EXPONENTIAL FINITE DIFFERENCE METHOD FOR NONLINEAR BLACK-SCHOLES EQUATION

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Approval of the Graduate School of Natural and Applied Sciences, Atılım University.

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ABSTRACT

Exponential Finite Difference Method For Nonlinear Black-Scholes Equation

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In this thesis, we investigate exponential finite difference method for nonlinear Black-Scholes equation arising in an illiquid market. Chapter 1 is devoted to the literature survey with some basic definitions and terminology of the option pricing problem. In Chapter 2 we review the Black-Scholes model and finite difference methods for Black-Scholes equation. In Chapter 3, an explicit finite difference method for a nonlinear Black-Scholes equation is studied with the monotonicity, stability and consistency results. In Chapter 4, we apply the exponential finite difference method to linear and nonlinear Black-Scholes equations. Moreover, we investigate consistency and convergence of the method. Numerical experiments are performed to verify theoretical results. Exponential finite difference method is compared with an explicit finite difference method proposed for linear and nonlinear Black-Scholes equation. Numerical results show that exponential finite difference method exhibits better performance then explicit method. Finally, we give a brief conclusion in Chapter 5.

Keywords: exponential finite difference method, nonlinear Black-Scholes equation, illiquid market

ÖZ

Doğrusal Olmayan Black-Scholes Denklemi İçin Üstel Sonlu Fark Yöntemi

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Bu tezde, likit olmayan bir piyasada ortaya çıkan doğrusal olmayan Black-Scholes denklemi için üstel sonlu fark yöntemi çalışılmıştır. 1. Bölüm opsiyon fiyatlandırması problemi terminolojisi, temel tanımlar ve literatür taramasına ayrılmıştır. 2. Bölümde Black-Scholes modeli ve Black-Scholes denklemi için sonlu fark yöntemleri gözden geçirilmiştir. 3. Bölümde doğrusal olmayan Black-Scholes denklemi için açık sonlu fark yöntemi, monotonluk, kararlılık ve tutarlılık sonuçları ile birlikte çalışılmıştır. 4. Bölümde doğrusal ve doğrusal olmayan Black-Scholes denklemleri için üstel sonlu fark yöntemi uygulanmıştır. Ayrıca, yöntemin tutarlılığı ve yakınsaklığı araştırılmıştır. Teorik sonuçları doğrulamak için sayısal örnekler verilmiştir. Sayısal sonuçlar, üstel sonlu fark yönteminin açık sonlu fark yönteminden daha iyi performans sergilediğini göstermiştir. 5. Bölüm sonuç kısmına ayrılmıştır.

Anahtar Kelimeler: üstel sonlu fark yöntemi, doğrusal olmayan Black-Scholes denklemi, likit olmayan piyasa

To my husband and my parents

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LIST OF SYMBOLS AND ABBREVIATIONS

$\mathbb{N}$	:	Set of natural numbers
$\mathbb{N}_0$	:	Set of non-negative integers
$\mathbb{R}$	:	Set of real numbers
$\mathbb{R}^n$	:	the n-dimensional real space
PDE	:	Partial Differential Equation
EFD	:	Exponential Finite Difference
FDM	:	Finite Difference Method
SDE	:	Stochastic Differential Equation
BSE	:	Black-Scholes Equation
NBSE	:	Nonlinear Black-Scholes Equation
$V(S,t)$	:	Value of European option
$C(S,t)$	:	Value of the European call option
$P(S,t)$	:	Value of the European put option
$S(t)$	:	price of the underlying asset at time t
t	:	time
r	:	interest rate
σ	:	Volatility of the underlying asset
μ	:	drift rate
E	:	Strike price (exercise price)
T	:	expiry, maturity or expiration time

CHAPTER 1

INTRODUCTION

Partial Differential Equations (PDEs) are widely used in many important phenomena and dynamics process in the fields of engineering, mechanics, physics, chemistry, biology, etc. Though the study of nonlinear PDEs is very important, many exact solutions are available depending on the boundary conditions, and the nonlinear term of the PDE under considerations [37]. A variety of powerful methods, such as Exp-function method [12], adomian decomposition method [63], bilinear transformation [40], Inverse Scattering Transform [62], the tanh-sech method [49, 66], the tanh-coth method [64], homogeneous balance method [31] and the sine-cosine methods [2, 65] have been widely used to search for various exact solutions of nonlinear PDEs. Many PDEs are modeled by using Cartesian coordinate system with one spatial dimension. When a PDE is modeled by using a different coordinate system or more than one spatial dimension is used in the model, the analytical solution of the PDE becomes extremely difficult or impossible. Nonlinear terms may also pose problems. The numerical methods are the only possibility to understand the behavior of the solution. Typically, the ability of the particular numerical method is tested by a numerically solved problem for which the exact solution is available. When the numerical results are a measure of the confidence, then the numerical method can be used to solve a problem whose exact solution does not exist or experimental tests are unavailable.

Mathematical finance is a field of applied mathematics which applies mathematical methods to the solution of problems in finance. It uses techniques from probability and statistics theory, stochastic processes, numerical analysis, partial differential equations, applied functional analysis and economic theory. Banks, insurance companies, industries such as energy, manufacturing and many financial institutions perform

the methods of mathematical finance to financial problems such as derivative pricing, portfolio selection, risk management, and simulation. In general, indeed, mathematical finance provides and extends the mathematical or numerical models without necessarily demonstrating a connection to financial theory.

In mathematical finance one of the important problems is the valuation of financial derivatives. In 1900, Bachelier introduced an option pricing model in his thesis "Théorie de la Spéculation"(Theory of Speculation) [7] and considered to be the first researcher who introduce Brownian motion and to apply it in modeling stock price, [1]. In 1973, Fischer Black and Myron Scholes [15] and independently Robert Merton [50] introduced the classical model for option pricing in mathematical finance theory. They obtained the so-called Black-Scholes partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{1.1}$$

which is a linear second order parabolic partial differential equation for the option value V as a function of the underlying asset S and time t, and solved explicitly by transforming it into the heat (diffusion) equation. The resulting solution is named as the Black-Scholes formula which has been utilized by financial analysts for more than forty years. In the literature, there are many results on the numerical methods of Black-Scholes equation, such as, binomial approximations, Monte-Carlo methods, finite difference approximations and finite element method, see for instance [22, 32, 36, 46, 59].

In a perfect financial market the value of an option can be found by Black-Scholes formula. In spite of the fact that the Black-Scholes model has been extensively employed by academics and by financial analysts, it has been also criticized by its unrealistic and restrictive assumptions for liquid, complete and idealized market, [10, 18]. There are many studies extending the framework of Black-Scholes model into general situations. Such extensions are introduced for options on multiple or dividend paying underlying assets, and to deterministic functions of volatilities and interest with respect to time, [22]. Further generalizations can be studied by weakening the assumptions of the model. For example, in [45] and [42] the interest rate and the volatility are not taken as deterministic functions and are modeled as stochastic, see also [23]. Instead of continuous stochastic processes in the Black-Scholes model, jump diffusion

models using Poisson or compound Poisson processes are also applicable, [51]. The assumption on the underlying asset of the option, which is lognormally distributed, is replaced by several distributions, some of them are the gamma distribution, fractional Brownian motion and Lévy processes, see [16, 14, 58].

In recent years, more general assumptions such as existence of transaction cost, incomplete or illiquid market have been introduced and several models are offered which reduce some of the assumptions, see [6, 9, 10, 27, 33, 34, 35, 47, 48, 55]. These models introduce nonlinear parabolic diffusion equations for the option value V as a function of the underlying asset S and the time t. One of assumptions of the Black-Scholes model that the underlying stock has a constant volatility contradicts to the observations in real market as many researches suggest. Thus, it is reasonable to consider a generalization of Black-Scholes model replacing the constant volatility in the model by a non constant volatility function. This leads to a nonlinear version of the Black-Scholes partial differential equation which has the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\hat{\sigma}(S,t,\frac{\partial^2 V}{\partial S^2})\right]^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{1.2}$$

Despite there are many results on the numerical methods for the linear Black-Scholes equation and some generalizations [11, 20, 22, 24, 41, 42, 36, 56, 58, 59, 61, 67] in the literature, a few results on the numerical solution of nonlinear Black-Scholes equations (NBSEs) have appeared. An explicit finite difference method is suggested in [17] for a nonlinear BSE of the form (1.2) arising in an illiquid market. In [5], the authors present a numerical method based on a nonstandard finite difference scheme to solve the equation (1.2) numerically. For a model assuming the existence of transaction cost, a semidiscretization technique is proposed in [18] and Crank-Nicolson method is used in [3, 4]. Implicit numerical schemes are studied for general type of equations (1.2) in [39].

The exponential finite difference method has been widely used for numerical solution of various nonlinear PDEs. The exponential finite difference (EFD) technique was first proposed by Bhattacharya [13] for one-dimensional conduction heat transfer in Cartesian coordinates. It was shown that EFD method with four multipliers performs better than the Crank-Nicolson method [21]. In [38] the method has been expanded to allow application to two and three dimensional unsteady heat condition

and the Burger's equation. Several PDEs have been solved by using the EFD method in the literature such as the Burgers-Huxley equation [43], the Nonlinear Klein Gordon equation [26], Korteweg-de Vries Burger (KdVB) equation [28, 29], Fitz Hugh-Nagumo equation [30], 1D Helmholtz equation [44], Korteweg-de Vries equation (KdV) [8], a nonlinear reaction diffusion equation and modified KdV equation [30]. There have been several higher order EFD method available in literature. Pandey [53] developed a new EFD method of order four for the numerical solution of second order elliptic equation. Pillai [54] proposed a fourth-order EFD method to solve one and two-dimensional convection diffusion type differential equations with constant and variable convection coefficients. Up to our knowledge, the method has not been used in the numerical solution of a PDE arising in finance. The objective of this thesis is to present, extend and compare the EFD method for the numerical solution of a PDE arising in finance which is of the form (1.2).

In the following, we give an overview of some fundamental definitions and results that will be used in further discussions.

1.1 Essential Concepts

In this section, we present some basic definitions and results used in assumptions of models for option pricing problem. In the following part, we start by presenting some of the financial terminology used in the theory of mathematical finance. For a detailed information, see [41, 61, 67].

1.1.1 Options

Stocks, bonds, currencies, commodities, and indices are called **assets**. A **financial derivative** is a contract between individuals or institutions whose value at the maturity date (or expiry date) T is uniquely determined by the value of an **underlying asset** (or assets) at time T or until time T. There are three types of financial derivatives: Options, Forwards and Futures, and Swaps.

An **option** is the right, but not the obligation, to buy or sell any asset at an agreed

price within a specified period. Options have a long history, and first used by the manufacturers and food producers to cover their risks. The option to buy an asset is known as a call option and the option to sell an asset is known as a put option. Options on stocks were first traded in an organized market in 1973 and they are traded in many markets throughout the world.

There are many types of options such as European, American and exotic options. In this thesis, we will concern with the valuation of European options under certain market assumptions. Thus, in the following we present the information about European options. For overview on the other type of options, see [67].

European call option is a contract on which, at a prescribed time in the future, known as the expiry date T, the holder of the option may buy a prescribed asset (the underlying asset), for a prescribed amount E, called as the exercise price or strike price. European put option gives its holder the right to sell an asset at the expiry date T for an exercise price E.

On the other party to the contract, which is known as the writer has the potential commitment, must sell the asset if the owner of the call option chose to buy.

The value of a European option is denoted by $V(S,t)$. If the distinction is important, European call and put options are denoted by $C(S,t)$ and $P(S,t)$, respectively. At time $t = T$ for exercise price E, the payoff functions are given by

$$C(S(T),T) = \max\{S(T) - E, 0\} = \begin{cases} S(T) - E & \text{if } S(T) \geq E \\ 0 & \text{if } S(T) < E \end{cases} \tag{1.3}$$

$$P(S(T),T) = \max\{E - S(T), 0\} = \begin{cases} E - S(T) & \text{if } S(T) \leq E \\ 0 & \text{if } S(T) > E \end{cases} \tag{1.4}$$

for the European call and put options, respectively. Figure 1.1 depicts the payoff diagrams for call and put options for the exercise price $E = 50$, and maturity date $T = 5/12$ years.

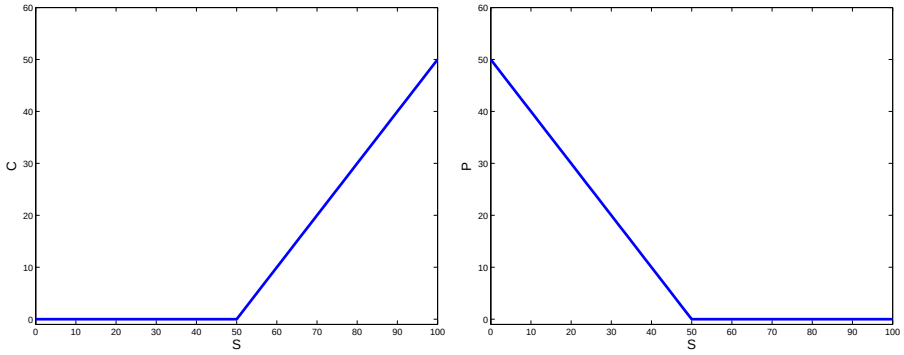

Figure 1.1: Payoff graphs for $E = 50, T = 5/12$.

1.1.2 Brownian Motion and Stochastic Differential Equations

In this part, we recall some basic concepts of stochastic calculus that will be used in the following chapters, see e.g. [41, 46, 56, 57] and the references therein. The basic notions of probability theory given in Appendix can be used to follow the subsequent descriptions.

Definition 1.1.1 *A* **stochastic process** *is a family of random variables* $\{X_t\}_{t\in T}$ *for* $T \subset [0,\infty)$ *which is defined on the same probability space* $(\Omega, \mathcal{F}, \mathbb{P})$. $\{X_t\}_{t\in T}$ *is said to be a discrete-time process if* $T = \mathbb{N}$ *or* $T = \mathbb{N}_0$, *a continuous-time process if* $T = [0,\infty)$.

Definition 1.1.2 Brownian motion $\{W_t\}_{t\geq 0}$ *is a continuous-time process for* $t \geq 0$ *with the following properties.*

(a) $W_0 = 0$,

(b) $W_t - W_s \sim N(0, t-s)$ *for all* $t \geq s$.

(c) The increments $W_{t_1} - W_{t_0}$, $W_{t_2} - W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}}$ *are independent for all* $0 = t_0 < t_1 < \cdots < t_n$.

(d) W_t *are continuous functions of* t.

The probability density function of W_t is

$$f(x,t) = \frac{1}{\sqrt{2\pi t}} e^{\frac{-x^2}{2t}}.$$

Definition 1.1.3 *An equation of the form*

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t; \tag{1.5}$$

where $a(X_t, t)$ and $b(X_t, t)$ are given, W_t is a Brownian motion and X_t is a stochastic process is called the **(Ito) stochastic differential equation (SDE).**

Definition 1.1.4 *A stochastic process S_t is said to follow* **geometric Brownian motion** *if it satisfies the stochastic differential equation of the form*

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1.6}$$

where μ (drift) and σ (volatility) are constants.

Lemma 1.1.5 (Itô lemma) *Let X_t for $t \geq 0$ be a stochastic process defined by*

$$dX_t = a(X_t, t)dt + b(X_t, t)dW_t,$$

and let $f(x, t)$ be a function for which the partial derivatives f_x, f_{xx}, f_t are defined and continuous.

Then, $Y_t := f(X_t, t)$ for every $t \geq 0$ is a stochastic process, and

$$dY_t = \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial x} + \frac{b^2}{2}\frac{\partial^2 f}{\partial x^2}\right)dt + \left(b\frac{\partial f}{\partial x}\right)dW_t$$

holds.

1.2 Outline

This thesis is organized as follows:

In Chapter 2, we review the Black-Scholes model with its original assumptions and the derivation of the Black-Scholes partial differential equation with the boundary and final conditions for the value of the European call and put options. Black-Scholes formula, which is the analytical solution of these problems is presented. Some of the finite difference methods are discussed for Black-Scholes equation and some examples are provided.

Chapter 3 is devoted to the study of nonlinear Black-Scholes equation which models illiquid market arising in option pricing. Firstly, a brief introduction is given for general nonlinear Black-Scholes equation under certain lack of assumptions of classical Black-Scholes model. Then, a model given by [48] in illiquid market is outlined and, following the article [17], an explicit scheme is derived. In addition, monotonicity, stability and consistency results are elaborated. Numerical examples reflecting the conditions of the results are presented. In this thesis, we use the MATLAB version 7.9.529 (R2009b) in all numerical implementations.

