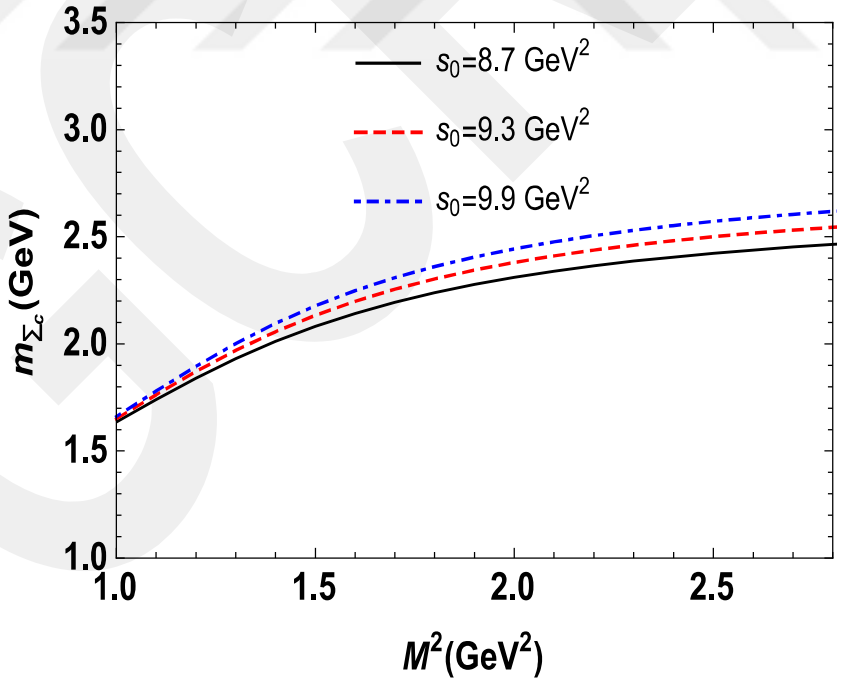


Figure 4.3: The mass obtained for Σ_c baryon for different s_0 values.

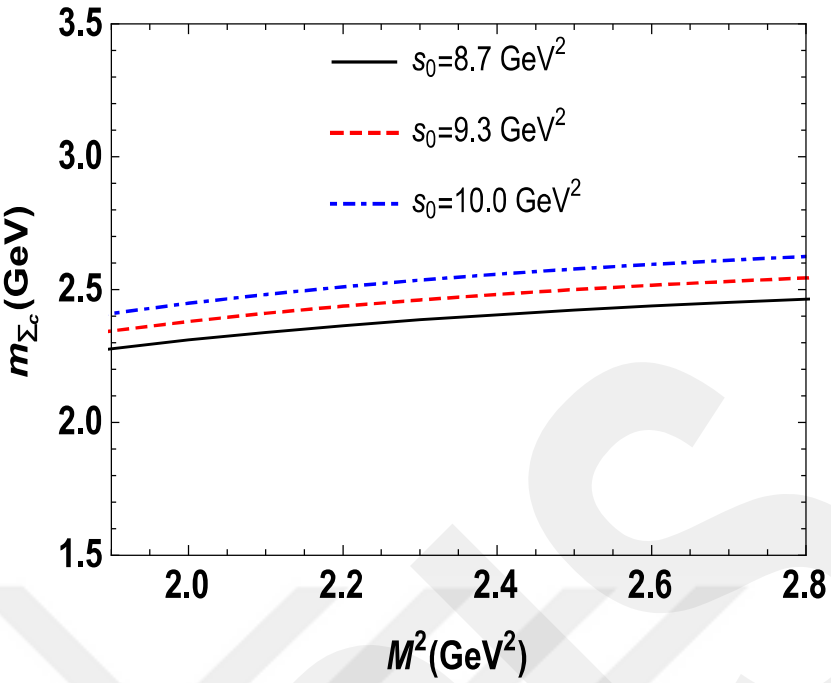


Figure 4.4: The mass obtained for Σ_c baryon as a function of M^2 for different s_0 values.

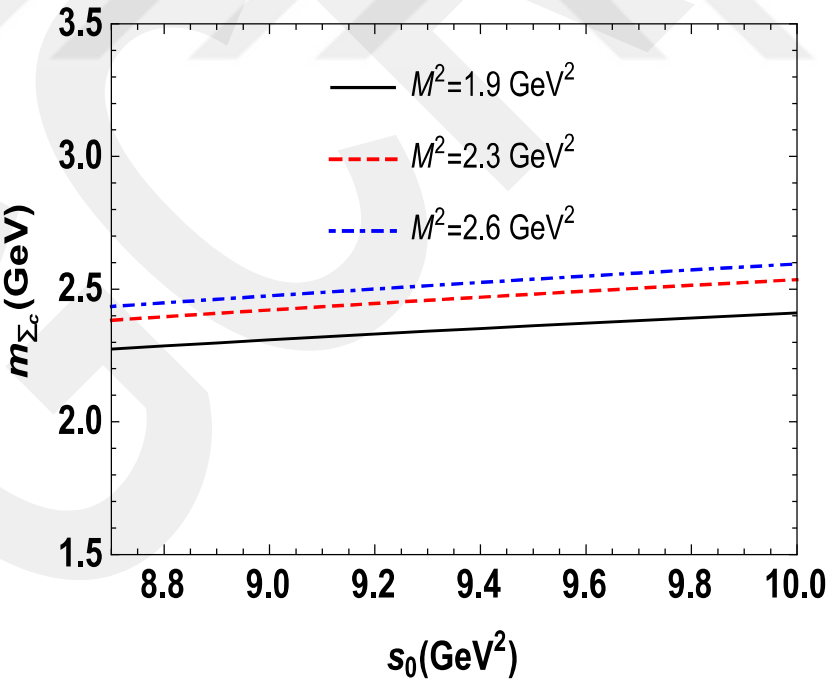


Figure 4.5: The mass obtained for Σ_c baryon as a function of s_0 for different M^2 values.

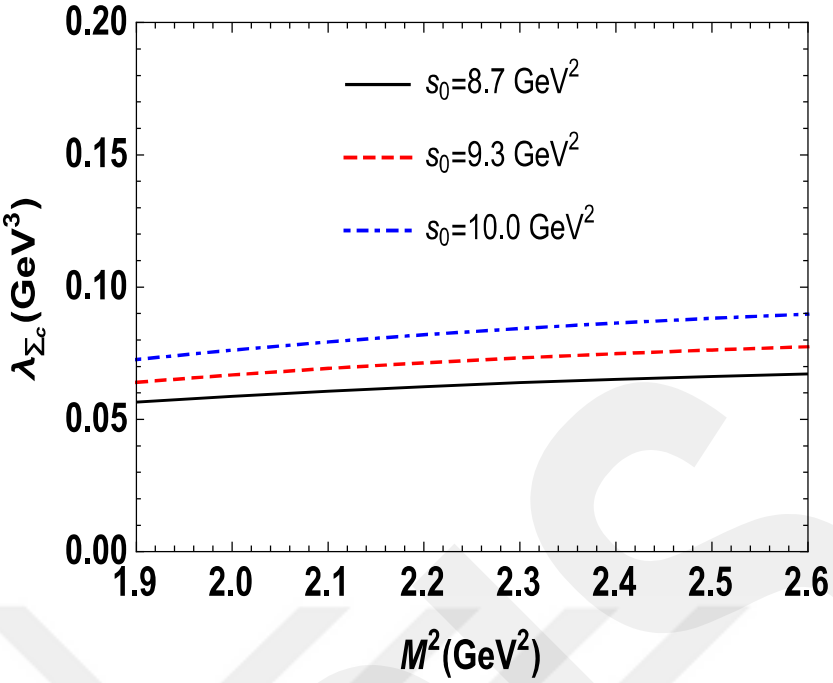


Figure 4.6: Residue λ_{Σ_c} as a function of M^2 for different s_0 values.

Using the input and obtained auxiliary parameters we also attain the corresponding residue for the Σ_c as

$$\lambda_{\Sigma_c} = 0.073^{+0.016}_{-0.017}\ \text{GeV}^3, \tag{4.12}$$

whose behaviors as functions of M^2 and s_0 are depicted in Figures 4.6 and Figure 4.7, respectively.

As for the $1P$ Σ_c baryon, since we don not know the energy of the next excited state, we decide the s_0 interval for our analyses from the pole dominance consideration and considering the stability of the results against the variations of the auxiliary parameters. The working interval of the threshold parameter is chosen as

$$11.0\ \text{GeV}^2 \leq s_0 \leq 13.0\ \text{GeV}^2, \tag{4.13}$$

for the analyses on the $1P$ Σ_c state. For the Borel parameter we use our previous determination given in Eq. (4.10). With these values of auxiliary parameters and the parameters given in Table. 4.1, we obtain the mass of the first orbital excitation of the Σ_c baryon as

$$\tilde{m}_{\Sigma_c} = 2.80^{+0.11}_{-0.12}\ \text{GeV}. \tag{4.14}$$

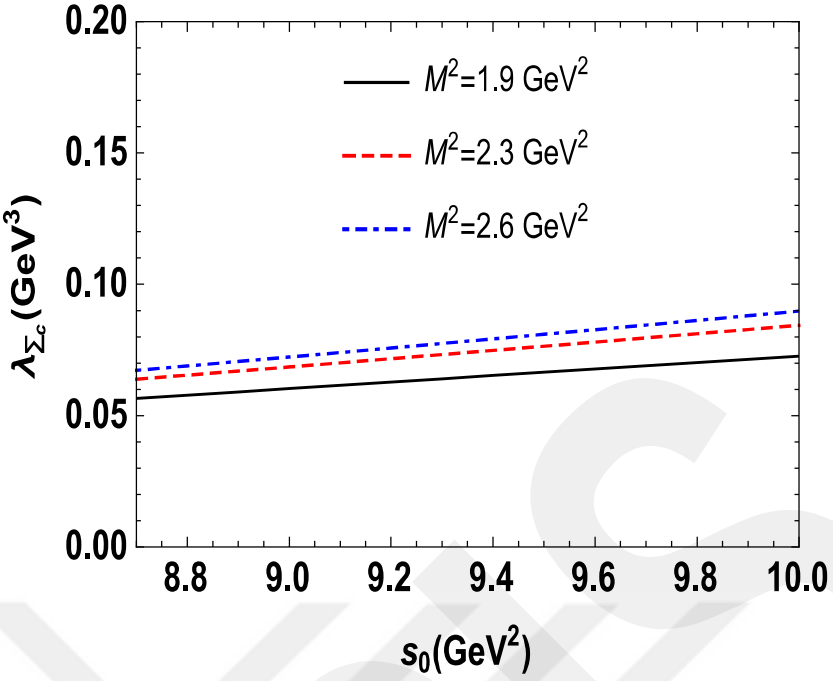


Figure 4.7: Residue λ_{Σ_c} as a function of s_0 for different M^2 values.

The Figures 4.8 and 4.9 show the variations of our mass result for $1P$ excitation with respect to the auxiliary parameters, M^2 and s_0, respectively.

The residue for the $1P$ excitation of the Σ_c state is obtained as

$$\tilde{\lambda}_{\Sigma_c} = 0.018^{+0.04}_{-0.04}\ \text{GeV}^3, \tag{4.15}$$

whose variations as functions of Borel mass and threshold parameter are presented in Figures 4.10 and 4.11.