In Chapter 4, we implement a numerical procedure called exponential finite difference method to solve numerically the Black-Scholes equation and nonlinear Black-Scholes equation arising from European option pricing problem in illiquid market discussed in Chapter 3. Firstly we describe the method in heat equation with the stability conditions. Next, the method is applied to the linear Black-Scholes equation for different choice of initial underlying stock prices. Some numerical results are compared with the analytical solutions. Finally, nonlinear Black-Scholes equation with final condition, is solved by using exponential finite difference schemes. Particular examples are studied and analyzed.

CHAPTER 2

FINITE DIFFERENCE METHODS FOR BLACK-SCHOLES EQUATION

2.1 Introduction

In this section, we first present the Black-Scholes model with its assumptions and derive the Black-Scholes equation, which is a linear second order parabolic partial differential equation. By defining the corresponding final and boundary conditions for Black-Scholes equation, we introduce European call and put options pricing problems. Next, the analytical solutions for these problems, called as Black-Scholes formulas for European call and put are presented. Finally, some finite difference methods are described for these problems. For the sake of completeness, we start with a brief review of partial differential equations and their classifications.

A partial differential equation (PDE) is a functional equation that contains both a function and some of its derivatives. In mathematical finance, the independent variables are usually the time t and a state variable S in a subset of $\mathbb{R}^n$ for $n \geq 1$. In this thesis, we are interested in partial differential equations having unknown functions of two variables, thus we will review the basic concepts for PDE's in that form.

Let A be a subset of $\mathbb{R}^2$, $V(S,t)$ be a function from A to $\mathbb{R}$. For a given function F, the equation

$$F\left(S,t,V,\frac{\partial V}{\partial t},\frac{\partial V}{\partial S},\ldots,\frac{\partial^n V}{\partial t^k \partial S^l}\right)=0 \tag{2.1}$$

for $k,l \in \mathbb{N}_0$, $k+l=n$ is called a partial differential equation in V of order n. If

$V : A \to \mathbb{R}$ is an n times differentiable function, it is a solution of (2.1) if it satisfies

$$F\left(S, t, V, \frac{\partial V}{\partial t}, \frac{\partial V}{\partial S}, \ldots, \frac{\partial^n V}{\partial t^k \partial S^l}\right) = 0 \text{ for all } (S, t) \in A.$$

(2.1) is called **linear** if it can be written as

$$\sum_{j=0}^{n} a_{kl}(S, t) \frac{\partial^j V}{\partial t^k \partial S^l} = f(S, t)$$

for some functions a_{kl}, $k, l \in \mathbb{N}_0$, $j \leq n, k + l = j$ and some function f. Otherwise, (2.1) is called **nonlinear**.

In this study, we will be concerned with linear and nonlinear second-order PDEs. In the linear case, an equation of the form

$$a_{20} \frac{\partial^2 V}{\partial t^2} + a_{11} \frac{\partial^2 V}{\partial t \partial S} + a_{02} \frac{\partial^2 V}{\partial S^2} + a_{10} \frac{\partial V}{\partial t} + a_{01} \frac{\partial V}{\partial S} + a_{00} V = f(S, t) \qquad (2.2)$$

is called **elliptic** if $a_{11}^2 < 4a_{20}a_{02}$; **hyperbolic** if $a_{11}^2 > 4a_{20}a_{02}$ and **parabolic** if $a_{11}^2 = 4a_{20}a_{02}$ holds.

2.2 Black-Scholes Partial Differential Equation

2.2.1 Black-Scholes Model and Derivation of Black-Scholes Partial Differential Equation

In this part, we will present the Black-Scholes-Merton model with its assumptions to use it as a basis to introduce the numerical methods we will consider for its nonlinear version.

The model was introduced in 1973 by Fischer Black and Myron Scholes [15] and previously by Robert Merton [50] to represent the dynamics of an asset price and price the European options. Merton and Scholes were awarded the Nobel Prize in Economics for this result in 1997.

The following assumptions were given in the Black-Scholes model:

- There are no dividends on the underlying asset.
- There are no transaction costs.

- Trading takes place continuously.
- One can borrow and lend cash at a constant risk-free interest rate.
- One can buy any fraction of a share of stock.
- There are no restrictions on short selling.
- The market is liquid.
- There is no-arbitrage, i.e., there is no instantaneous riskless profit.
- The stock price follows a geometric Brownian motion given by (1.6) with constant expected return μ (drift) and volatility σ.

Let $V = V(S,t)$ be the value of a European option, for which the partial derivatives V_S, V_{SS} and V_t are continuous for $S \geq 0$, $0 \leq t \leq T$. Then, from the Itô Lemma (1.1.5), we get

$$dV = \left(\frac{\partial V}{\partial S}\mu S + \frac{\partial V}{\partial t} + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial V}{\partial S}\sigma S\, dW. \tag{2.3}$$

Let us consider the portfolio (sometimes called as delta-hedge portfolio)

$$\Pi = V - \Delta S$$

where Δ is to be determined. Change in the value of this portfolio in the time interval dt becomes

$$d\Pi = dV - \Delta dS \tag{2.4}$$

where $dS = \mu S\, dt + \sigma S\, dW$. Substituting (2.3) into (2.4) gives

$$\begin{aligned} d\Pi &= dV - \Delta dS \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \mu S\frac{\partial V}{\partial S} - \Delta\mu S\right)dt + \left(\sigma S\frac{\partial V}{\partial S} - \Delta\sigma S\right)dW. \end{aligned} \tag{2.5}$$

If

$$\Delta = \frac{\partial V}{\partial S}$$

holds, the coefficient of dW becomes zero and thus the random part is eliminated. Then, (2.5) becomes

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt. \tag{2.6}$$

The no-arbitrage assumption (see for example [41]) gives

$$d\Pi = r\Pi dt, \tag{2.7}$$

from which it follows that

$$d\Pi = r(V - \Delta S)dt = \left(rV - rS\frac{\partial V}{\partial S}\right) dt. \tag{2.8}$$

(2.6) and (2.8) give the equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2.9}$$

which is called the **Black-Scholes partial differential equation (BSE)**. It can be observed that, (2.9) is a second-order linear parabolic partial differential equation.

To find the unique solution of this equation in the cases of European call and put options, one needs to give the corresponding boundary and final conditions.

If $C(S,t)$ is the value of the European call option at time t having strike price E and expiry date T, the final condition of the call option at time $t = T$ is the payoff.

$$C(S,T) = \max\{S - E, 0\}. \tag{2.10}$$

If the value of the underlying asset is 0, then the call option does not worth to be paid, i.e. the value becomes zero, which can be represented as

$$C(0,t) = 0. \tag{2.11}$$

Conversely, if the price of the underlying asset becomes arbitrarily large, i.e. as $S \to \infty$, the call option will have the value asymptotically equivalent to the asset value at time t, that is,

$$C(S,t) \sim S - Ee^{-r(T-t)} \quad \text{as} \quad S \to \infty. \tag{2.12}$$

In the case of a put option, the final condition of $P(S,t)$ will be the payoff

$$P(S,T) = \max\{E - S, 0\}. \tag{2.13}$$

If $S = 0$, the value of the European put option will be the discounted value of the exercise price E at time t which can be expressed as

$$P(0,t) = Ee^{-r(T-t)} \tag{2.14}$$

with the interest rate r. Besides, as $S \to \infty$, a put option is unlikely to be exercised and therefore

$$P(S,t) \to 0 \quad \text{as} \quad S \to \infty \tag{2.15}$$

holds.

In the next section, we will briefly outline the procedure to find the analytical solution of BSE (2.9) with the corresponding boundary conditions for call and put, through the transformation of the problem into an initial-boundary value problem for heat equation. Moreover, sensitivities of the option with respect to certain variables, which are called Greeks in the literature, will be defined shortly.

2.2.2 Analytical Solution of BSE and Definitions of Greeks

In this part, we will discuss the analytical solution of European call and put options by transforming the BSE into heat equation by using some change of variables. We first take the problem for European call option.

For the European call option $C(S,t)$, we have the following equation and conditions:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0 \quad \text{for} \quad t \in (0,T) \ , \ S \in (0,\infty), \tag{2.16}$$

$$C(S,T) = \max\{S - E, 0\} \quad \text{for} \quad S \in (0,\infty), \tag{2.17}$$

$$C(0,t) = 0, \tag{2.18}$$

$$C(S,t) \sim S - Ee^{-r(T-t)} \quad \text{as} \quad S \to \infty \quad \text{for} \quad t \in [0,T]\,. \tag{2.19}$$

Consider the following change of variables:

$$S = Ee^x, \ t = T - \frac{2\tau}{\sigma^2}, \ C(S,t) = Ev(x,\tau)\,.$$

Then, we get the following partial derivatives:

$$\frac{\partial C}{\partial S} = e^{-x}\frac{\partial v}{\partial x},\ \frac{\partial^2 C}{\partial S^2} = \frac{e^{-2x}}{E}\left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}\right),\ \frac{\partial C}{\partial t} = -\frac{E\sigma^2}{2}\frac{\partial v}{\partial \tau}$$

Now, substituting these derivatives into equation (2.16) gives

$$\frac{E\sigma^2}{2}\frac{\partial v}{\partial \tau} = \frac{\sigma^2 E}{2}(\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x}) + rE\frac{\partial v}{\partial x} - rEv\ .$$

For $k = \frac{2r}{\sigma^2}$ and transforming the final and boundary conditions the following equation and conditions in terms of $v(x,\tau)$ are obtained:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv \ \text{ for } \ x \in (-\infty, \infty) \ \text{ and } \ \tau \in (0, T\sigma^2/2)\ , \tag{2.20}$$

$$v(x,0) = \max\{e^x - 1, 0\} \ \text{ for } \ x \in (-\infty, \infty)\ , \tag{2.21}$$

$$v(x,\tau) = 0 \quad as \quad x \to -\infty \ \text{ for } \ \tau \in (0, T\sigma^2/2)\ , \tag{2.22}$$

$$v(x,\tau) \sim e^x - e^{2\tau/\sigma^2} \quad as \quad x \to \infty \text{ for } \ \tau \in (0, T\sigma^2/2)\ . \tag{2.23}$$

Using another transformation such as,

$$v(x,\tau) = e^{\alpha x + \beta\tau}u(x,\tau)$$

the corresponding partial derivatives become

$$\frac{\partial v}{\partial \tau} = e^{\alpha x+\beta\tau}\left(\beta u(x,\tau) + \frac{\partial u}{\partial \tau}\right)\ ,\ \frac{\partial v}{\partial x} = e^{\alpha x+\beta\tau}\left(\alpha u(x,\tau) + \frac{\partial u}{\partial x}\right)$$

$$\frac{\partial^2 v}{\partial x^2} = e^{\alpha x+\beta\tau}\left(\alpha^2 u(x,\tau) + 2\alpha\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right)$$

with

$$\alpha = \frac{1-k}{2}\ ,\ \beta = -\frac{(k+1)^2}{4},$$

so that the following problem for heat equation is obtained:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \ \text{ for } \ x \in (-\infty, \infty) \ \text{ and } \ \tau \in (0, T\sigma^2/2)\ , \tag{2.24}$$

$$u(x,0) = \max\{e^{(\frac{(k+1)}{2})x} - e^{(\frac{k-1}{2})x}, 0\} \ \text{ for } \ x \in (-\infty, \infty)\ , \tag{2.25}$$

$$u(x,\tau) \to 0 \quad \text{as} \quad x \to -\infty \ \text{ for } \ \tau \in (0, T\sigma^2/2)\ , \tag{2.26}$$

$$u(x,\tau) \sim e^{\frac{(k+1)}{2}(x+(k+1)\tau/2)} \quad \text{as} \quad x \to \infty \ \text{ for } \ \tau \in (0, T\sigma^2/2)\ . \tag{2.27}$$

Solving the problem (2.24)-(2.27) see [15], and transforming the original problem gives the formula for European call option. The put-call parity for the European options [41], and the solution for the value of a European call option follows the value of a European put option. The following theorem gives these formulas.

Theorem 2.2.1 *The Black-Scholes formula for a European call option is*

$$C(S,t) = S\,N(d_1) - Ee^{-r(T-t)}N(d_2), \tag{2.28}$$

for a European put option is

$$P(S,t) = Ee^{-r(T-t)}N(-d_2) - S\,N(d_1), \tag{2.29}$$

where

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t},$$

and $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$ *is the cumulative distribution function for the standard normal distribution.*

2.2.2.1 Greeks

Sensitivities of the price of options to the parameters of the options are denoted by Greek letters and they are called Greeks. Measuring these sensitivities is important especially in risk management. Most common Greeks are the following:

- **Delta:** $\Delta = \frac{\partial V}{\partial S}$: measures the rate of change of the option value with respect to changes in the underlying asset's price S.
- **Gamma:** $\Gamma = \frac{\partial^2 V}{\partial S^2}$: measures the rate of change in the delta with respect to changes in the underlying asset's price S.
- **Theta:** $\theta = \frac{\partial V}{\partial t}$: measures the sensitivity of the value of the option to the time t.
- **Vega:** $\nu = \frac{\partial V}{\partial \sigma}$: measures sensitivity to volatility σ.

- **Rho:** $\rho = \dfrac{\partial V}{\partial r}$: measures sensitivity to the interest rate r.

The following section is devoted to the study of numerical solution of the European option valuation problem using finite difference method. This method will also be used for nonlinear option pricing problem in the next chapter.

2.3 Numerical Solution of Black-Scholes Equation Using Finite Difference Methods

Finite difference schemes for solving the partial differential equation formulation of European options can be found in [24, 61, 56, 67]. Also, there are many studies on the application of finite difference methods in option valuation problems, see for example, [20, 19].

In this section, we will present explicit, implicit and Crank-Nicolson finite difference methods to solve Black-Scholes PDE briefly. For detailed analysis with several numerical experiments, see [61]. First, we will review finite difference approximations to be used in finite difference methods.

2.3.1 Finite Difference Approximations

Finite difference is a technique where the derivatives of the given functions are approximated by differences in the values of the function between a given value of the independent variable x, and a small increment $x + h$.

Let $f(x)$ be a smooth function of any order, Then, Taylor's theorem states that

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3 f'''(x) + O(h^4).$$

If the terms of order h^2 and higher are neglected, the derivative $f'(x)$ can be written as

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

or

$$f'(x) \approx \frac{1}{h}\left[f(x+h) - f(x)\right].$$

This finite difference approximation is called a forward difference. Similarly, $f'(x)$ can be written by using

$$f(x-h) = f(x) - hf'(x) + \frac{1}{2}h^2 f''(x) - \frac{1}{6}h^3 f'''(x) + O(h^4).$$

Solving for $f'(x)$ we get

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h),$$

or

$$f'(x) \approx \frac{1}{h}\left[f(x) - f(x-h)\right],$$

which is called the backward difference approximation.

$f'(x)$ can also be obtained by using Taylor series for $f(x+h)$ and $f(x-h)$:

$$f'(x) = \frac{1}{2h}\left[f(x+h) - f(x-h)\right] + O(h^2).$$

Now,

$$f'(x) \approx \frac{1}{2h}\left[f(x+h) - f(x-h)\right]$$

is called the central difference formula for $f'(x)$. For other finite difference formulas to approximate $f'(x)$, see for example [61].

Using Taylor's expansions of $f(x+h)$ and $f(x-h)$. We compute

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4)$$

so that, $f''(x)$ can be written as

$$f''(x) = \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right] + O(h^2),$$

or

$$f''(x) \approx \frac{1}{h^2}\left[f(x+h) - 2f(x) + f(x-h)\right],$$

which is called the central difference approximation for the second-derivative f'' at x.

Finite difference approximation for partial derivatives can be derived in a similar way by using Taylor expansion in two dimension. For $u = u(x,y)$ the approximation for the partial derivative $u_x(x,y)$ is

$$u_x(x,y) = \frac{1}{h}\left[u(x+h,y) - u(x,y)\right] + O(h),$$

and it is called a forward difference formula for $u = u(x, y)$, where $h = \Delta x$.

Central difference formula for the second partial derivative of u with respect to x can be written as

$$u_{xx}(x, y) = \frac{1}{h^2}\left[u(x + h, y) - 2u(x, y) + u(x - h, y)\right] + O(h^2).$$

The other partial derivatives of any order may be obtained in a similar manner.