4.4 Numerical Analysis for Σ_b States

We make similar analyses, as in the calculation of the masses and residues of the Σ_c states, for Σ_b states, also. The analyses again start by the determination of the threshold parameters. We use similar requirements as in previous section. With the same consideration the threshold parameter for the analysis of the ground state Σ_b baryon is chosen as

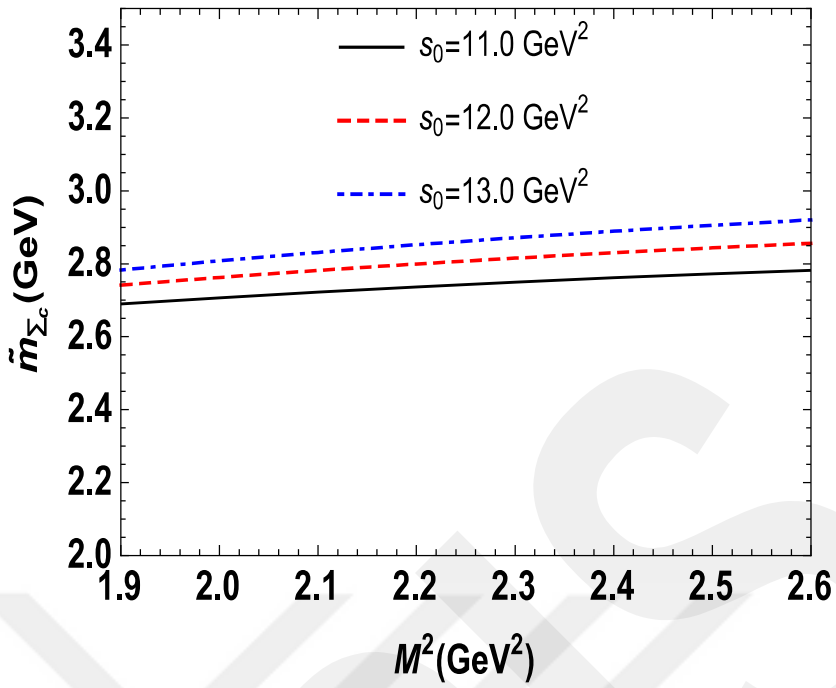


Figure 4.8: The mass obtained for the $1P$ excitation of the Σ_c baryon as a function of M^2 for different s_0 values.

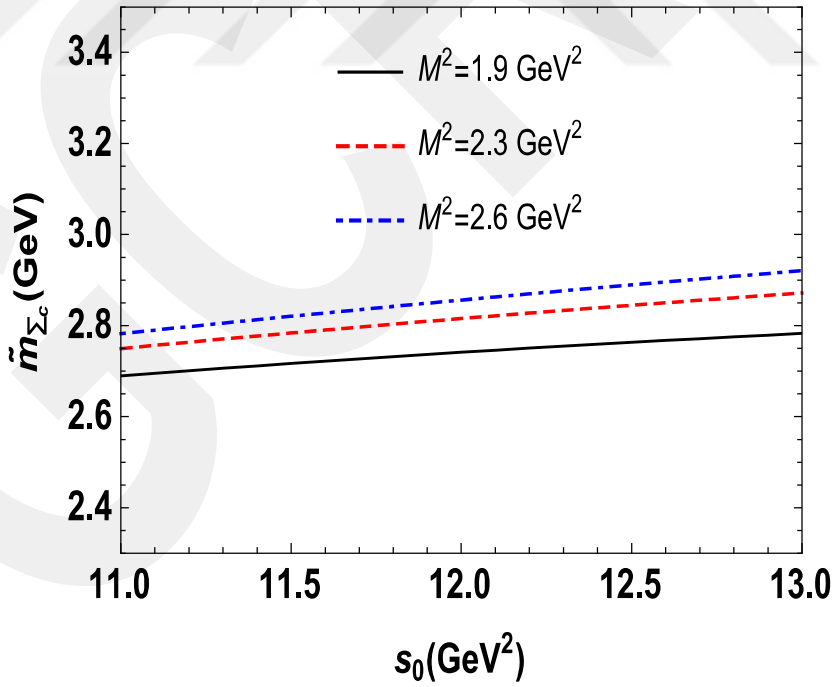


Figure 4.9: The mass obtained for the $1P$ excitation of the Σ_c baryon as a function of s_0 for different M^2 values.

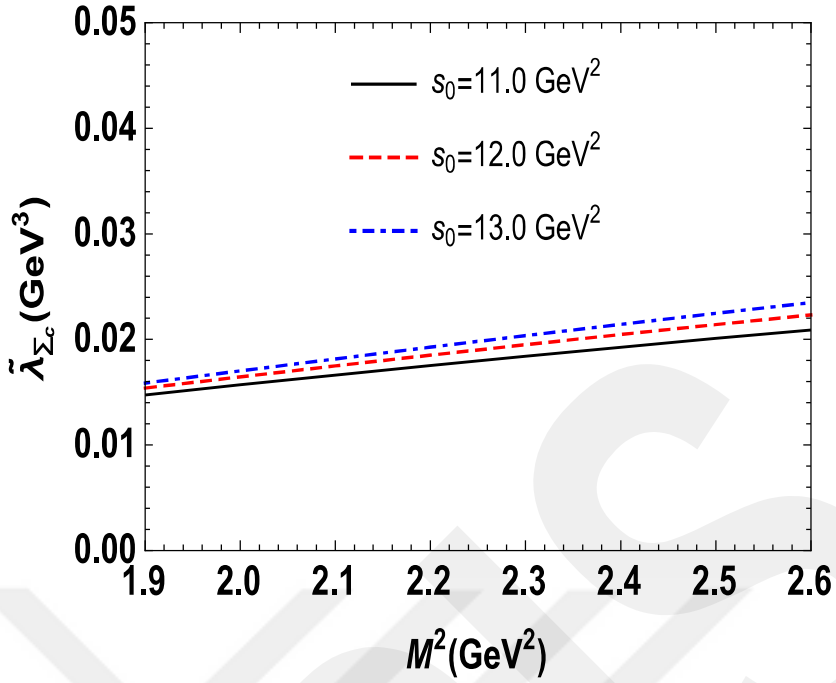


Figure 4.10: The residue obtained for the $1P$ excitation of the Σ_c baryon as a function of M^2 for different s_0 values.