2.3.2 Finite Difference Methods for Black-Scholes Equation

2.3.2.1 The Grid

In the following, we will present the explicit, implicit and Crank-Nicolson finite different methods for solving Black-Scholes equation following the discussion in [61], see also [56, 67] and references there in. We start with discretizing the solution domain $S \geq 0$, $0 \leq t \leq T$. Let us define the step sizes

$$\Delta t = \frac{T}{M} \quad \text{and} \quad \Delta S = \frac{S_{max} - S_{min}}{N}.$$

for

$$S_{min} \leq S \leq S_{max} \quad \text{and} \quad 0 \leq t \leq T,$$

and

$$S_j = S_{min} + j\Delta S \quad \text{and} \quad t_n = n\Delta t,$$

where $j = 0, 1, \ldots, N$ and $n = 0, 1, \ldots, M$ and S_{min} and S_{max} are small and large values of stock prices for the European options, respectively. We represent the approximation of the value $V(S_j, t_n)$ as V_j^n, i.e., $V(S_j, t_n) \approx V_j^n$.

2.3.2.2 Explicit Method

The **explicit method** uses the backward difference approximation for time derivative and the central difference approximations for space derivatives for BSE (2.9), i.e.,

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \tag{2.30}$$

since the call and put problems contain final conditions. That is,

$$\frac{\partial V}{\partial t} \approx \frac{V_j^n - V_j^{n-1}}{\Delta t}, \tag{2.31}$$

$$\frac{\partial V}{\partial S} \approx \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S}, \tag{2.32}$$

$$\frac{\partial^2 V}{\partial S^2} \approx \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{(\Delta S)^2}. \tag{2.33}$$

Substituting the approximations (2.31), (2.32) and (2.33) into the Black-Scholes equation (2.30) we get

$$\frac{V_j^n - V_j^{n-1}}{\Delta t} = rV_j^n - rS_j \frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S} - \frac{1}{2}\sigma^2 S_j^2 \frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{(\Delta S)^2}. \tag{2.34}$$

This method is of order $O(\Delta t) + O((\Delta S)^2)$. Rewriting (2.34) gives

$$V_j^{n-1} = \alpha_j V_{j-1}^n + \beta_j V_j^n + \gamma_j V_{j+1}^n, \tag{2.35}$$

where

$$\alpha_j = \frac{1}{2}\Delta t \left\{ \sigma^2 (\frac{S_j}{\Delta S})^2 - r\frac{S_j}{\Delta S} \right\}, \tag{2.36}$$

$$\beta_j = 1 - \Delta t \left\{ \sigma^2 (\frac{S_j}{\Delta S})^2 + r \right\}, \tag{2.37}$$

$$\gamma_j = \frac{1}{2}\Delta t \left\{ \sigma^2 (\frac{S_j}{\Delta S})^2 + r\frac{S_j}{\Delta S} \right\} \tag{2.38}$$

for every $j = 1, \ldots, N-1$ and $n = 1, \ldots, M$. Since (2.35) gives an explicit formula for V_j^{n-1} this method is called explicit method.

Defining

$$V^{(n)} = \begin{bmatrix} V_1^n \\ \vdots \\ V_{N-1}^n \end{bmatrix} \tag{2.39}$$

the system (2.35) can be written as

$$V^{(n-1)} = AV^{(n)} + b^{(n)}, \quad n = M, M-1, \ldots, 1, \tag{2.40}$$

for which

$$A = \begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_{N-2} & \gamma_{N-2} \\ 0 & 0 & 0 & \dots & \alpha_{N-1} & \beta_{N-1} \end{bmatrix}, \quad b^{(n)} = \begin{bmatrix} \alpha_1 V_0^n \\ 0 \\ \vdots \\ 0 \\ \gamma_{N-1} V_N^n \end{bmatrix}.$$

Using the final conditions

$$C(S,T) = \max\{S - E, 0\}, \quad P(S,T) = \max\{E - S, 0\}$$

and the boundary conditions

$$C(S_{min}, t) = 0, \quad C(S_{max}, t) = S_{max} - Ee^{-r(T-t)},$$

$$P(S_{min}, t) = Ee^{-r(T-t)} - S_{min}, \quad P(S_{max}, t) = 0,$$

for a European call and put options $C(S,t)$ and $P(S,t)$, respectively, one can solve (2.40) directly and find solution for European call and put values for given exercise price E and maturity date T.

Stability, Consistency and Convergence of the Explicit Method

Stability of a finite difference scheme concerns the unstable growth or stable decay of errors in the arithmetic operations to solve finite difference equations. For a stable finite difference method, errors, in principle, can be made arbitrarily small. Stability analysis depends on the propagation of the round off errors in the method. A scheme is stable if the accumulated round off error is bounded, if it is unbounded then the method is said to be unstable. Usually, the terms ”conditional” or ”unconditional” are used to refer to the stability of a finite difference approximation. A conditionally stable finite difference algorithms have restrictive bounds on the temporal and spatial step sizes $\Delta t, \Delta x, \Delta y, \cdots$, above which the numerical solution is unstable and below which the numerical solution is stable. It is well known that most explicit finite difference approximations are unstable, unless certain conditions are satisfied. On the other hand most implicit approximations are unconditionally stable meaning that there is

no restrictive bound on the mesh widths $\Delta t, \Delta x, \Delta y, \cdots$. These facts are not always true. For example, DuFort-Frankel method for heat equation is an unconditionally stable explicit finite difference approximation. Experiences indicate that an unstable scheme is not convergent. Local truncation errors are measures of the error by which the exact solution of a differential equation does not satisfy the difference equation at the grid points and are obtained by substituting the exact solution of the problem into the numerical scheme. A necessary condition for the convergence of the numerical solutions to the continuous solution is that the local truncation error tends to zero as the step size goes to zero. In this case the method is said to be consistent. Consistency is only a necessary but not a sufficient condition for convergence. The relationship between the stability and the convergence of a finite difference scheme is investigated by the Lax equivalence theorem, which states that for a given properly posed initial or boundary value parabolic PDE and a consistent finite difference approximation to that equation, stability is the necessary and sufficient condition for convergence.

In the literature, to consider the stability of the methods described for solving BSE, usually stability is discussed for the heat equation obtained from the transformation of BSE, see [67]. In this part, we will discuss the stability of the explicit method directly for the BSE.

Assume that we have two different sets of the values V_j^n and $\hat{V}_j^n$, $j = 0, 1, \ldots, N-1$ of the approximate solution at the nth time level, for $n = 1, 2, \ldots, M$. Then the values of the $(n-1)$th level, computed according to the explicit finite difference method (2.35) should satisfy the equation

$$V_j^{n-1} - \hat{V}_j^{n-1} = \alpha_j(V_{j-1}^n - \hat{V}_{j-1}^n) + \beta_j(V_j^n - \hat{V}_j^n) + \gamma_j(V_{j+1}^n - \hat{V}_{j+1}^n), \tag{2.41}$$

$j = 0, 1, \ldots, N-1$. Let $|V_j^n - \hat{V}_j^n| < \epsilon$, that is $\epsilon > 0$ be the maximal difference of the values of V and $\hat{V}$ at the nth time level, then

$$|V_j^{n-1} - \hat{V}_j^{n-1}| \leq (|\alpha_j| + |\beta_j| + |\gamma_j|)\epsilon$$

holds for $j = 0, 1, \ldots, N-1$. Notice that, the coefficients $\alpha_j, \beta_j, \gamma_j$ given in (2.36), (2.37), (2.38) satisfy

$$\alpha_j + \beta_j + \gamma_j = 1 - r\Delta t.$$

Thus, if $\alpha_j, \beta_j, \gamma_j$ are all non negative, the maximal error in the $(n-1)$th level that becomes bounded by $(1 - r\Delta t)\epsilon$, and so that the errors are not increasing and the

system is stable. In case of at least one of $\alpha_j, \beta_j, \gamma_j$ is negative, $|\alpha_j| + |\beta_j| + |\gamma_j|$ may becomes larger than 1 which results in big errors at the end and the method becomes unstable. Hence, it is important to choose the step sizes N and M so that $\alpha_j, \beta_j, \gamma_j$ are all positive.

For the completeness of the discussions on finite difference methods applied for BSE, we will present the implicit, Crank-Nicolson and θ-averaged methods in the following part.

2.3.2.3 Implicit and Crank-Nicolson Methods

The **implicit method** differs from explicit method since it uses the forward difference approximation

$$\frac{\partial V}{\partial t} \approx \frac{V_j^{n+1} - V_j^n}{\Delta t} \tag{2.42}$$

for time derivative. As in the explicit method, the space derivatives are approximated by using central difference approximations (2.32) and (2.33). Substituting these derivatives into the equation (2.30) and rearranging the terms give

$$a_j V_{j-1}^n + b_j V_j^n + c_j V_{j+1}^n = V_j^{n+1}, \tag{2.43}$$

where

$$\begin{aligned} a_j &= \frac{1}{2}\Delta t \left\{ r\frac{S_j}{\Delta S} - \sigma^2 (\frac{S_j}{\Delta S})^2 \right\}, \\ b_j &= 1 + \Delta t \left\{ \sigma^2 (\frac{S_j}{\Delta S})^2 + r \right\}, \\ c_j &= -\frac{1}{2}\Delta t \left\{ r\frac{S_j}{\Delta S} + \sigma^2 (\frac{S_j}{\Delta S})^2 \right\} \end{aligned}$$

for all $j = 1, \ldots, N-1$ and $n = 0, 1, \ldots, M-1$. This method is called implicit method and it is of order $O(\Delta t) + O((\Delta S)^2)$.

Now, the system of equations (2.43) can be written as a matrix equation as

$$BV^{(n)} = V^{(n+1)} + d^{(n+1)}, \quad n = M-1, M-2, \ldots, 0, \tag{2.44}$$

where $V^{(n)}$ is given by (2.39) and

$$B = \begin{bmatrix} b_1 & c_1 & 0 & \dots & 0 & 0 \\ a_2 & b_2 & c_2 & \dots & 0 & 0 \\ 0 & a_3 & b_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & b_{N-2} & c_{N-2} \\ 0 & 0 & 0 & \dots & a_{N-1} & b_{N-1} \end{bmatrix}, \quad d^{(n+1)} = \begin{bmatrix} -a_1 V_0^n \\ 0 \\ \vdots \\ 0 \\ -c_{N-1} V_N^n \end{bmatrix}.$$

To solve the system (2.43) or equation (2.44), an iterative method successive over relaxation (SOR) or a direct method LU-decomposition can be used, respectively. For detailed information about these techniques, see [67].

The **Crank-Nicolson method** is defined by taking the arithmetic average of the methods (2.35) and (2.43) which results in

$$-p_j V_{j-1}^{n-1} + (1 - q_j) V_j^{n-1} - r_j V_{j+1}^{n-1} = p_j V_{j-1}^n + (1 + q_j) V_j^n + r_j V_{j+1}^n, \quad (2.45)$$

where

$$p_j = \frac{1}{4}\Delta t \left\{ \sigma^2 \left(\frac{S_j}{\Delta S} \right)^2 - r \frac{S_j}{\Delta S} \right\},$$
$$q_j = -\frac{1}{2}\Delta t \left\{ \sigma^2 \left(\frac{S_j}{\Delta S} \right)^2 + r \right\},$$
$$r_j = \frac{1}{4}\Delta t \left\{ \sigma^2 \left(\frac{S_j}{\Delta S} \right)^2 + r \frac{S_j}{\Delta S} \right\}$$

for every $j = 1, 2, \dots, N-1$ and $n = M-1, M-2, \dots, 1$ which corresponds to the matrix equation

$$CV^{(n-1)} = D^{(n)} + k^{(n)}, \quad n = M-1, M-2, \dots, 1, \quad (2.46)$$

where $V^{(n)}$ is given by (2.39),

$$k^{(n)} = \begin{bmatrix} p_1 (V_0^{n-1} + V_0^{n-1}) \\ 0 \\ \vdots \\ 0 \\ r_{N-1} (V_N^{n-1} + V_N^n) \end{bmatrix},$$

$$C = \begin{bmatrix} 1-q_1 & -r_1 & 0 & \dots & 0 & 0 \\ -p_2 & 1-q_2 & -r_2 & \dots & 0 & 0 \\ 0 & -p_3 & 1-q_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1-q_{N-2} & -r_{N-2} \\ 0 & 0 & 0 & \dots & -p_{N-1} & 1-q_{N-1} \end{bmatrix},$$

$$D = \begin{bmatrix} 1+q_1 & r_1 & 0 & \dots & 0 & 0 \\ p_2 & 1+q_2 & r_2 & \dots & 0 & 0 \\ 0 & p_3 & 1+q_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1+q_{N-2} & r_{N-2} \\ 0 & 0 & 0 & \dots & p_{N-1} & 1+q_{N-1} \end{bmatrix}.$$

Crank-Nicolson method is also an implicit method which has order $O((\Delta t)^2)+O((\Delta S)^2)$. The system (2.45) or equation (2.46) can also be solved using LU decomposition or SOR iteration technique.

Investigating the conditions for the coefficients of the Implicit and Crank-Nicolson schemes (2.43) and (2.46) for the stability, the Implicit and Crank-Nicolson methods become both unconditionally stable, see [25, 61].

Remark 2.3.1 The θ-method *is a convex combination of the explicit and the implicit methods.*

For $0 \le \theta \le 1$, *the method is defined as*

$$\begin{aligned} \frac{V_j^n - V_j^{n-1}}{\Delta t} &= \theta\left(rV_j^n - rS_j\frac{V_{j+1}^n - V_{j-1}^n}{2\Delta S}\right) \\ &\quad - \frac{1}{2}\theta\left(\sigma^2 S_j^2\frac{V_{j+1}^n - 2V_j^n + V_{j-1}^n}{(\Delta S)^2}\right) \\ &\quad + (1-\theta)\left(rV_j^{n+1} - rS_j\frac{V_{j+1}^{n-1} - V_{j-1}^{n-1}}{2\Delta S}\right) \\ &\quad - \frac{1}{2}(1-\theta)\left(\sigma^2 S_j^2\frac{V_{j+1}^{n-1} - 2V_j^{n-1} + V_{j+1}^{n-1}}{(\Delta S)^2}\right). \end{aligned} \tag{2.47}$$

From (2.47), we can obtain

- *the explicit method (2.35) for* $\theta = 1$,
- *the implicit method (2.43) for* $\theta = 0$,
- *the Crank-Nicolson method (2.45) for* $\theta = \frac{1}{2}$.

CHAPTER 3

AN EXPLICIT METHOD FOR NONLINEAR BLACK-SCHOLES EQUATION ARISING IN ILLIQUID MARKETS

3.1 Introduction

As we discussed in Chapter 2, Black-Scholes model for the value of a European option $V(S,t)$ is based on some assumptions. Indeed, these assumptions are given for the case of complete idealized market and they corresponds to a unique price for the option. We have discussed this unique price analytically and numerically in the previous section.

In reality, the market is incomplete and hence some of the assumptions of the Black-Scholes model may not hold; indeed, Black-Scholes model simplifies market mechanisms, e.g. transaction costs, market illiquidity, etc. If the completeness property of the market is dropped, several models are introduced with different numerical approaches, see [6, 10, 33, 35, 47], for example. Changing the assumptions of the Black-Scholes model leads to nonlinear extensions of the Black-Scholes PDE. These equations are known as **nonlinear Black-Scholes equation (NBSE)** in the literature which takes the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\left[\hat{\sigma}(S,t,\frac{\partial^2 V}{\partial S^2})\right]^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0. \tag{3.1}$$

In different models, $\hat{\sigma}$ function associates varied kinds of nonlinearity. It can be observed that, in the case of $\hat{\sigma} = \sigma$, Eqn. (3.1) gives the linear Black-Scholes equation (2.9).

In the literature there are several recent studies on models for option valuation under the assumption of the existence of transaction cost, uncertain volatility and market illiquidity.

The aim of this thesis is to discuss the numerical methods for nonlinear Black-Scholes equation arising from the illiquid market assumption. Therefore, in the next section we will give a detailed literature survey for such models. In the following, we briefly discuss nonlinear Black-Scholes equations arising from the assumptions of existing transaction cost under different approaches. For the models with the uncertain volatility supposition, see [6].