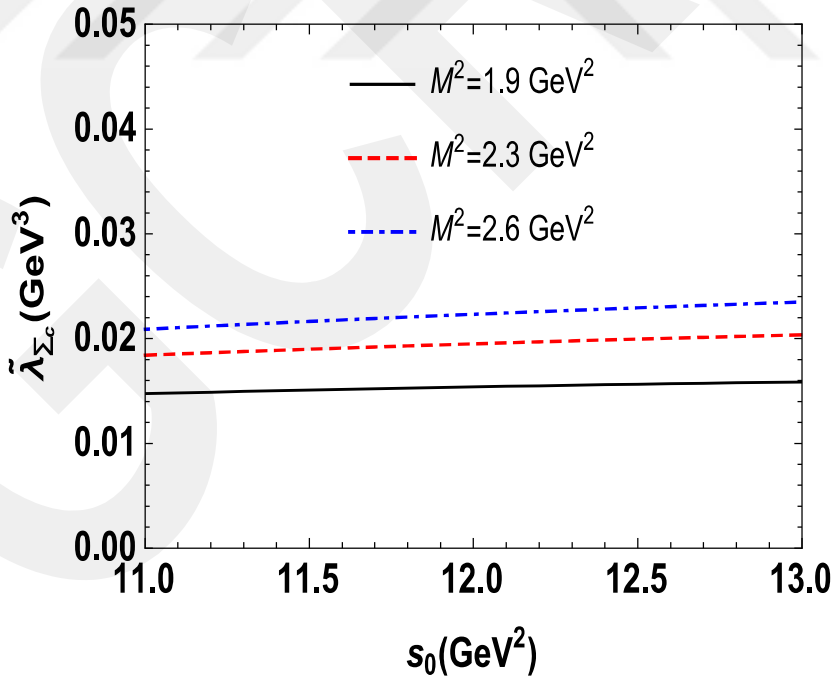


Figure 4.11: The residue obtained for the $1P$ excitation of the Σ_c baryon as a function of s_0 for different M^2 values.

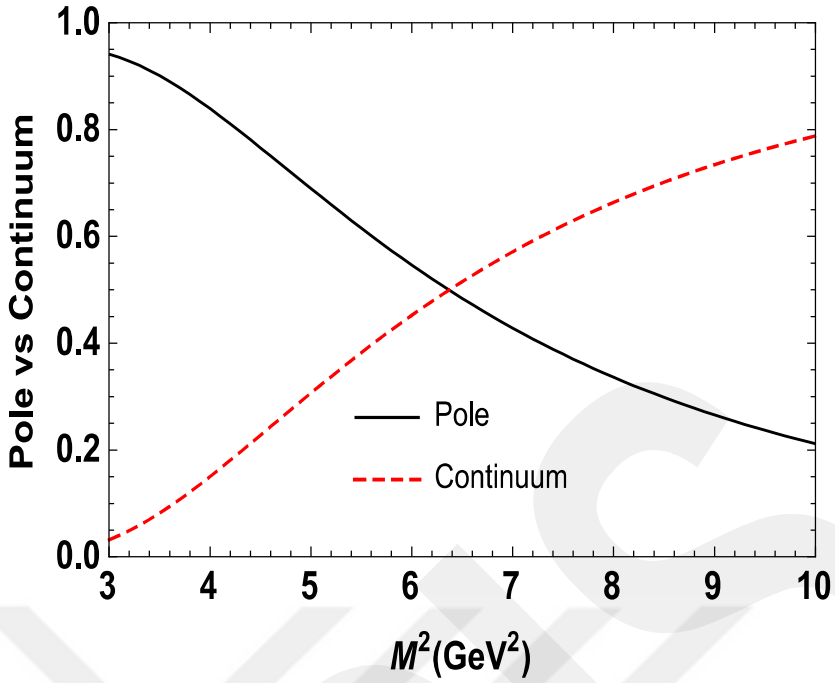


Figure 4.12: The pole and continuum contributions obtained from Σ_b calculations for $s_0 = 43.5\ \text{GeV}^2$.

$$41.5\ \text{GeV}^2 \leq s_0 \leq 43.5\ \text{GeV}^2. \tag{4.16}$$

To fix the working interval for the Borel parameter, we, again, take into account the convergence of the OPE calculation and the dependency of the results on the threshold parameters, demanding their relative weak variation with variation of the M^2. We firsly plot the graph depicting pole versus continuum contribution presented by Figure 4.12. Considering again the dominance of the pole contribution over continuum and perturbative contribution to nonperturbative ones, and also, taking that the pole contribution between % 75 and % 50 of the total, we determine the working window of the Borel parameter as

$$4.6\ \text{GeV}^2 \leq M^2 \leq 6.2\ \text{GeV}^2. \tag{4.17}$$

Using the intervals of the auxiliary parameters, we obtain the final value for the mass of the ground state Σ_b baryon, from the QCD sum rule calculations obtained using the $\not{q}$ structure, as

$$m_{\Sigma_b} = 5.81^{+0.15}_{-0.11}\ \text{GeV}, \tag{4.18}$$

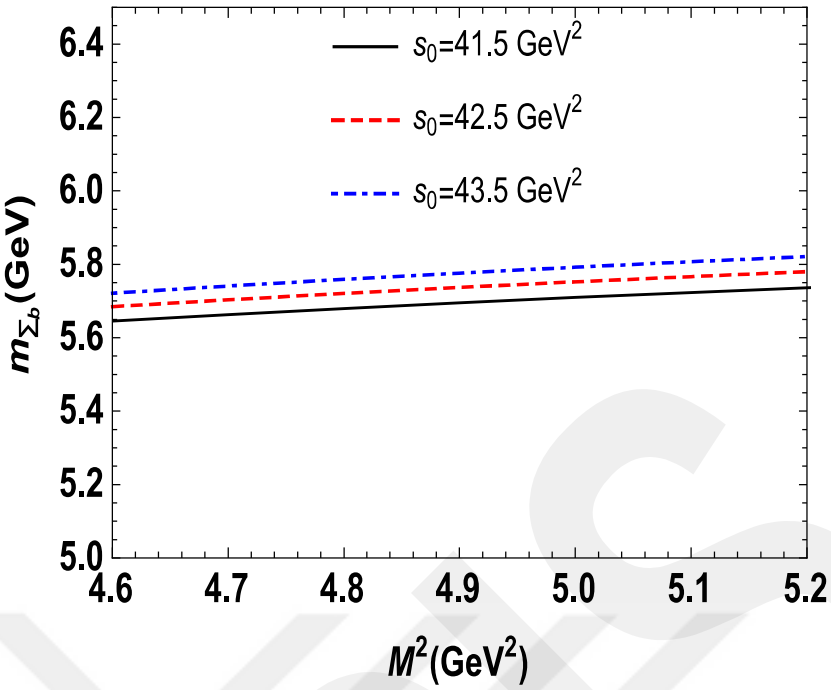


Figure 4.13: The mass obtained for Σ_b baryon as a function of M^2 for different s_0 values.

where the + and − parts give the corresponding errors resulted from the variations of the auxiliary parameters and the input parameters, as well. The Figures 4.13 and 4.14 present the final mass results as functions of Borel parameter M^2 and threshold parameter s_0, respectively.

Using the input and obtained auxiliary parameters we find the residue for the Σ_b as

$$\lambda_{\Sigma_b} = 0.096^{+0.038}_{-0.035}\ \text{GeV}^3, \tag{4.19}$$

whose behaviors as functions of M^2 and s_0 are depicted in Figures 4.15 and Figure 4.16, respectively.

After obtaining the results for ground state Σ_b baryon, we again study its $1P$ excitation. With the lack of knowledge on the energy of its excited state, we fix the s_0 interval for our analyses from pole dominance consideration and considering the stability of the results as functions of the auxiliary parameters. For the threshold parameter, we choose the working intervals as

$$45.5\ \text{GeV}^2 \leq s_0 \leq 47.5\ \text{GeV}^2, \tag{4.20}$$

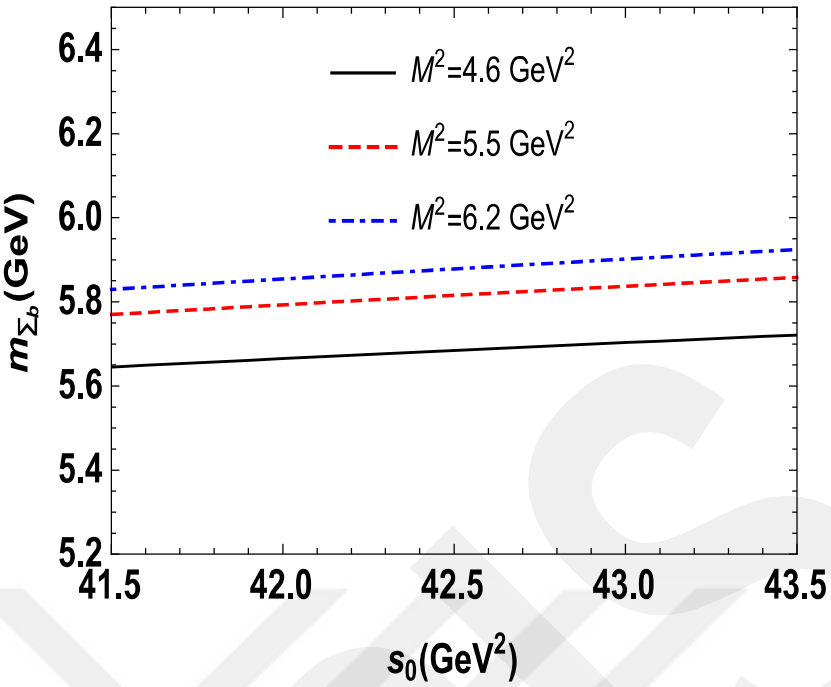


Figure 4.14: The mass obtained for Σ_b baryon as a function of s_0 for different M^2 values.

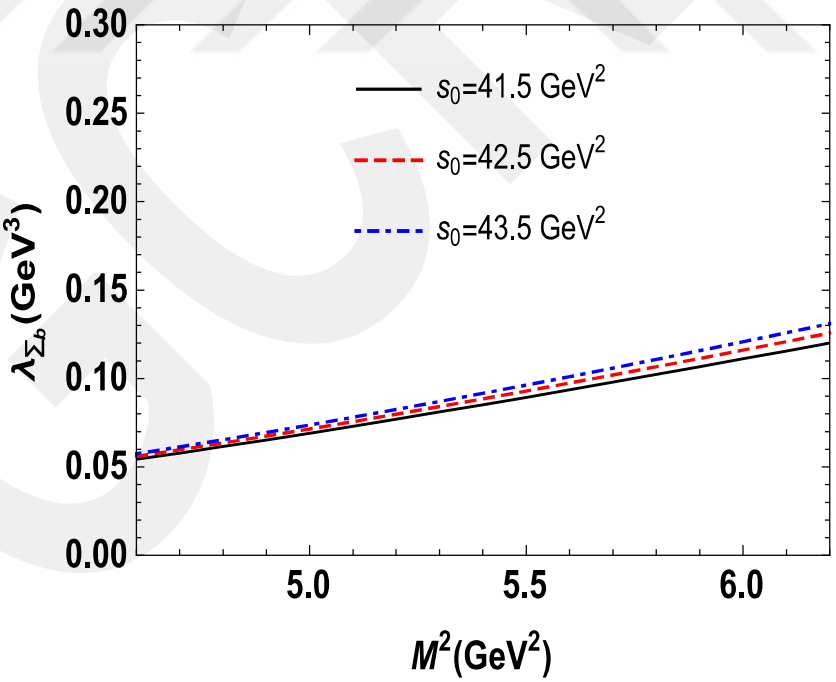


Figure 4.15: Residue λ_{Σ_b} as a function of M^2 for different s_0 values.