In the real market, transaction costs arise in trading financial assets which are not considered in the Black-Scholes model. Many studies show that such costs cause the increase in the option price. Therefore, actual option prices become higher than the computed value from the Black-Scholes formula. If transaction costs are considered, one of the leading models is the Leland's model [47], which assumes that the transaction cost $\frac{K}{2}\Delta S$ arises for the assets bought or sold. Here K is the return of the cost of transaction cost per unit dollar of the transaction and Δ is the number of asset that have been purchased ($\Delta \geq 0$) or sold ($\Delta \leq 0$) at price S. Model concludes that the option price is the solution of the non linear B-S equation (3.1) with the volatility adjustment:

$$\hat{\sigma}^2 = \sigma^2(1 + A\text{sign}(V_{SS})) \tag{3.2}$$

where A is the Leland number given by

$$A = \sqrt{\frac{2}{\pi}}\frac{K}{\sigma\sqrt{\delta t}}$$

and σ is the volatility of the underlying and δt is the time between adjustments of portfolio. Alternatively, Bales and Soner derived a more complex model [10] by utility maximization, where

$$\hat{\sigma}^2 = \sigma^2(1 + \Psi(e^{r(T-t)}a^2s^2V_{SS})) \tag{3.3}$$

with the parameter for the transaction costs and risk aversion $a = \frac{K}{\sqrt{\epsilon}}$. Here, $\Psi(x)$ is the solution of the nonlinear ordinary differential equation

$$\Psi'(x) = \frac{\Psi(x) + 1}{2\sqrt{x\Psi(x)} - x}$$

for $x \neq 0$ with $\Psi(0) = 0$.

3.2 Nonlinear Black-Scholes Equation in Illiquid Markets

3.2.1 A Literature Survey for Option Pricing in Illiquid Markets

Most of the financial models in the literature assume that the supply and demand in the market are perfectly elastic, which means, traders do not have any impact on financial market. In real market conditions, however, the powerful traders having large number of shares can influence the price of the traded shares by using their market power in order to control market prices in their favors.

Perfect liquid market assumption of the Black-Scholes model which assumes that an investor can trade large amounts of assets without affecting its price is accepted as an unrealistic assumption in the literature, see [35, 55]. It is natural to think that there will be a feedback, and this requires an extensive research on financial models to illiquid markets. Indeed, market liquidity is one of the most crucial factors in investment decisions and derivative pricing. It is believed that, liquidity affects the asset dynamics and trading strategies. Especially, risk management companies highly concern with the market liquidity issue.

There are recent studies concentrated on the pricing problem introduced by the market illiquidity. Frey [33, 34], Schönbucher and Wilmott [55], Frey and Patie [35], Bank and Baum [9] and Liu and Yong [48] introduced closely related models.

In [33, 35, 34] Frey at al. discuss the models assuming illiquidity of the market and the following NBSE is derived:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2\left(1 - \rho\lambda(S)\frac{\partial^2 V}{\partial S^2}\right)^2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where $\rho > 1$ is a constant and $\lambda(S) > 1$ is a strictly convex function. Liu and Yong [48] extend the Frey model by finding an expression for ρ and they obtain the equation

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2(1 - \lambda\left(S,t\right)S\frac{\partial^2 V}{\partial S^2}\right)^2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

where $\lambda(s,t)$ is given by

$$\lambda(s,t) = \begin{cases} \frac{\gamma}{S}(1 - e^{-\beta(T-t)}), & \underline{S} \leq S \leq \overline{S}, \\ 0, & \text{otherwise} \end{cases}$$

where $\gamma \geq 0$ measures the price impact per stock trading price and $\underline{S}$ and $\overline{S}$ represent the lower and upper limit of the stock price in presence of a price impact, respectively. They also found sufficient conditions for existence and uniqueness of solution of this nonlinear Black-Scholes equation under certain conditions.

3.2.2 A Model for Illiquid Market

In this study, we consider the following problem proposed by Liu and Yong [48]:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2\left(1 - \lambda(S,t)S\frac{\partial^2 V}{\partial S^2}\right)^2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \quad (S,t) \in (0,\infty)\times(0,T) \tag{3.4}$$

$$V(S,T) = f(S) \;\; 0 \leq S < \infty \tag{3.5}$$

where V is the price of the option depending on the underlying asset S and the time t, $\sigma > 0$ is the constant volatility, r is the interest rate, T is the expiry date for the option, $f(S)$ is the payoff function for which $f(e^x)$ is Lipschitz continuous and $e^{-\alpha\sqrt{1+x^2}}f(e^x)$ is bounded for some $\alpha \geq 0$. The price impact function $\lambda(S,t)$ is given by

$$\lambda(S,t) = \begin{cases} \frac{\gamma}{S}(1 - e^{-\beta(T-t)}), & \underline{S} \leq S \leq \overline{S}, \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

for which $\gamma > 0$ is the constant price impact coefficient which measures the price impact per traded share, $\underline{S}$ and $\overline{S}$ are the lower and upper limits of stock price in the presence of a price impact, β is a constant. In their numerical analysis, Liu and Yong assumed that $\beta = 100$, $\underline{S} = 20$ and $\overline{S} = 80$. We will also use this values in our numerical experiments.

In this study, we concern with the European options whose payoff functions $f(S)$ are of the form (1.3)-(1.4), that is, $f(S) = \max\{S - E, 0\}$ or $f(S) = \max\{E - S, 0\}$, in the cases of call or put respectively, which are piecewise linear and continuous functions, see also Fig.1.1. Thus the requirements of the above model for the payoff function are satisfied.

In the following an explicit numerical method is studied for the above problem (3.4) and (3.5) elaborating the article [17]. Before starting to construct the numerical scheme, the terminal value problem (3.4) and (3.5) is transformed into an initial value

problem via the change of variables $\tau = T - t$, $u(S,\tau) = V(S,t)$. The partial derivatives

$$\frac{\partial V}{\partial t} = -\frac{\partial u}{\partial \tau}, \quad \frac{\partial V}{\partial S} = \frac{\partial u}{\partial S}, \quad \frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 u}{\partial S^2}$$

are substituted into the problem (3.4) and (3.5) and the transformed equation

$$\frac{\partial u}{\partial \tau} - \frac{\sigma^2 S^2}{2\left(1 - \lambda(S, T-\tau)S\frac{\partial^2 u}{\partial S^2}\right)^2}\frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru = 0 \tag{3.7}$$

for $(S,\tau) \in (0,\infty) \times (0,T)$, with initial condition

$$u(S,0) = f(S), \quad 0 \leq S < \infty \tag{3.8}$$

is obtained.

3.2.3 An Explicit Numerical Scheme

In this part, we present an explicit finite difference method studied in the article [17]. In order to compute a numerical solution of the transformed problem (3.7) and (3.8), the interval $[0,b]$ is taken as the domain for the asset variable S for which b is chosen satisfying $0 < \underline{S} < \overline{S} < b$. Thus, the domain for the approximation becomes

$$(S,\tau) \in [0,b] \times [0,T]$$

with

$$S_j = jh, \; h = \Delta S, \; \tau_n = nk, \; k = \Delta\tau$$

for $0 \leq j \leq N, 0 \leq n \leq l, Nh = b, lk = T$.

The numerical approximation of the solution $u(S_j,\tau_n)$ is denoted by U_j^n. Let us introduce the approximations of the partial derivatives as:

$$\frac{\partial u}{\partial \tau}(S_j,\tau_n) = \frac{U_j^{n+1} - U_j^n}{k} + O(k) \tag{3.9}$$

$$\frac{\partial^2 u}{\partial S^2}(S_j,\tau_n) = \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2} + O(h^2) = \Delta_j^n(U) + O(h^2) \tag{3.10}$$

$$\frac{\partial u}{\partial S}(S_j,\tau_n) = \frac{U_{j+1}^n - U_{j-1}^n}{2h} + O(h^2) = \nabla_j^n(U) + O(h^2). \tag{3.11}$$

Taking outside nodes $S_{-1} = -h$ and $S_{N+1} = (N+1)h$, U_{-1}^n and U_{N+1}^n are obtained by

$$U_{-1}^n = 2U_0^n - U_1^n, \quad U_{N+1}^n = 2U_N^n - U_{N-1}^n \tag{3.12}$$

for $0 \leq n \leq l$ using linear interpolation. Moreover, using the approximations for the partial derivatives give

$$\Delta_0^n = \Delta_N^n = 0. \tag{3.13}$$

Now writing all these approximations into the equation (3.7) the numerical scheme takes the form

$$\frac{U_j^{n+1} - U_j^n}{k} - \frac{\sigma^2 S_j^2}{2\left(1 - \lambda(S_j, T - \tau_n) S_j \Delta_j^n\right)^2} \frac{U_{j-1}^n - 2U_j^n + U_{j+1}^n}{h^2} - rS_j \frac{U_{j+1}^n - U_{j-1}^n}{2h} + rU_j^n = 0.$$

Rearrangement of the terms yields the numerical scheme

$$\begin{aligned} U_j^{n+1} &= \left(1 - kr - \frac{k}{h^2}\beta_j^n\right) U_j^n \\ &+ \frac{k}{2h^2}\left[(\beta_j^n - jh^2 r) U_{j-1}^n + (\beta_j^n + jh^2 r) U_{j+1}^n\right] \end{aligned} \tag{3.14}$$

for $0 \leq j \leq N-1$ where

$$\beta_j^n = \frac{\sigma^2 S_j^2}{[1 - v_j^n]^2} \geq 0, \tag{3.15}$$

with $0 \leq j \leq N$, $0 \leq n \leq l$, and

$$v_j^n = \begin{cases} \gamma(1 - e^{-\beta nk}) \Delta_j^n(U), & \underline{S} \leq S_j \leq \overline{S} \\ 0, & \text{otherwise} \end{cases}. \tag{3.16}$$

Now, the scheme (3.14) and equalities (3.12) give

$$U_0^{n+1} = U_0^n = \ldots\ldots = U_0^0 = f(0), \tag{3.17}$$

$$U_N^{n+1} = (1 + (N-1)kr) U_N^n - NkrU_{N-1}^n. \tag{3.18}$$

To write the scheme (3.14) in vector form in $\mathbb{R}^{N+1}$, denote

$$U^n = [U_0^n \; U_1^n \ldots U_N^n]^T \tag{3.19}$$

and let A, $B(n)$, C, and D be the matrices in $\mathbb{R}^{(N+1)\times(N+1)}$ given by

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix}, \quad B(n) = \text{diag}(\beta_0^n, \ldots, \beta_N^n),$$

$$C = \begin{bmatrix} -2 & 2 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & \cdots & -1 & 0 & 1 \\ 0 & \cdots & 0 & -2 & 2 \end{bmatrix}, \quad D = \text{diag}(0, 1, 2, \ldots, N).$$

Then the scheme (3.14) can be written in the vector form as

$$U^{n+1} = \left[(1 - kr)\, I + \frac{k}{2h^2} B(n) A + \frac{kr}{2} DC \right] U^n \tag{3.20}$$

for $0 \leq n \leq l - 1$, satisfying the initial condition

$$U^0 = [f(0)\ f(h) \ldots f(Nh)]^T .$$

Here, I is identity matrix in $\mathbb{R}^{(N+1)\times(N+1)}$. To analyze the gamma Greek u_{SS} in the model, the scheme (3.14) can be rewritten for the vector $\Delta^n(U)$ in $\mathbb{R}^{N+1}$ defined by

$$\Delta^n(U) = [\Delta_0^n u\ \Delta_1^n u \ldots \Delta_N^n]^T$$

for $0 \leq n \leq l$. Definitions of $\Delta_j^n U$, U^n and $\Delta^n(U)$ give

$$\Delta^n(U) = \frac{1}{h^2} A U^n$$

and thus from the (3.20) it follows that

$$\begin{aligned} \Delta^{n+1}(U) = \frac{1}{h^2} A U^{n+1} &= \frac{1}{h^2} A \left[(1 - kr)\, I + \frac{k}{2h^2} B(n) A + \frac{kr}{2} DC \right] U^n \\ &= (1 - kr)\Delta^n(U) + \frac{k}{2h^2} AB(n)\Delta^n(U) + \frac{kr}{2h^2} ADCU^n, \end{aligned}$$

where

$$ADC = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 2 & -2 & -2 & -2 & 0 & \cdots & \cdots & \vdots \\ -1 & 4 & -2 & -4 & 3 & 0 & \cdots & \vdots \\ 0 & -2 & 6 & -2 & -6 & 4 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & & & \\ \vdots & & -(j-1) & 2j & -2 & -2j & j+1 & \vdots \\ \vdots & & \ddots & \ddots & \vdots & & & \\ \vdots & & & -(N-3) & 2(N-2) & -2 & -2(N-2) & (N-1) \\ \vdots & & & & -(N-2) & 2(N-1) & -(N+2) & 2 \\ 0 & \cdots & & & 0 & 0 & 0 & 0 \end{bmatrix}.$$

By (3.13) and (3.14) we have

$$\Delta_0^{n+1} = \Delta_0^n = 0, \Delta_N^{n+1} = \Delta_N^n = 0 \tag{3.21}$$

and (3.14) becomes

$$\Delta_j^{n+1} = \left(1 - \frac{k}{h^2}\beta_j^n\right)\Delta_j^n + \frac{k}{2h^2}\left[\left(\beta_{j-1}^n - (j-1)h^2 r\right)\Delta_{j-1}^n + (\beta_{j+1}^n + (j+1)h^2 r)\Delta_{j+1}^n\right] \tag{3.22}$$

for $1 \leq j \leq N-1$.

In the following section, we study the monotonicity, stability and consistency properties of the numerical scheme (3.14), (3.17) and (3.18). We introduce these properties following [17] in a detailed complete form.

3.2.4 Monotonicity, Stability and Consistency

Before starting with the analysis of the properties of the scheme (3.14), we will give a lemma giving order property of Δ_j^n which appears in the coefficient β_j^n given by (3.15) by means of v_j^n in (3.16). Now, let us first introduce the notation

$$\sum(n) := \sum_{j=0}^{N} \Delta_j^n.$$

By (3.13), we can also write $\sum(n) = \sum_{j=1}^{N-1} \Delta_j^n$. Equations given by (3.21) and (3.22) and the scheme (3.14) imply

$$\begin{aligned}
\sum(n+1) &= \sum_{j=1}^{N-1} \Delta_j^{n+1} \\
&= \sum_{j=1}^{N-1}\left((1 - \frac{k}{h^2}\beta_j^n)\Delta_j^n + \frac{k}{2h^2}\left[\left(\beta_{j-1}^n - (j-1)h^2 r\right)\Delta_{j-1}^n + (\beta_{j+1}^n + (j+1)h^2 r)\Delta_{j+1}^n\right]\right) \\
&= \sum_{j=1}^{N-1}(1 - \frac{k}{h^2}\beta_j^n)\Delta_j^n + \frac{k}{2h^2}\left(\sum_{j=0}^{N-2}(\beta_j^n - jh^2 r)\Delta_j^n + \sum_{j=2}^{N}(\beta_j^n + jh^2 r)\Delta_j^n\right).
\end{aligned}$$

Rearranging the terms gives

$$\begin{aligned}\sum(n+1) &= \sum_{j=1}^{N-1}(1-\frac{k}{h^2}\beta_j^n)\Delta_j^n \\ &+ \frac{k}{2h^2}\left(\sum_{j=1}^{N-1}(\beta_j^n - jh^2r)\Delta_j^n + \beta_0^n\Delta_0^n - (\beta_{N-1}^n - (N-1)h^2r)\Delta_{N-1}^n\right) \\ &+ \frac{k}{2h^2}\left(\sum_{j=1}^{N-1}(\beta_j^n + jh^2r)\Delta_j^n - (\beta_1^n + h^2r)\Delta_1^n\right) \\ &= \sum(n) - \frac{k}{2h^2}\left[(\beta_1^n + h^2r)\Delta_1^n + (\beta_{N-1}^n - (N-1)h^2r)\Delta_{N-1}^n\right]. \quad (3.23)\end{aligned}$$