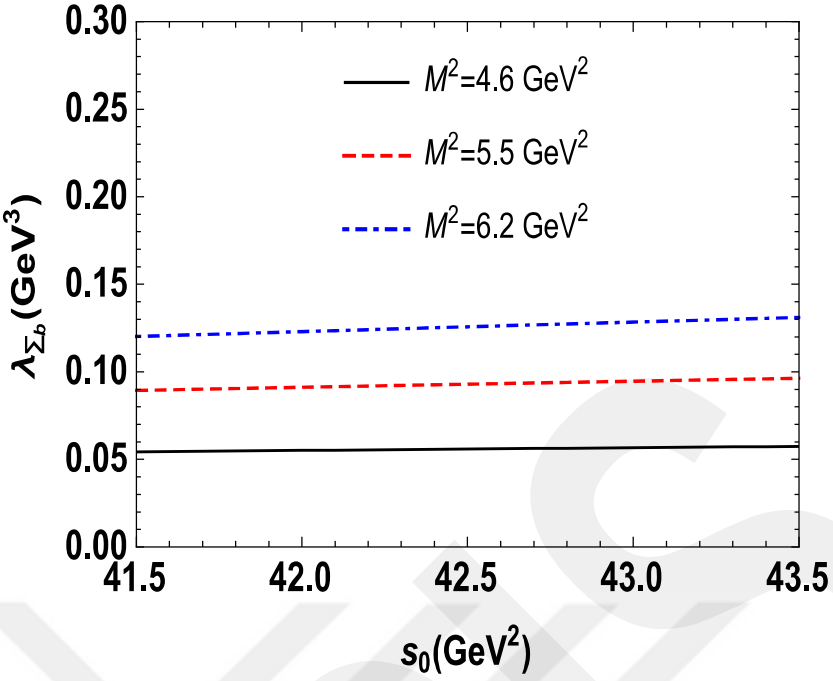


Figure 4.16: Residue λ_{Σ_b} as a function of s_0 for different M^2 values.

for the analyses on the $1P$ Σ_b state. For the Borel parameter we use our previous determination given in Eq. (4.17). With these values of auxiliary parameters and the parameters given in Table. 4.1, we obtain the mass of the first orbital excitation of the Σ_b baryon as

$$\tilde{m}_{\Sigma_b} = 6.090^{+0.066}_{-0.062}\ \text{GeV}. \tag{4.21}$$

The behavior of our results for $1P$ excitation as functions of M^2 and s_0 are depicted via Figures 4.17 and 4.18, respectively.

After getting the mass value for the $1P$ excitation of the Σ_b state we calculate the corresponding residue using the ground state's mass as input parameter. The residue is obtained as

$$\tilde{\lambda}_{\Sigma_b} = 0.0087^{+0.0035}_{-0.0033}\ \text{GeV}^3. \tag{4.22}$$

The variations of the residue of the $1P$ excitation as functions of Borel mass and threshold parameter are depicted in Figures 4.19 and 4.20.

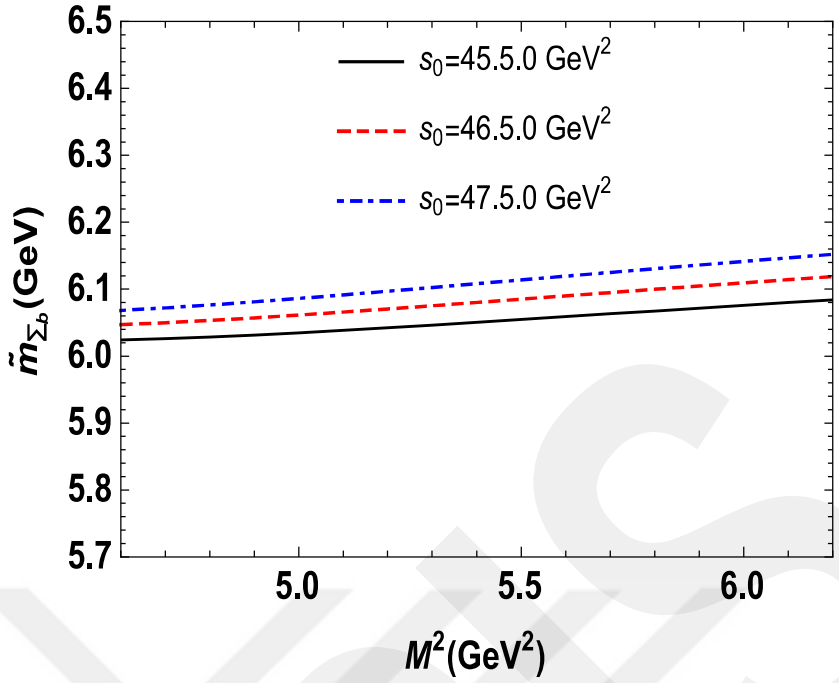


Figure 4.17: The mass obtained for the $1P$ excitation of the Σ_b baryon as a function of M^2 for different s_0 values.

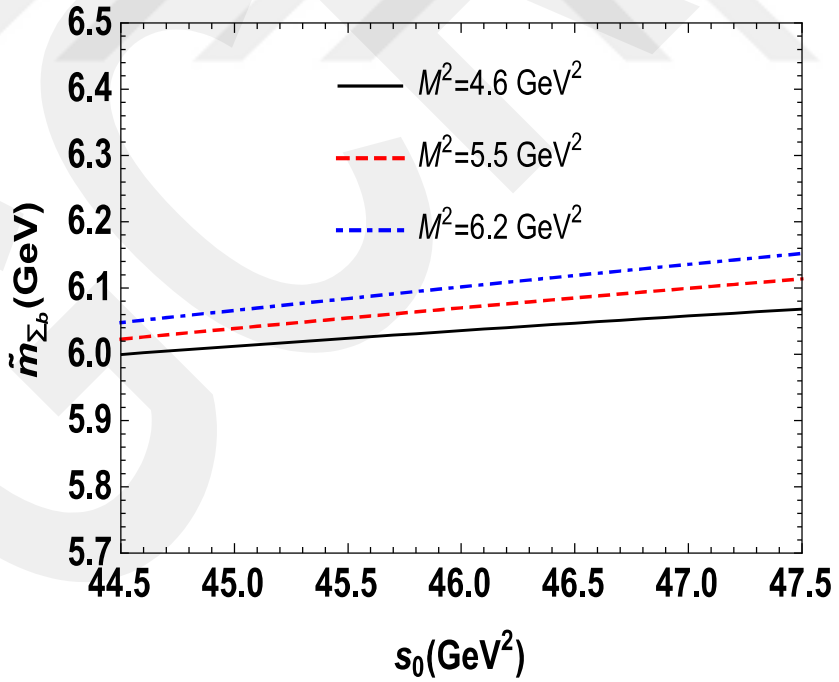


Figure 4.18: The mass obtained for the $1P$ excitation of the Σ_b baryon as a function of s_0 for different M^2 values.

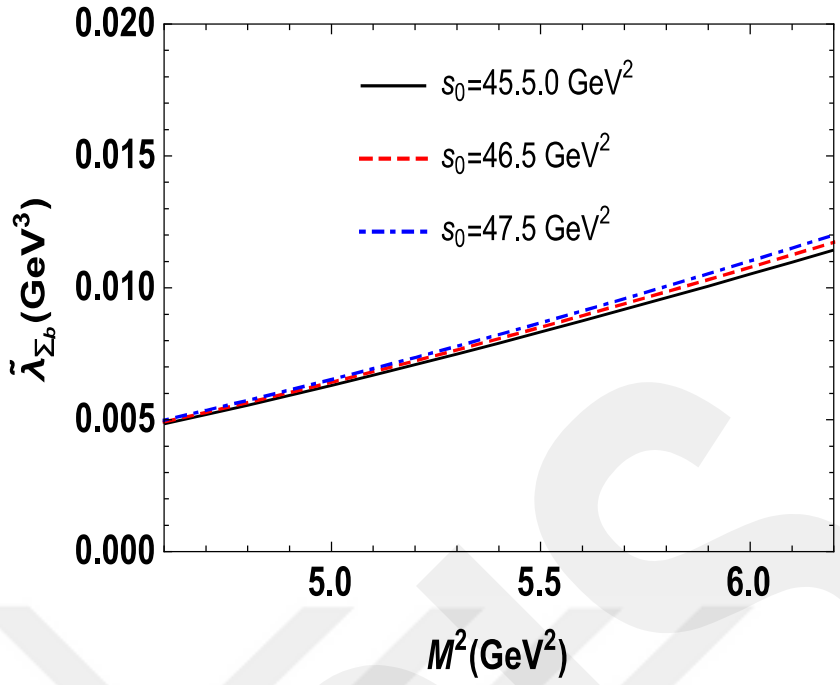


Figure 4.19: The residue obtained for the $1P$ excitation of the Σ_b baryon as a function of M^2 for different s_0 values.

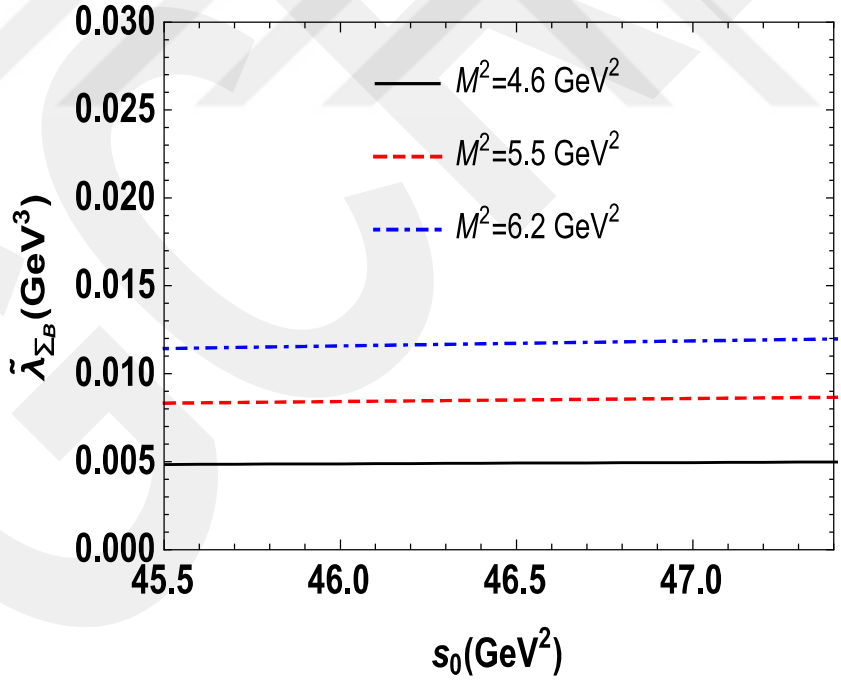


Figure 4.20: The residue obtained for the $1P$ excitation of the Σ_b baryon as a function of s_0 for different M^2 values.

CHAPTER 5

CONCLUSION

In this work we study the spectroscopic properties, such as masses and current coupling strengths, of the Σ_Q baryons and their first orbital excitations considering $J^P = \frac{1}{2}^{+(-)}$ quantum numbers. To calculate the masses and current coupling strengths we use a powerful nonperturbative method, namely, QCD sum rule method. We summarize our results in Table 5.1 The experimental values for the ground state masses are

Table 5.1: Results obtained from the QCD sum rule analyses for Σ_Q states.

State	Mass (GeV)	Decay Constant λ (GeV^3)
$\Sigma_c(1S)$	$2.44^{+0.17}_{-0.15}$	$0.073^{+0.016}_{-0.017}$
$\Sigma_c(1P)$	$2.80^{+0.11}_{-0.12}$	$0.018^{+0.04}_{-0.04}$
$\Sigma_b(1S)$	$5.81^{+0.15}_{-0.11}$	$0.096^{+0.038}_{-0.035}$
$\Sigma_b(1P)$	$6.09^{+0.07}_{-0.06}$	$0.0087^{+0.0035}_{-0.0033}$

as follows [75]:

$$\begin{aligned} m_{\Sigma_c} &= 2453.97 \pm 0.14 \text{ MeV} \\ m_{\Sigma_b} &= 5811.3^{+0.9}_{-0.8} \pm 1.7 \text{ MeV} \end{aligned} \tag{5.1}$$

As it can be seen from the table our results are in agreement with those of the experimental results.

Though almost all of the ground state baryons were experimentally observed, we have limited data on their excited states. Among these observations is the $\Sigma_c(2800)$ whose quantum numbers are not clear yet. The mass of the $\Sigma_c(2800)$ is [75] $m_{\Sigma_c(2800)^{++}} = 2801^{+4}_{-6}$ MeV. If we compare our result obtained for the $1P$ excitation of the ground

state Σ_c, it can be seen that our mass value is in consistency with experimental result within the errors. The mass result obtained in this study considering $J^P = \frac{1}{2}^-$ is also in agreement with the predictions obtained in Refs. [33, 34, 37, 38]. However in these works there are different predictions for its quantum numbers as either its being $J^P = \frac{1}{2}^-$ or $J^P = \frac{3}{2}^-$.