Lemma 3.2.1 *Let r be the riskless interest rate, σ the volatility and γ the impact factor parameter of the illiquid market given in the equation (3.7). Assume that the payoff function f in (3.8) is convex, i.e.,*

$$f(ax_1 + bx_2) \le af(x_1) + bf(x_2) \text{ for } 0 \le a, b \le 1.$$

If $\eta = \gamma(1 - e^{-\beta\tau})\sum(0)$, $S \in [0, b]$, $\sigma^2 \ge r$, $0 \le \eta \le 1$,

$$\frac{k}{h^2} \le \frac{(1-\eta)^2}{\sigma^2 b} \quad (3.24)$$

then

(a) $\Delta_j^n \ge 0,\ 0 \le j \le N,\ 0 \le n \le l,$

(b) $\sum(n+1) \le \sum(n),\ 0 \le n \le l-1.$

Proof. Convexity of the function f implies that

$$f(S_j) = f\left(\frac{1}{2}S_{j-1} + \frac{1}{2}S_{j+1}\right) \le \frac{1}{2}f(S_{j-1}) + \frac{1}{2}f(S_{j+1})$$

and thus we have

$$\Delta_j^0 = \frac{f(S_{j-1}) - 2f(S_j) + f(S_{j+1})}{h^2} \ge 0 \quad (3.25)$$

for all $1 \le j \le N-1$. By (3.13), $\Delta_0^0 = \Delta_N^0 = 0$, and thus $\Delta_j^0 \ge 0$ for all $0 \le j \le N$. Assume that $\Delta_j^n \ge 0$ holds for $0 \le n \le l, 0 \le j \le N-1$. We will prove that $\Delta_j^{n+1} \ge 0$

is true. Observe that,

$$\begin{aligned}\beta_j^n - h^2 r j &= \frac{\sigma^2 S_j^2}{(1-v_j^n)^2} - h^2 r j \\ &= \frac{\sigma^2 h^2 j^2}{(1-v_j^n)^2} - h^2 r j \\ &= h^2 j\left(\frac{\sigma^2 j}{(1-v_j^n)^2} - r\right) \\ &\geq h^2 j(\sigma^2 j - r) \geq 0 \end{aligned} \tag{3.26}$$

for $1 \leq j \leq N$, by the condition $\sigma^2 \geq r$. Thus,

$$\beta_{N-1}^n - h^2 r(N-1) \geq 0 \tag{3.27}$$

holds. Now, the relation (3.23) implies

$$\sum(n+1) \leq \sum(n). \tag{3.28}$$

Notice that, by (3.28) and assumption $\Delta_j^n \geq 0$, v_j^n defined in (3.16) satisfies

$$\begin{aligned} 0 \leq v_j^n &\leq \gamma(1-e^{-\beta nk})\Gamma_j^n \\ &= \gamma(1-e^{-\beta nk})\Gamma_j^n \sum(n) \\ &\leq \gamma(1-e^{-\beta nk})\Gamma_j^n \sum(0) \\ &= \eta. \end{aligned} \tag{3.29}$$

Relations (3.24) and (3.29) imply

$$\begin{aligned} 0 \leq \beta_j^n &= \frac{\sigma^2 S_j^2}{[1-v_j^n]^2} \\ &\leq \frac{\sigma^2 b^2}{[1-\eta]^2} \\ &\leq \frac{h^2}{k}. \end{aligned} \tag{3.30}$$

As a result, the scheme (3.22) implies that $\Delta_j^{n+1} \geq 0$. □

3.2.4.1 Monotonicity

Definition 3.2.2 *The scheme $F(U_j^n) = 0$, for $j, n \in \mathbb{N}_0$ is said to be monotonicity-preserving if $U_{j+1}^n \geq U_j^n$ then $U_{j+1}^{n+1} \geq U_j^{n+1}$ holds for all j.*

Theorem 3.2.3 *Assume that the payoff function f is convex, $\eta = \gamma(1 - e^{-\beta\tau})\sum(0)$, $S \in [0, b]$, $\sigma^2 \geq r$, $0 \leq \eta \leq 1$,*

$$\frac{k}{h^2} \leq \frac{(1-\eta)^2}{\sigma^2 b} \tag{3.31}$$

and $k \leq h^2 L(h)$ for

$$L(h) = \frac{(1-\eta)^2}{\sigma^2 b^2 + \frac{1}{2}(1-\eta)^2 r h^2}. \tag{3.32}$$

Then the scheme (3.14-3.18) is monotonicity preserving for $0 \leq j \leq N, 0 \leq n \leq l$.

Proof. In the proof, we will use the induction principle. Let us assume that

$$U^n_{j+1} \geq U^n_j \tag{3.33}$$

holds for $1 \leq j \leq N-1, 0 \leq n \leq l-1$. We will prove that $U^{n+1}_{j+1} \geq U^{n+1}_j$ holds.

Let us start with writing the difference $U^{n+1}_{j+1} - U^{n+1}_j$ as

$$U^{n+1}_{j+1} - U^{n+1}_j = \left(U^{n+1}_{j+1} - (1-kr)U^n_{j+1}\right) + (1-kr)(U^n_{j+1} - U^n_j) - (U^{n+1}_j - (1-kr)U^n_j). \tag{3.34}$$

The scheme (3.14) gives

$$\begin{aligned}
U^{n+1}_j - (1-kr)U^n_j &= -\frac{k}{h^2}\beta^n_j U^n_j + \frac{k}{2h^2}\left[(\beta^n_j - jh^2 r)U^n_{j-1} + (\beta^n_j + jh^2 r)U^n_{j+1}\right] \\
&\leq -\frac{k}{h^2}\beta^n_j U^n_j + \frac{k}{2h^2}\left[(\beta^n_j - jh^2 r)U^n_j + (\beta^n_j + jh^2 r)U^n_{j+1}\right] \\
&= (-\frac{k}{h^2}\beta^n_j + \frac{k}{2h^2}\beta^n_j - \frac{k}{2h^2}jh^2 r)U^n_j + \frac{k}{2h^2}(\beta^n_j + jh^2 r)U^n_{j+1} \\
&= \frac{k}{2h^2}\left[(U^n_{j+1} - U^n_j)(\beta^n_j + jh^2 r)\right]
\end{aligned} \tag{3.35}$$

for $1 \leq j \leq N-1$. Then we have

$$-(U^{n+1}_j - (1-kr)U^n_j) \geq -\frac{k}{h^2}[(U^n_{j+1} - U^n_j)(\beta^n_j + jh^2 r)]. \tag{3.36}$$

In the other way, we obtain

$$\begin{aligned}
U^{n+1}_j - (1-kr)U^n_j &\geq -\frac{k}{h^2}\beta^n_j U^n_j + \frac{k}{2h^2}[(\beta^n_j - jh^2 r)U^n_{j-1} + (\beta^n_j + jh^2 r)U^n_j] \\
&= (-\frac{k}{h^2}\beta^n_j + \frac{k}{2h^2}\beta^n_j + \frac{k}{2h^2}jh^2 r)U^n_j + \frac{k}{2h^2}(\beta^n_j - jh^2 r)U^n_{j-1} \\
&\geq \frac{k}{2h^2}(U^n_j - U^n_{j-1})(\beta^n_j - jh^2 r)
\end{aligned} \tag{3.37}$$

for $1 \leq j \leq N-1$, which also may be written as

$$U^{n+1}_{j+1} - (1-kr)U^n_{j+1} \geq \frac{k}{2h^2}(U^n_{j+1} - U^n_j)(\beta^n_{j+1} - (j+1)h^2 r). \tag{3.38}$$

Now, using (3.36) and (3.38), (3.34) takes the form

$$\begin{aligned} U_{j+1}^{n+1} - U_j^{n+1} &= \left(U_{j+1}^{n+1} - (1-kr)U_{j+1}^n\right) + (1-kr)(U_{j+1}^n - U_j^n) - (U_j^{n+1} - (1-kr)U_j^n) \\ &\geq \left(1 - \frac{k}{2h^2}(\beta_{j+1}^n + \beta_j^n) - \frac{kr}{2}\right)(U_{j+1}^n - U_j^n) \end{aligned} \tag{3.39}$$

for $1 \leq j \leq N-2$. If we denote

$$\beta_{max} = \frac{\sigma^2 b^2}{(1-\eta)^2},$$

from the condition (3.15) we can write

$$0 \leq \beta_j^n \leq \frac{\sigma^2 S_j^2}{(1-\eta)^2} \leq \frac{\sigma^2 b^2}{(1-\eta)^2} = \beta_{max}$$

and thus

$$1 - \frac{k}{2h^2}(\beta_{j+1}^n + \beta_j^n) - \frac{kr}{2} \geq 1 - \frac{k}{2h^2} 2\beta_{max} - \frac{kr}{2} = 1 - \frac{k}{h^2}\beta_{max} - \frac{kr}{2}$$

holds. By (3.31) and (3.32) we get

$$\frac{k}{h^2} \leq L(h) = \frac{h^2(1-\eta)^2}{\sigma^2 b^2 + \frac{1}{2}(1-\eta)^2 r h^2}.$$

Multiplying both sides by the positive number $\frac{\sigma^2 b^2}{(1-\eta)^2}$ we get

$$\frac{k}{h^2}\frac{\sigma^2 b^2}{(1-\eta)^2} \leq \frac{(1-\eta)^2}{\sigma^2 b^2 + \frac{1}{2}(1-\eta)^2 r h^2}\frac{\sigma^2 b^2}{(1-\eta)^2},$$

that is,

$$-\frac{k}{h^2}\beta_{max} \geq \frac{-\sigma^2 b^2}{\sigma^2 b^2 + \frac{1}{2}(1-\eta)^2 r h^2}.$$

Adding $1 - \frac{kr}{2}$ to both sides of the above inequality and taking $0 < k < \frac{1}{r}$ gives

$$1 - \frac{k}{h^2}\beta_{max} - \frac{kr}{2} \geq 1 - \frac{\sigma^2 b^2}{\sigma^2 b^2 + \frac{1}{2}(1-\eta)^2 r h^2} - \frac{kr}{2} \geq 0.$$

Now, from (3.39) it follows that

$$U_{j+1}^{n+1} - U_j^{n+1} \geq 0$$

for all $1 \leq j \leq N-2$.

In the case of $j = 0$, by (3.38) we have

$$\begin{aligned} U_1^{n+1} - U_0^{n+1} &= U_1^{n+1} - (1 - kr)U_1^n + (1 - kr)(U_1^n - U_0^n) \\ &\geq -\frac{k}{2h^2}(U_1^n - U_0^n)(\beta_1^n - h^2 r) + (1 - kr)(U_1^n - U_0^n) \\ &= (U_1^n - U_0^n)(1 - kr - \frac{k}{2h^2}\beta_1^n + \frac{kr}{2}) \\ &\geq (1 - \frac{k}{2h^2}\beta_{max} - \frac{kr}{2})(U_1^n - U_0^n) \geq 0 \end{aligned} \tag{3.40}$$

Similarly, for $j = N - 1$ we get

$$U_N^{n+1} - U_{N-1}^{n+1} \geq \left(1 + \frac{Nkr}{2} - \left(\frac{k}{2h^2}\beta_{max} + \frac{kr}{2}\right)\right)(U_N^n - U_{N-1}^n) \geq 0.$$

Thus the proof is completed. □

The following result confirms the positivity of the scheme (3.14).

Corollary 3.2.4 *If all the hypotheses of Theorem 3.2.3 hold and also the payoff function $f(S)$ is nondecreasing and nonnegative with $f(0) = 0$, then the scheme (3.14),(3.17), (3.18) is nonnegative and nondecreasing in variable j for each n, that is,*

$$0 = U_0^n \leq U_1^n \leq \cdots \leq U_j^n \leq U_{j+1}^n \leq \cdots \leq U_N^n \tag{3.41}$$

for all $0 \leq n \leq l$.

Proof. By (3.17), $U_0^n = f(0) = 0$. Monotonicity of the (3.14) implies (3.41). □

3.2.4.2 Stability

Definition 3.2.5 *The numerical scheme (3.20) in vector form for U^n given in (3.19) is said to be $\|.\|_\infty$-stable in the fixed station sense in the domain $[0, b] \times [0, T]$, if for given τ with $0 \leq \tau \leq T$, every partition with $k = \Delta\tau$, $h = \Delta S$ with $\tau = lk$, and every N with $Nh = b$,*

$$\|U^n\|_\infty \leq C$$

holds for some C independent of h ,k and N. Here, $\|.\|_\infty$ denotes the supremum norm in R^{N+1}.

Theorem 3.2.6 *Under condition of Theorem 3.2.3, the numerical scheme (3.20) with a convex nondecreasing and nonnegative piecewise linear payoff function f is $\|.\|_\infty$-stable.*

Proof. Writing $j = N-1$ in the scheme (3.14) we obtain

$$U_{N-1}^{n+1} = (1-kr-\frac{k}{h^2}\beta_{N-1}^n)U_{N-1}^n + \frac{k}{2h^2}[(\beta_{N-1}^n-(N-1)h^2r)U_{N-2}^n + (\beta_{N-1}^n+(N-1)h^2r)U_N^n]$$

and using (3.18) given by

$$U_N^{n+1} = (1+(N-1)kr)U_N^n - NkrU_{N-1}^n$$

we get

$$U_N^{n+1} - U_{N-1}^{n+1} = U_N^n - U_{N-1}^n - \frac{k}{2}\Delta_{N-1}^n(\beta_{N-1}^n - h^2r(N-1)) \tag{3.42}$$

for $0 \le n \le l-1$. By (3.26) we have $\beta_{N-1}^n - h^2r(N-1) \ge 0$ and by Lemma 3.2.1 $\Delta_{N-1}^n \ge 0$ holds. Hence, using (3.42) we get

$$U_N^{n+1} - U_{N-1}^{n+1} \le U_N^n - U_{N-1}^n \tag{3.43}$$

for $0 \le n \le l-1$. Now (3.43) and (3.18) results in

$$U_N^{n+1} = (1-kr)U_N^n + krN(U_N^n - U_{N-1}^n) \le (1-kr)U_N^n + krN(U_N^0 - U_{N-1}^0).$$

Inductively one can find

$$U_N^n \le (1-kr)^nU_N^0 + Nkr(U_N^0 - U_{N-1}^0)(1+(1-kr)+\cdots+(1-kr)^{n-1}). \tag{3.44}$$

For $0 \le k \le \frac{1}{r}$, $\sum_{j=0}^{\infty}(1-kr)^j = \frac{1}{kr}$ holds and (3.44) becomes

$$U_N^n \le U_N^0 + (U_N^0 - U_{N-1}^0)N. \tag{3.45}$$

If m is the slope of the last linear part of the piecewise linear payoff function f, then

$$U_N^0 - U_{N-1}^0 = mh \tag{3.46}$$

is satisfied. Now, (3.45), (3.46) and Corollary 3.2.4 imply

$$\begin{aligned} \|U^n\|_\infty &= U_N^n \\ &\le U_N^0 + (U_N^0 - U_{N-1}^0)N \\ &= U_N^0 + mhN = f(Nh) + mhN = f(b) + mb \\ &= C \end{aligned} \tag{3.47}$$

for $1 \le n \le l$ and $C = f(b) + mb$, which is independent of h, k and N. Thus the proof is completed. □

3.2.4.3 Consistency

Let us recall the consistency definition.

Definition 3.2.7 *A numerical scheme in the form $F(u_j^n) = 0$ is said to be consistent with a PDE $L(u) = 0$, if the exact solution of the scheme approximates the exact solution of the PDE as local truncation error*

$$T_j^n(u) = F(u_j^n) - L(u_j^n)$$

tends to zero as $h \longrightarrow 0$, $k \longrightarrow 0$, where u_j^n is the value of the analytical solution of $L(u) = 0$ at the corresponding mesh point.

Theorem 3.2.8 *The numerical scheme (3.14) is consistent with the Eq.3.7 and the local truncation error T_J^n satisfies*

$$T_j^n(u) = O(h^2) + O(k). \tag{3.48}$$

Proof. We write the scheme given in (3.14) as

$$u_j^{n+1} = (1 - kr - \frac{k}{h^2}\beta_j^n)u_j^n + \frac{k}{2h^2}[(\beta_j^n - jh^2r)u_{j-1}^n + (\beta_j^n + jh^2r)u_{j+1}^n]$$

in the form

$$F(u_j^n) = \frac{u_j^{n+1} - u_j^n}{k} - \frac{1}{2}\beta_j^n(u)\Delta_j^n(u) - rS_j\nabla_j^n(u) + ru_j^n = 0 \tag{3.49}$$

where Δ_j^n, ∇_j^n and β_j^n are given by (3.10), (3.11) and (3.15), respectively. We will show that the local truncation error

$$T_j^n(u) = F(u_j^n) - L(u_j^n) \tag{3.50}$$

satisfies

$$T_j^n(u) \to 0, \text{ as } h = \Delta S \to 0, k = \Delta\tau \to 0 \tag{3.51}$$

where u_j^n is the value of the analytical solution of

$$L(u) = \frac{\partial u}{\partial \tau} - \frac{\sigma^2 S^2}{2(1 - \lambda(S, T - \tau)S(\frac{\partial^2 u}{\partial S^2})^2}\frac{\partial^2 u}{\partial S^2} - rS\frac{\partial u}{\partial S} + ru = 0$$

at the mesh point (S_j, τ_n), $S_j = jh$, $\tau_n = nk$.

Assume that u has four times continuous partial derivatives with respect to S and twice continuous partial derivatives with respect to τ. Using Taylor's expansion about (S_j, τ_n) we have

$$\begin{aligned} u_{j+1}^n &= u(S_J, \tau_n) + h\frac{\partial u}{\partial S}(S_J, \tau_n) + \frac{1}{2}h^2\frac{\partial^2 u}{\partial S^2}(S_J, \tau_n) \\ &+ \frac{1}{6}h^3\frac{\partial^3 u}{\partial S^3}(S_J, \tau_n) + \frac{1}{24}h^4\frac{\partial^4 u}{\partial S^4}(S_J, \tau_n) + O(h^5) \end{aligned}$$

and similarly writing u_{j-1}^n and substituting them into the ∇_j^n and Δ_j^n we get

$$\nabla_j^n(u) = \frac{\partial u}{\partial S}(S_j, \tau_n) + h^2 E_j^n(1) \tag{3.52}$$

where

$$E_j^n(1) = \frac{1}{6}\frac{\partial^3 u}{\partial S^3}(\zeta_1, \tau_n), \quad S_j - h \le \zeta_1 \le S_j + h$$

which satisfies

$$\left|E_j^n(1)\right| \le \frac{1}{6}\max\left\{\left|\frac{\partial^3 u}{\partial S^3}(S, \tau_n)\right|; 0 \le S \le b\right\} := \frac{1}{6}\left|W^n(1)\right|_{max}$$

.

$$\Delta_j^n(u) = \frac{\partial^2 u}{\partial S^2}(S_j, \tau_n) + h^2 E_J^n(2) \tag{3.53}$$

where

$$E_j^n(2) = \frac{1}{12}\frac{\partial^4 u}{\partial S^4}(\zeta_2, \tau_n), \quad S_j - h \le \zeta_2 \le S_j + h$$

which satisfies

$$\left|E_j^n(2)\right| \le \frac{1}{12}\max\left\{\left|\frac{\partial^4 u}{\partial S^4}(S, \tau_n)\right|; 0 \le S \le b\right\} := \frac{1}{12}\left|W^n(2)\right|_{max}.$$

Also, we obtain

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{\partial u}{\partial \tau}(S_j, \tau_n) + kE_j^n(3) \tag{3.54}$$

where

$$E_j^n(3) = \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2}(S_j, \tau), \tau_n \le \tau \le \tau_{n+1}$$

satisfying

$$\left|E_j^n(3)\right| \le \frac{1}{2}\max\left\{\left|\frac{\partial^2 u}{\partial \tau^2}(S, \tau_n)\right|; \tau_n \le \tau \le \tau_{n+1}\right\} := \frac{1}{2}\left|W_j^n(3)\right|_{max}.$$

By using (3.49), (3.50) and (3.52) we obtain

$$\begin{aligned} T_j^n(u) &= F(u_j^n) - L(u_j^n) \\ &= \frac{\partial u}{\partial \tau} + kE_j^n(3) - \frac{1}{2}\beta_j^n(u)\Delta_j^n(u) - rS_j(\frac{\partial u}{\partial S} + h^2E_j^n(1)) + ru \\ &- \frac{\partial u}{\partial \tau} + \frac{\sigma^2 S_j^2}{2(1-\lambda S_j)\frac{\partial^2 u}{\partial S^2}}\frac{\partial^2 u}{\partial S^2} + rs\frac{\partial u}{\partial S} - ru \\ &= kE_j^n(3) - rS_j h^2 E_j^n(1) \\ &- \frac{1}{2}\left(\beta_j^n(u)\Delta_j^n(u) - \frac{\sigma^2 S_j^2}{(1-\lambda(S_j, T-\tau_n)S_j\frac{\partial^2 u}{\partial S^2}(S_j,\tau_n)}\frac{\partial^2 u}{\partial S^2}(S_j,\tau_n)\right) \end{aligned} \quad (3.55)$$

where

$$\beta_j^n(u) = \frac{\sigma^2 S_j^2}{[1-\nu_j^n]^2}, \quad \nu_j^n = \begin{cases} \gamma(1-e^{-\beta nk})\Delta_j^n(u), & \underline{S} \le S_j \le \overline{S}, \\ 0, & \text{otherwise} \end{cases}.$$

Note that, by the definition of $\lambda(S,t)$ given in (3.6), we can write

$$\beta_j^n(u) = \frac{\sigma^2 S_j^2}{(1-S_j\lambda(S_j, T-\tau_n)\Delta_j^n(u))^2}. \quad (3.56)$$

Define the function

$$G(S,\tau,x) = \frac{x}{(1-\lambda(S,T-\tau)Sx)^2}$$

which is continuously differentiable with respect to x in the domain $\lambda(S,T-\tau)Sx \le 1$. Taking the partial derivative of G with respect to x we get

$$\frac{\partial G}{\partial x} = \frac{(1-\lambda Sx)^2 - 2x(1-\lambda Sx)(-\lambda S)}{(1-\lambda Sx)^4} = \frac{1+\lambda Sx}{(1-\lambda Sx)^3}.$$

Let us define $\delta_0 > 0$ as it is suggested in [48] satisfying

$$1 - \lambda(S,t)u_{SS} \ge \delta_0 > 0$$

so that

$$\delta_0 - \frac{h_0^2}{12}|W^n(2)|_{max}\gamma(1-e^{-\beta\tau_n}) = \delta_1^n > 0$$

holds for small enough $h_0 > 0$.

Taking $x = \dfrac{\partial^2 u}{\partial s^2}(S_j, \tau_n)$ and $\Delta x = h^2 E_j^n(2)$ where $0 \le h \le h_0$ and $x + \Delta x = \Delta_j^n(u)$ and using the mean value theorem, we obtain:

$$\begin{aligned}
&\beta_j^n(u)\Delta_j^n(u) - \frac{\sigma^2 S_j^2}{(1-\lambda(S_j, T-\tau_n)S_j \frac{\partial^2 u}{\partial S^2}(S_j,\tau_n))^2}\frac{\partial^2 u}{\partial S^2}(S_j,\tau^n)\\
= &\frac{\sigma^2 S_j^2}{(1-S_j\lambda(S_j,T-\tau^n)\Delta_j^n)^2}\Delta_j^n - \frac{\sigma^2 S_j^2}{(1-\lambda(S_j,T-\tau_n)S_j\frac{\partial^2 u}{\partial S^2}(S_j,\tau_n))^2}\frac{\partial^2 u}{\partial S^2}(S_j,\tau_n)\\
= &\sigma^2 S_j^2\left\{\frac{x+\Delta x}{(1-S_j\lambda(S_j,T-\tau_n)(x+\Delta x))^2} - \frac{x}{(1-S_j\lambda(S_j,T-\tau_n)x)^2}\right\}\\
= &\sigma^2 S_j^2\Big(G(S_j,\tau,x+\Delta x) - G(S_j,\tau,x)\Big)\\
= &\sigma^2 S_j^2\frac{\partial G(c)}{\partial x}\Delta x \qquad (3.57)
\end{aligned}$$

where $c \in (x, x+\Delta x)$. Now, (3.55) can be rewritten as

$$T_j^n(u) = kE_j^n(3) - rS_j h^2 E_j^n(1) - \frac{1}{2}\sigma^2 S_j^2\frac{\partial G(c)}{\partial x}\Delta x.$$

Denote

$$\left|W_j^n(4)\right|_{max} = \max\left\{\left|\frac{\partial^2 u}{\partial S^2}(S,\tau_n)\right|; 0 \le S \le b\right\}$$

in the further calculations. We take absolute value of $T_j^n(n)$ which gives

$$\left|T_j^n(u)\right| = |kE_j^n(3)| + |rS_j h^2 E_j^n(1)| + \left|\frac{1}{2}\sigma^2 S_j^2\frac{\partial G(c)}{\partial x}\Delta x\right|. \qquad (3.58)$$

Observe that

$$|kE_j^n(3)| \le \frac{k}{2}|W^n(3)|_{max}, \qquad (3.59)$$

$$|rS_j h^2 E_j^n(1)| \le \frac{rbh^2}{6}|W^n(1)|_{max} \qquad (3.60)$$

and for $c \in (x, x+\Delta x)$

$$\begin{aligned}
\left|\frac{1}{2}\sigma^2 S_j^2\frac{\partial G(c)}{\partial x}\Delta x\right| &\le \frac{1}{2}\sigma^2 b^2\left|\frac{1+\lambda(S_j,T-\tau_n)S_j c}{(1-\lambda(S_j,T-\tau_n)S_j c)^3}\right|\Delta x\\
&\le \frac{1}{2\delta_0^3}\sigma^2 b^2|1+\lambda(S_j,T-\tau_n)S_j c|h^2 E_j(2)\\
&\le \frac{1}{2\delta_0^3}\sigma^2 b^2\left|\frac{1+\gamma}{S_j}(1-e^{-\beta\tau_n})S_j\left(\frac{\partial^2 u}{\partial S^2}(S_j,T-\tau_n)+\theta h^2 E_j^n(2)\right)\right|\frac{h^2}{12}|W^n(2)|_{max}
\end{aligned}$$

where $0 < \theta < 1$. Let us define

$$C(n,h) = \frac{\sigma^2 b^2}{12(\sigma_1^n)^3}\left[1+\gamma(1-e^{\beta\tau_n})\left(|W^n(4)|_{max} + \frac{h^2}{12}|W^n(2)|_{max}\right)\right].$$

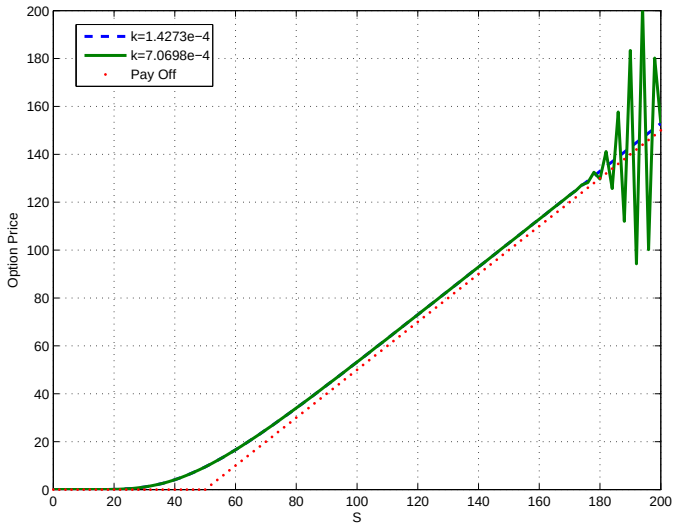

Figure 3.1: Numerical solutions for several temporal step size value.

Now we have

$$\left|T_j^n(u)\right| \leq \left(\frac{rb}{6}\left|W^n(1)\right|_{max} + C(n,h)\left|W^n(2)\right|_{max}\right)h^2 + \frac{1}{2}\left|W_j^n(3)\right|_{max} k$$

which gives (3.48). Thus the proof is completed. □.

3.3 Numerical Experiments

In this section we discuss the numerical method (3.14) with (3.17)-(3.18). The following parameters are used in this test:

$$E = 50, r = 0.06, \sigma = 0.4, T = 1, \underline{S} = 20, \overline{S} = 80, \beta = 100, \gamma = 1, h = 2.$$

For $k = 1.4273 \times 10^{-4}$ and $k = 7.0698 \times 10^{-4}$, the values of the option are shown in Fig. 3.1. The condition in Theorem 3.2.3 is satisfied for $k = 1.4273 \times 10^{-4}$ and thus the stability is obtained. In the case of $k = 7.0698 \times 10^{-4}$ however, the stability condition is not satisfied and numerical solution for the option price oscillates. For the different choices of the price impact coefficient γ, the changes in the option price are demonstrated in Fig. 3.2. It can be seen that, price of the option increases as γ increases. The case of $\gamma = 0$ corresponds to the value of the call option which is the numerical solution of the linear Black-Scholes equation.

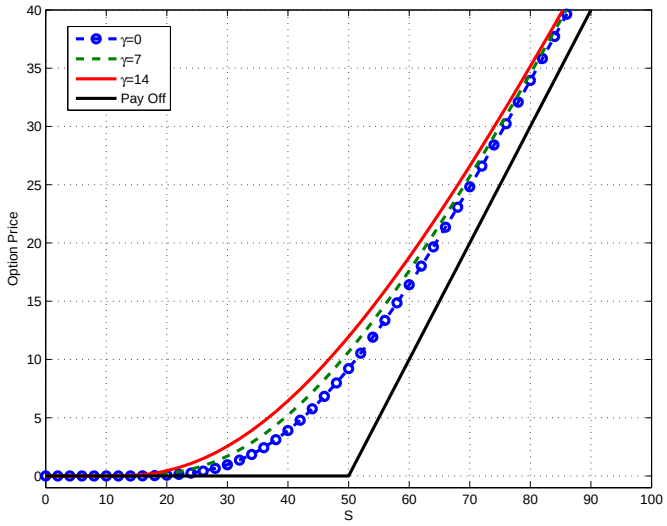

Figure 3.2: Option price for different values of γ.

The equation (3.4) contains the terms $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$ and it is reasonable to investigate the sensitivity Δ of the option price with respect to the price of the underlying asset S and sensivity Γ of the Δ with respect to S. Change in the Greek Δ for several values of γ is shown in Fig. 3.3. Change in Γ with respect to γ is shown in 3.4. We observe that the Δ grows for $S < E$ and diminishes for $S > E$ as γ increases. We also see that, maximum value of Γ is reached for smaller values of the asset price as the parameter γ becomes larger.

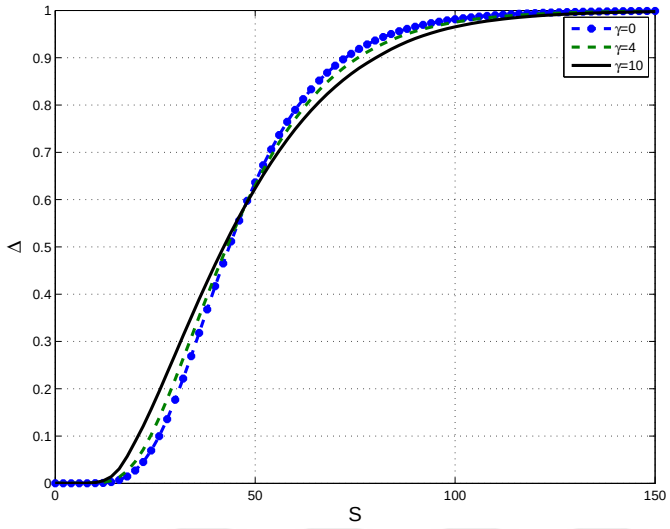

Figure 3.3: Effect of γ on $\Delta = \frac{\partial V}{\partial S}$.

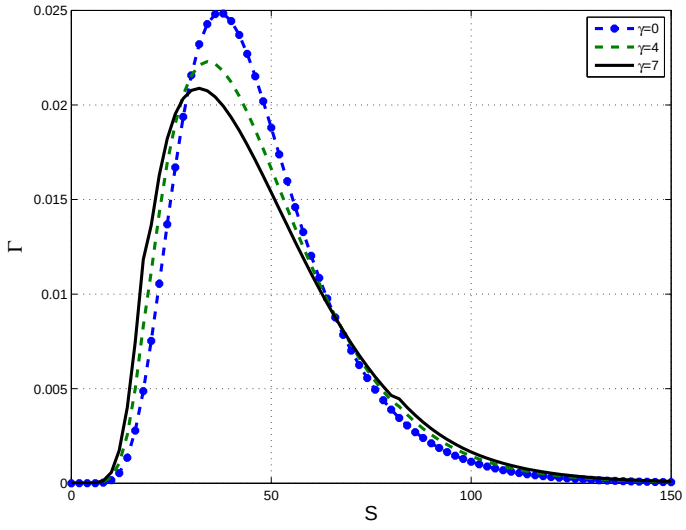

Figure 3.4: Effect of γ on $\Gamma = \frac{\partial^2 V}{\partial S^2}$.

CHAPTER 4

EXPONENTIAL FINITE DIFFERENCE METHOD

The objective of this chapter is to present, extend and compare the EFD method for the numerical solution of a PDE arising in finance.