As for the excited states of the bottom baryons, among the recent observations are $\Sigma_b(6097)^+$ and $\Sigma_b(6097)^-$ with their reported masses and decay widths given as [27] $m_{\Sigma_b(6097)^-} = 6098.0 \pm 1.7 \pm 0.5$ MeV, $\Gamma(\Sigma_b(6097)^-) = 28.9 \pm 4.2 \pm 0.9$ MeV and $m_{\Sigma_b(6097)^+} = 6095.0 \pm 1.7 \pm 0.4$ MeV, $\Gamma(\Sigma_b(6097)^+) = 31.0 \pm 5.5 \pm 0.7$ MeV, respectively. From our analyses we obtain a mass value for $1P$ excitation of the Σ_b^+ state, and our result for this mass is in agreement with the observed mass value. However in literature there are works supporting the quantum number for these state being $J^P = \frac{3}{2}^-$ or $\frac{5}{2}^-$ [65, 63, 64, 66]. Especially in Ref. [66] QCD sum rule calculation was performed using a more general form of current than that we used in this work, and both mass and strong decay for the considered state were studied. The presented mass results for spin-$\frac{1}{2}$ and -$\frac{3}{2}$ considerations showed degeneracy, which are in agreement with our prediction also. Therefore using the strong decay width analyses the quantum numbers for $\Sigma_b(6097)^\pm$ were concluded to be $J^P = \frac{3}{2}^-$ in Ref. [66].

To sum up, in this work we calculated the masses of the ground state Σ_Q states having quantum numbers $J^P = \frac{1}{2}^+$ and their $1P$ excited states having $J^P = \frac{1}{2}^-$. Our results gave the mass values that are consistent with the experimental findings for ground states. As for the calculations of $1P$ excitations our masses are in consistency with masses of experimentally observed states $\Sigma_c(2800)^{++}$ and $\Sigma_b(6097)^\pm$. However, considering the present theoretical findings on the $\Sigma_c(2800)$ and $\Sigma_b(6097)^+$ in literature, we should mention that, though mass calculations are necessary in identification of the quantum numbers of these baryons, they are not enough to make conclusive statement, and these results should be endorsed by means of the analyses of other properties of these baryons, such as their strong decay width investigations, which are out of the scope of this work. In a future work the strong decay of the $1P$ excitation of Σ_b state to Λ_b and π final states can be calculated using the interpolating current of this work, to make a comparison with the experimental observation and provide a further statement on the possibility of the quantum numbers of the observed state being spin-$\frac{1}{2}$ or not.

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APPENDIX A

SOME INTEGRALS AND EQUATIONS

This appendix presents some useful integrals and equations used in this thesis. Feynman parameters:

$$\frac{1}{A_1 A_2 ... A_n} = \int_0^1 dx_1 ... dx_n \delta(\sum x_i - 1) \frac{(n-1)!}{[x_1 A_1 + x_2 A_2 + ... x_n A_n]^n} \tag{A.1}$$

The D dimensional integrals in Minkowski space:

$$\begin{aligned}
\int \frac{d^D\ell}{(2\pi)^D} \frac{1}{(\ell^2-\Delta)^n} &= \frac{(-1)^n i}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)} \Big(\frac{1}{\Delta}\Big)^{n-\frac{D}{2}} , \\
\int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^2}{(\ell^2-\Delta)^n} &= \frac{(-1)^{n-1} i}{(4\pi)^{\frac{D}{2}}} \frac{D}{2} \frac{\Gamma(n-\frac{D}{2}-1)}{\Gamma(n)} \Big(\frac{1}{\Delta}\Big)^{n-\frac{D}{2}-1} , \\
\int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu}{(\ell^2-\Delta)^n} &= \frac{(-1)^{n-1} i}{(4\pi)^{\frac{D}{2}}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n-\frac{D}{2}-1)}{\Gamma(n)} \Big(\frac{1}{\Delta}\Big)^{n-\frac{D}{2}-1} , \\
\int \frac{d^D\ell}{(2\pi)^D} \frac{(\ell^2)^2}{(\ell^2-\Delta)^n} &= \frac{(-1)^n i}{(4\pi)^{\frac{D}{2}}} \frac{D(D+2)}{4} \frac{\Gamma(n-\frac{D}{2}-1)}{\Gamma(n)} \Big(\frac{1}{\Delta}\Big)^{n-\frac{D}{2}-2} , \\
\int \frac{d^D\ell}{(2\pi)^D} \frac{\ell^\mu \ell^\nu \ell^\rho \ell^\sigma}{(\ell^2-\Delta)^n} &= \frac{(-1)^n i}{(4\pi)^{\frac{D}{2}}} \frac{\Gamma(n-\frac{D}{2}-2)}{\Gamma(n)} \Big(\frac{1}{\Delta}\Big)^{n-\frac{D}{2}-2} \\
&\quad \times \frac{1}{4}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}) .
\end{aligned} \tag{A.2}$$

For a converging integral one takes the $D = 4$ from the beginning, if the integral does not converge the following expansion is used to get its behavior near $D = 4$

$$\Big(\frac{1}{\Delta}\Big)^{2-\frac{D}{2}} = 1 - (2 - \frac{D}{2}) log\Delta + ... \tag{A.3}$$

The expansion for $\Gamma(x)$ has the following form

$$\Gamma(x) = \frac{1}{x} - \gamma + O(x) , \tag{A.4}$$

near $x = 0$ where the γ is the Euler-Mascheroni constant, $\gamma \approx 0.5772$. We also applied the following equation in the present work in calculations:

$$\frac{\Gamma(2-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}}\Big(\frac{1}{\Delta}\Big)^{2-\frac{D}{2}} = \frac{2}{\epsilon} - (log\Delta + \gamma - log(4\pi)) + O(\epsilon)\,, \tag{A.5}$$

with $\epsilon = 4 - D$.

We also used the following expression in D-dimension in the calculations

$$\frac{1}{(x^2)^n} = \int \frac{d^D p}{(2\pi)^D} e^{-ipx} i(-1)^{n+1} 2^{D-2n} \pi^{\frac{D}{2}} \frac{\Gamma(\frac{D}{2}-n)}{\Gamma(n)} (-\frac{1}{p})^{\frac{D}{2}-n}. \tag{A.6}$$