4.1 Exponential Finite Difference Method For Heat Conduction Problem

In this section, the exponential finite difference method is briefly summarized. The content of this section is mainly based on [13].

One-dimensional heat transfer in a homogenous material can be modeled by the diffusion equation

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}, \tag{4.1}$$

where c is the thermal diffusivity constant, $u(x,t)$ is the temperature at the point x where $-L < x < L$ and at the time t where $0 < t < T$. The equation (4.1) describes all kinds of one-dimensional heat flow. Clearly, the temperature $u(x,t)$ depends on the initial temperature

$$u(x,0) = f(x), \qquad -L < x < L. \tag{4.2}$$

We shall assume boundary conditions of the form

$$u(-L,t) = T_1, \quad u(L,t) = T_2, \quad 0 < t < T. \tag{4.3}$$

The conditions (4.2) and (4.3) determines the unique solution of the diffusion equation (4.1).

We assume that the diffusion equation (4.1) has a solution for $u(x,t)$ of the form

$$u(x,t) = X(x)T(t) \tag{4.4}$$

where $X(x)$ and $T(t)$ are functions in x and t, respectively. For convenience, the initial condition

$$u(x,0) = X(x)T(0) = X(x), \quad \text{i.e.} \quad T(0) = 1 \tag{4.5}$$

is imposed. Then we can write

$$\frac{\partial u}{\partial t} = X(x)\frac{\partial T}{\partial t} \tag{4.6}$$

and

$$\frac{\partial^2 u}{\partial x^2} = T(t)\frac{\partial^2 X}{\partial x}. \tag{4.7}$$

Substitution of these derivatives into the equation (4.1) gives

$$\frac{c}{X(x)}\frac{\partial^2 X}{\partial x^2} = \frac{1}{T(t)}\frac{\partial T}{\partial t}. \tag{4.8}$$

Note that in (4.8) the x variable is separated from the t variable, that is the left hand side of (4.8) depends only on x and the right hand side of (4.8) depends only on t. Since (4.8) is true for all $0 < t < T$ and for all $x \in (-L, L)$, we can fix t while varying x. From this we see that the left side of (4.8) must be constant. Similarly, if we fix x while vary t, then the right side of (4.8) must be equal to the same constant, that is

$$\frac{c}{X(x)}\frac{\partial^2 X}{\partial x^2} = \frac{1}{T(t)}\frac{\partial T}{\partial t} = -K. \tag{4.9}$$

Here, we consider the case for cooling where K is a positive constant. This leads to two equations to solve:

$$\frac{c}{X(x)}\frac{\partial^2 X}{\partial x^2} = -K, \tag{4.10}$$

and

$$\frac{1}{T(t)}\frac{\partial T}{\partial t} = -K. \tag{4.11}$$

Multiplication of both sides of the equation (4.11) by $X(x)$ yields

$$\frac{X(x)}{T(t)}\frac{\partial T}{\partial t} = -X(x)K, \tag{4.12}$$

i.e.

$$\frac{1}{u(x,t)}\frac{\partial u}{\partial t} = -K, \tag{4.13}$$

which has a solution for non-zero value of u. The solution of (4.13) for the time step Δt can be written as

$$u(x, \Delta t) = u(x, 0) \exp(-K\Delta t). \tag{4.14}$$

Now multiply both sides of the equation (4.10) by $T(t)$ to get

$$c\frac{T(t)}{X(x)}\frac{\partial^2 X}{\partial x^2} = -T(t)K, \tag{4.15}$$

i.e.

$$c\frac{T(t)}{u(x,t)}\frac{\partial^2 X}{\partial x^2} = -K. \tag{4.16}$$

At the time $t = 0$, we obtain by using the equation (4.7)

$$\frac{c}{u(x,0)}\frac{\partial^2 u}{\partial x^2} = -K. \tag{4.17}$$

Substitution of the value K from the equation (4.17) into the equation (4.14) yields

$$u(x, \Delta t) = u(x, 0) \exp\left(\frac{c\Delta t}{u(x,0)}\frac{\partial^2 u}{\partial x^2}\right). \tag{4.18}$$

In the equation (4.18), a finite difference approximation can be replaced for the derivative $\frac{\partial^2 u}{\partial x^2}$. For instance, consider the second-order symmetric discretization at the node $(x_j, 0)$

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u_{j+1}(0) - 2u_j(0) + u_{j-1}(0)}{(\Delta x)^2}.$$

We denote $u(x_j, 0)$ at node (j, n) and $u(x_j, \Delta t)$ at node $(j, n + 1)$ by U_j^n and U_j^{n+1}, respectively. Then from the equation (4.18), we have

$$U_j^{n+1} = U_j^n \exp\left(\frac{c\Delta t}{U_j^n}\ \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}\right) \tag{4.19}$$

which is the exponential finite difference method for the heat equation (4.1). It may be noted that the use of (4.19) can be improved by subdividing the main time step Δt into equal subintervals. Suppose that the time interval $\Delta t = t_{n+1} - t_n$ is divided into two equal subintervals $[t_n, t_{n+1/2}] = [t_n, t_n + \frac{\Delta t}{2}]$ and $[t_n + \frac{\Delta t}{2}, t_{n+1}]$. Thus, for the first subinterval $[t_n, t_{n+1/2}]$, we have

$$U_j^{n+1/2} = U_j^n \exp\left(\frac{r}{2}Z_j^n\right) \tag{4.20}$$

where $r = c\Delta t/(\Delta x)^2$, and

$$Z_j^n = \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{U_j^n}. \tag{4.21}$$

For the second subinterval $[t_{n+1/2}, t_{n+1}]$, we use $U_j^{n+1/2}$ as initial value and obtain

$$U_j^{n+1} = U_j^{n+1/2} \exp\left(\frac{r}{2} Z_j^{n+1/2}\right) \tag{4.22}$$

where

$$Z_j^{n+1/2} = \frac{U_{j+1}^{n+1/2} - 2U_j^{n+1/2} + U_{j-1}^{n+1/2}}{U_j^{n+1/2}}. \tag{4.23}$$

Note that the temperature values $U_{j+1}^{n+1/2}$ and $U_{j-1}^{n+1/2}$ should be generated for the above equation (4.23). The values $U_{j+1}^{n+1/2}$ and $U_{j-1}^{n+1/2}$ can be obtained directly from (4.20) as

$$\begin{aligned} U_{j-1}^{n+1/2} &= U_{j-1}^{n} \exp\left(\tfrac{r}{2} Z_{j-1}^{n}\right) \\ U_{j+1}^{n+1/2} &= U_{j+1}^{n} \exp\left(\tfrac{r}{2} Z_{j+1}^{n}\right). \end{aligned} \tag{4.24}$$

Substitution of (4.24) and (4.20) into (4.23) yields

$$Z_j^{n+1/2} = \frac{U_{j-1}^{n} \exp\left(\frac{r}{2} Z_{j-1}^{n}\right) - 2U_j^{n} \exp\left(\frac{r}{2} Z_j^{n}\right) + U_{j+1}^{n} \exp\left(\frac{r}{2} Z_{j+1}^{n}\right)}{U_j^{n} \exp\left(\frac{r}{2} Z_j^{n}\right)} \tag{4.25}$$

Thus, we can arrive the temperature U_j^{n+1} from U_j^n in two equal subintervals of length $\Delta t/2$, by using (4.20) and (4.22) as

$$\begin{aligned} U_j^{n+1} &= U_j^{n+1/2} \exp\left(\tfrac{r}{2} Z_j^{n+1/2}\right) \\ &= U_j^{n} \exp\left(\tfrac{r}{2} Z_j^{n}\right) \exp\left(\tfrac{r}{2} Z_j^{n+1/2}\right) \\ &= U_j^{n} \exp\left(\tfrac{r}{2} \left\{Z_j^{n} + Z_j^{n+1/2}\right\}\right) \end{aligned} \tag{4.26}$$

where Z_j^n and $Z_j^{n+1/2}$ are defined in (4.21) and (4.25), respectively. The process can be extended if the time step is sub-divided into $k+1$ intervals. It is shown that the exponential finite difference method with 5 sub-intervals performs better than the Crank-Nicolson method for the heat equation [13].

A practical alternative derivation of the exponential finite difference method (4.19) is available in the literature (see [30], and references therein). Assume that $F(u)$ is a continuous differentiable function. Multiply both sides of the heat equation (4.1) by $\dfrac{\partial F(u)}{\partial u}$ to get

$$\frac{\partial F(u)}{\partial u} \frac{\partial u}{\partial t} = c \, \frac{\partial F(u)}{\partial u} \frac{\partial^2 u}{\partial x^2}. \tag{4.27}$$

The chain rule yields

$$\frac{\partial F(u)}{\partial t} = c \, \frac{\partial F(u)}{\partial u} \frac{\partial^2 u}{\partial x^2}. \tag{4.28}$$

We discretize the derivative $\frac{\partial F(u)}{\partial t}$ by using the forward difference, and obtain

$$\frac{F(U^{n+1}) - F(U^n)}{\Delta t} = c \left(\frac{\partial F(u)}{\partial u}\right)^n \frac{\partial^2 U^n}{\partial x^2}. \tag{4.29}$$

If we choose $F(u) = \ln(u)$, we get

$$\frac{\ln(U^{n+1}) - \ln(U^n)}{\Delta t} = c \left(\frac{1}{U^n}\right) \frac{\partial^2 U^n}{\partial x^2} \tag{4.30}$$

or

$$\ln\left(\frac{U^{n+1}}{U^n}\right) = \Delta t \, c \frac{1}{U^n} \frac{\partial^2 U^n}{\partial x^2}. \tag{4.31}$$

We solve for U^{n+1} and get

$$U^{n+1} = U^n \ \exp\left(\frac{\Delta t \, c}{U^n} \frac{\partial^2 U^n}{\partial x^2}\right). \tag{4.32}$$

When the derivative $\frac{\partial^2 U^n}{\partial x^2}$ is replaced by the second-order symmetric finite difference approximation, we obtain the exponential finite difference method (4.19).

We note that the choice of F is crucial. If we choose $F(u) = \exp(u)$, we obtain logarithmic finite difference method [28]

$$U_j^{n+1} = U_j^n + \ln\left(1 + c\Delta t \ \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}\right). \tag{4.33}$$

4.2 Stability of the Exponential Finite Difference Method

In this section, the stability of the exponential finite difference scheme (4.19) will be discussed. There are various methods that have been used for the stability of a finite difference scheme, such as matrix stability method, von Neumann stability method [60]. The conventional stability analysis discusses the growth or decay of the "amplification factor" ξ which for the exponential finite difference scheme (4.19) is the ratio

$$\xi = \frac{U^{n+1}}{U^n} = \exp\left(\frac{c\Delta t}{(\Delta x)^2} \ \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{U_j^n}\right). \tag{4.34}$$

For stability $\lim_{\Delta t \to 0} |\xi|$ must be less than or equal to unity. For the equation (4.34), this condition is satisfied for

$$\frac{c\Delta t}{(\Delta x)^2} \ \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{U_j^n} \leq 0$$

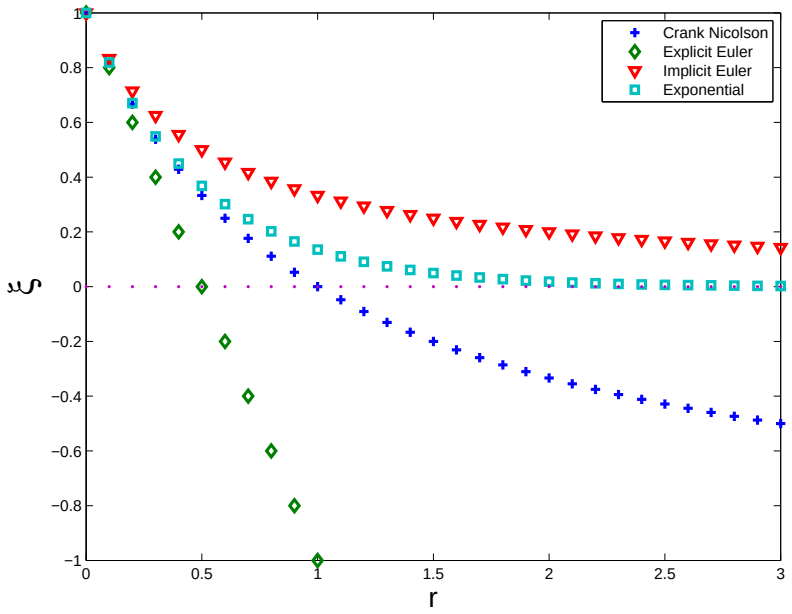

Figure 4.1: Stability of finite difference schemes for a one-node model $U_j^{n+1} = \xi U_j^n$, $r = c\Delta t/(\Delta x)^2$.

or

$$\frac{U_{j+1}^n + U_{j-1}^n}{U_j^n} \leq 2U_j^n. \tag{4.35}$$

However the condition (4.35) does not give any relation between the spatial step size Δx and the temporal step size Δt. Experience reveals that the exponential finite difference scheme (4.19) remains stable for values of $\Delta t \leq 0.52\Delta x^2$ [13]. Instead, one-node difference equation is studied by converting all difference equations into equations of a common form [52]. Following this fact, the equation (4.35) can be written as

$$\xi = \exp\left(r\frac{-2U_j^n}{U_j^n}\right) = \exp(-2r), \qquad r = \frac{c\Delta t}{(\Delta x)^2} \tag{4.36}$$

by putting $U_{j-1}^{n+1} = U_{j+1}^{n+1} = U_{j-1}^n = U_{j+1}^n = 0$. These substitutions for some other finite difference methods are summarized below for heat equation (4.1):

Explicit Euler	$\xi = 1 - 2r,$
Implicit Euler	$\xi = 1/(1 + 2r),$
Crank Nicolson	$\xi = (1 - r)(1 + r),$
Exponential Fin.Diff.	$\xi = \exp(-2r)$

Figure 4.1 illustrates the behavior of various finite difference equations in terms of the amplification factor and the time step functional variable r. From the figure it is clear

that all schemes except explicit Euler remain stable as r increases. On the other hand, explicit Euler shows conditional stability.

4.3 Order, Consistency, Linear Stability Domain and Convergence of the Exponential Finite Difference Method

This section is devoted to the order, consistency and convergence of the exponential finite difference method. First we will show how to apply the exponential finite difference method to the ODE

$$\frac{dy}{dt} = f(t,y). \tag{4.37}$$

Multiplication of both sides by $\frac{dF(y)}{dy}$ yields

$$\begin{aligned} \frac{dF(y)}{dy}\frac{dy}{dt} &= \frac{dF(y)}{dy} f(t,y) \\ \frac{d}{dt}F(y) &= \frac{dF(y)}{dy} f(t,y). \end{aligned} \tag{4.38}$$

Use forward difference approximation for the first-order derivative to get

$$\frac{F(y_{n+1}) - F(y_n)}{\Delta t} = \frac{dF(y_n)}{dy} f(t_n, y_n). \tag{4.39}$$

Now we choose $F(y) = \ln y$, and we obtain:

$$\begin{aligned} \ln y_{n+1} - \ln y_n &= \frac{\Delta t}{y_n} f(t_n, y_n) \\ \ln \left| \frac{y_{n+1}}{y_n} \right| &= \frac{\Delta t}{y_n} f(t_n, y_n). \end{aligned} \tag{4.40}$$

Thus we obtain the exponential finite difference method

$$y_{n+1} = y_n \exp\left(\frac{\Delta t}{y_n} f(t_n, y_n) \right) \tag{4.41}$$

for the ODE (4.37).

Recall that the local truncation error (or local discretization error) T_j^n is the amount by which the solution of the differential equation (4.37) fails to satisfy the finite difference equation (4.41). Therefore plugging the exact solution of (4.37) into the finite

difference solution (4.41) gives the local truncation error

$$\begin{aligned} T_j^n &= y(t_{n+1}) - y(t_n)\exp\left(\frac{\Delta t}{y(t_n)}f(t_n, y(t_n))\right) \\ &= \left[y(t_n) + \Delta t y'(t_n) + \frac{\Delta t^2}{2!}y''(t_n) + O(\Delta t^3)\right] \\ &\quad -y(t_n)\left[1 + \frac{\Delta t}{y(t_n)}f(t_n, y(t_n)) + \frac{1}{2!}\left(\frac{\Delta t}{y(t_n)}f(t_n, y(t_n))\right)^2 + O(\Delta t^3)\right] \\ &= \frac{\Delta t^2}{2!}\left(y''(t_n) + \frac{f^2(t_n, y(t_n))}{y^2(t_n)}\right) + O(\Delta t^3) \end{aligned} \tag{4.42}$$

which indicates the first-order convergence of the exponential finite difference method (4.41). The method (4.41) is consistent with the differential equation (4.37), since the principal part of the truncation error

$$\frac{\Delta t^2}{2!}\left(y''(t_n) + \frac{f^2(t_n, y(t_n))}{y^2(t_n)}\right)$$

vanishes as $\Delta t \to 0$.

Let us consider the scalar linear equation

$$\frac{dy}{dt} = \lambda y, \quad t \geq 0,\ y(0) = 1, \tag{4.43}$$

where $\lambda \in \mathbb{C}$. We apply the exponential finite difference method (4.41) to the linear equation (4.43) and obtain the sequence of solution

$$y_n = \exp(n\lambda\Delta t), \quad n = 0, 1, 2, \cdots . \tag{4.44}$$

Hence

$$\lim_{n\to\infty} y_n = 0 \tag{4.45}$$

if and only if $\mathrm{Re}(\lambda) < 0$. Thus we conclude that the exponential finite difference method (4.41) is an A-stable method. In other words, the exponential finite difference method (4.41) mimics the asymptotic stability of linear ODE without any need to decrease the step size. According to the Lax theorem, the method (4.41) is convergent since it is consistent and stable.

In order to show the accuracy and the expected rate of convergence of the exponential finite difference method (4.41), we perform some numerical tests. Consider the initial value problem

$$\frac{dy}{dt} = \frac{1}{\sqrt{x+2}}, \qquad y(2) = 4. \tag{4.46}$$

Table 4.1: The L_∞ and L_2 errors and convergence rate at time $t = 3$.

Δt	L_∞	order	L_2	order
0.1	0.00534699	—	0.00343760	—
0.05	0.00265630	1.009	0.00165271	1.056
0.025	0.00132389	1.004	0.00080993	1.028
0.0125	0.00066088	1.002	0.00040087	1.014

The equation (4.46) has the exact solution $y(t) = \sqrt{x+2}$. Let y_n be denote the numerical solution at t_n of the equation (4.46) by using the exponential finite difference method (4.41). Assume $y(t_n)$ is the exact solution at the mesh point t_n.

The rate of convergence for the scheme (4.41) is calculated using the formula

$$\text{order} = \frac{\log\left[E(\Delta t)/E(\Delta t/2)\right]}{\log(2)} \tag{4.47}$$

where $E(\Delta t)$ is L_∞ and L_2 errors given by

$$\begin{aligned} L_\infty &= \max_{1\le n\le N} |y(t_n) - y_n|, \\ L_2 &= \sqrt{\Delta t \sum_{n=1}^{N} (y(t_n) - y_n)^2}. \end{aligned} \tag{4.48}$$

Table 4.1 shows the errors and the rate of convergence of the scheme (4.41). From the table we can see that first-order accuracy of the scheme is achieved. This verifies the theoretical results (4.42) that the scheme (4.41) has first order accuracy.

4.4 Exponential Finite Difference Method for Linear Black-Scholes Equation

In this section we will address the linear European option pricing problem (2.30), i.e.

$$V_t + \frac{\sigma^2 S^2}{2} V_{SS} + rS V_S - rV = 0. \tag{4.49}$$

Although the exponential finite difference method will be used in the next section for the nonlinear Black-Scholes model, we check the method here for the linear case because the exact solution of the linear model is known. First we should point out one problem that may arise in the denominator of the exponent in the exponential finite difference method. When the final condition becomes zero ($V(S,T) = 0$) the denominator of the exponent in the exponential finite difference method ensures the problem.

This problem is not encountered in physical models such as KdV equation and Burger equation, but encountered in financial model because the value of European call option $C(S,t)$ and the value of European put option $P(S,t)$ at some $t = T$ value are zero (see Fig. 1.1). To overcome this difficulty, we define transformation

$$V(S,t) = \alpha - \overline{V}(S,t), \qquad \alpha < 0, \tag{4.50}$$

substitute into (4.49) and obtain

$$-\overline{V}(S,t) - \frac{\sigma^2 S^2}{2}\overline{V}_{SS} - rS\overline{V}_S - r(\alpha - \overline{V}) = 0. \tag{4.51}$$

For the case of European call option for instance, the final condition and the boundary conditions are transformed into

$$\overline{C}(S,T) = \alpha - \max\{S - E, 0\}, \tag{4.52}$$

$$\begin{aligned} \overline{C}(S_{max},t) &= \alpha - S_{max} + Ee^{-(T-t)}, \\ \overline{C}(S_{min},t) &= \alpha. \end{aligned} \tag{4.53}$$

Now following Sec.4.1, we multiply both sides of (4.51) by $-\frac{\partial F(\overline{V})}{\partial \overline{V}}$ and get

$$\frac{\partial F(\overline{V})}{\partial \overline{V}}\frac{\partial \overline{V}}{\partial t} = -\frac{\partial F(\overline{V})}{\partial \overline{V}}\left\{\frac{\sigma^2 S^2}{2}\overline{V}_{SS} + rS\overline{V}_S + r(\alpha - \overline{V})\right\} \tag{4.54}$$

or

$$\frac{\partial F(\overline{V})}{\partial t} = -\frac{\partial F(\overline{V})}{\partial \overline{V}}\left\{\frac{\sigma^2 S^2}{2}\overline{V}_{SS} + rS\overline{V}_S + r(\alpha - \overline{V})\right\} \tag{4.55}$$

Now let

$$F(\overline{V}) = \ln\overline{V},$$

then

$$\frac{\partial F(\overline{V})}{\partial \overline{V}} = \frac{1}{\overline{V}}.$$

Using the backward Euler method for time derivative and the second order central difference for space, we get the finite difference representation of (4.49) as

$$\frac{F(\overline{V}_j^n) - F(\overline{V}_j^{n-1})}{\Delta t} = -\frac{1}{\overline{V}_j^n}\left[\frac{\sigma^2 S^2}{2}\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{\Delta S^2} + rS\frac{\overline{V}_{j+1}^n - \overline{V}_{j-1}^n}{2\Delta S} + r\alpha - r\overline{V}_j^n\right] \tag{4.56}$$

or using $F(\overline{V}) = \ln\overline{V}$

$$\ln(\overline{V}_J^n) - \ln(\overline{V}_j^{n-1}) = -\frac{\Delta t}{\overline{V}_j^n}\left[\frac{\sigma^2 S^2}{2}\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{\Delta S^2} + rS\frac{\overline{V}_{j+1}^n - \overline{V}_{j-1}^n}{2\Delta S} + r\alpha - r\overline{V}_j^n\right]. \tag{4.57}$$

The property of the logarithm $\ln(x/y) = \ln x - \ln y$ leads to

$$\ln\left|\frac{\overline{V}_j^n}{\overline{V}_j^{n-1}}\right| = -\frac{\Delta t}{\overline{V}_j^n}\left[\frac{\sigma^2 S^2}{2}\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{\Delta S^2} + rS\frac{\overline{V}_{j+1}^n - \overline{V}_{j-1}^n}{2\Delta S} + r\alpha - r\overline{V}_j^n\right],$$

or equivalently

$$\frac{\overline{V}_j^n}{\overline{V}_j^{n-1}} = \exp\left\{-\frac{\Delta t}{\overline{V}_j^n}\left[\frac{\sigma^2 S^2}{2}\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{\Delta S^2} + rS\frac{\overline{V}_{j+1}^n - \overline{V}_{j-1}^n}{2\Delta S} + r\alpha - r\overline{V}_j^n\right]\right\}.$$

Thus, we get the explicit exponential finite difference scheme

$$\overline{V}_j^{n-1} = \overline{V}_j^n\left\{\exp\left\{\frac{\Delta t}{\overline{V}_j^n}\left[\frac{\sigma^2 S^2}{2}\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{\Delta S^2} + rS\frac{\overline{V}_{j+1}^n - \overline{V}_{j-1}^n}{2\Delta S} + r\alpha - r\overline{V}_j^n\right]\right\}\right\} \tag{4.58}$$

for the European option pricing problem (4.49). The final condition (4.52) and the boundary condition (4.53) will be used for the difference scheme (4.58).

4.4.1 Numerical Results

In this section we will check the properties of the numerical scheme (4.58) for the linear Black-Scholes equation (4.51). Notice that under transformation $V = \alpha - \overline{V}$ in (4.50) the auxiliary variable α appears in the final condition (4.52) and boundary conditions (4.53) of the transformed equation (4.51). Therefore one may expect that the choice of α may change the solution. We run the scheme (4.58) for different α values with $\Delta S = 2$, $\Delta t = 1/1700$ and see that the different choice of α values does not yield significant change in the solution. For this we consider the European Call option so that $f(S) = \max\{S - E, 0\}$ with the following parameters:

$$E = 50,\ \sigma = 0.4,\ r = 0.1,\ T = 5/12,\ S_{min} = 0,\ S_{max} = 150.$$

For these parameters the exact value $C(S,0)$ in (2.28) of the linear equation (4.49) is $C(S,0) = 6.11508$. The explicit finite difference solution (2.35) yields the solution $C(S,0) = 6.100815$ with absolute error 0.015693. Table 4.2 yields exponential finite difference solution (4.58) for the same set of parameters. From the table we see that the best choice of parameter α is $\alpha = -0.029$. For this reason, otherwise stated, this value of α will be used for the further experiments. In addition we see that exponential

Table 4.2: European Call option value $C(S,0)$ for various α values.

α	C_0	$\lvert Error\rvert$
-1	6.104209	0.0123
-0.1	6.109050	0.0075
-0.02	6.126925	0.0011
-0.029	6.116384	0.0001
-0.0299	6.115930	0.0006
-0.03	6.115883	0.0006

Table 4.3: Comparison of the exponential finite difference scheme (4.58) (EFD) with the explicit finite difference (Explicit FD) method (2.35) at different current stock prices.

S_0	Exact	Explicit FD	EFD
40	1.600448	1.590457	1.602728
50	6.116508	6.100815	6.116384
70	22.512829	22.508751	22.514371

finite difference scheme gives smaller error than the explicit finite difference scheme (2.35) for all choice of α in Table 4.2.

Table 4.3 represents a comparison between the exponential finite difference scheme (4.58) with the explicit finite difference method (2.35) at different current stock prices. From the table we see that approximation to the current value $C(S,0)$ of the exponential finite difference scheme is always better than the explicit scheme (2.35).

Fig 4.2 represents the absolute error between the exact solution (2.28) and the exponential finite difference method (4.58). The following parameters are used in Fig 4.2

$$E = 50; \sigma = 0.4; r = 0.1; T = 5/12; S_{min} = 0; S_{max} = 150; \Delta S = 2.$$

We see that error in the current value European call option decreases as the temporal mesh size decreases.

To indicate the convergence rate in space of the proposed numerical scheme (4.58), we perform some numerical experiments for various values of ΔS and fixed value of Δt. In these experiments we choose $\Delta t = 1.0 \times 10^{-4}$ to ensure that the temporal error is negligible. Furthermore, we make the calculations under the assumption that the method is a convergent method in space. The rate of convergence for the scheme

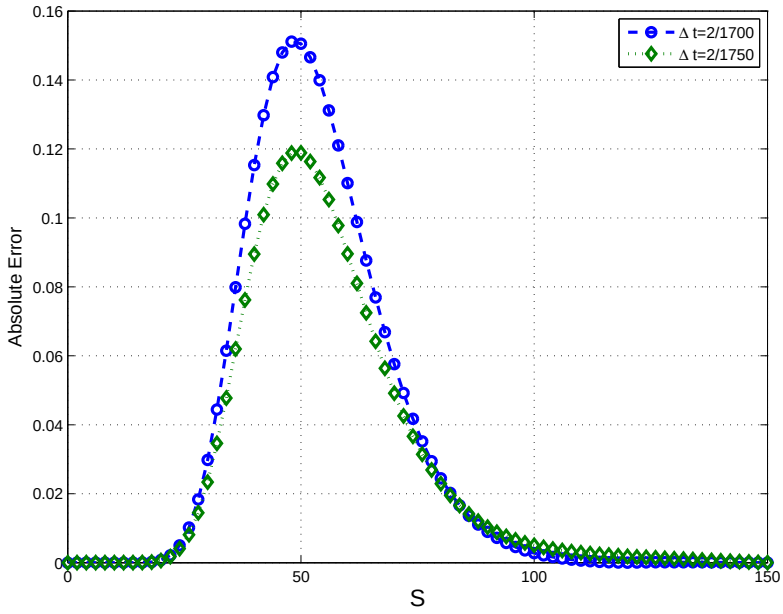

Figure 4.2: Numerical solutions for several temporal step size.

Table 4.4: Rate of convergence of the exponential finite difference method (4.58) for $\Delta t = 1.0 \times 10^{-4}$ at $T = 5/12$.

ΔS	L_∞	order
10	0.436 028	—
5	0.093 979	2.214
2.5	0.021 377	2.136
1.25	0.003 599	2.570
0.625	0.000 996	1.853

(4.58) is calculated using the formula

$$\text{order} = \frac{\log\left[L_\infty(\Delta S)/L_\infty(\Delta S/2)\right]}{\log(2)} \tag{4.59}$$

where L_∞ error is given by

$$L_\infty = \max_{1\le j\le J} |V(S_j, t_n) - V_j^n|. \tag{4.60}$$

The errors and the rate of convergence is tabulated in the Table 4.4. This verifies the second order convergence in space of the scheme (4.58), since we have used second order central difference approximations for $\frac{\partial V}{\partial S}$ and $\frac{\partial^2 V}{\partial S^2}$.

Figure 4.3 represents a double logarithmic plot of the errors. From the figure, we see that the exponential finite difference method (4.58) is second order convergence in S.

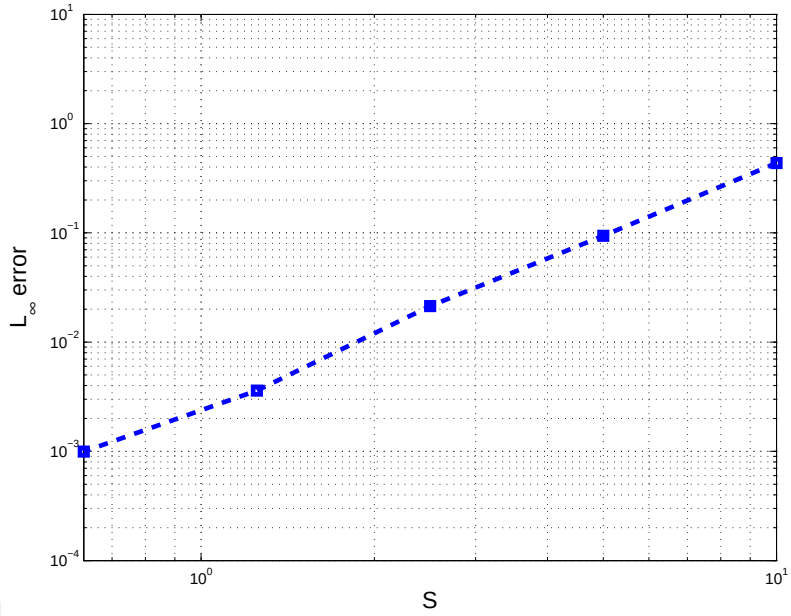

Figure 4.3: Double logarithmic plot of error versus space step S, indicating second order convergence in S of the exponential finite difference scheme (4.58).

4.5 Exponential Finite Difference Method for Nonlinear BSE

In this section we will consider the nonlinear Black-Scholes equation (3.4)-(3.6), i.e. the illiquid market problem

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2\left(1 - \lambda(S,t)S\frac{\partial^2 V}{\partial S^2}\right)^2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0, \quad (S,t) \in (0,\infty)\times(0,T) \tag{4.61}$$

$$V(S,T) = f(S) \;\; 0 \le S < \infty \tag{4.62}$$

with the price impact function $\lambda(S,t) > 0$ is given by

$$\lambda(S,t) = \begin{cases} \frac{\gamma}{S}(1 - e^{-\beta(T-t)}), & \underline{S} \le S \le \overline{S} \\ 0, & \text{otherwise} \end{cases} \tag{4.63}$$

where V is the price of the option depending on the underlying asset S and the time t, $\sigma > 0$ is the constant volatility, r is the interest rate, T is the expiry date for the option, $f(S)$ is the payoff function (see Sec.3.2.2). We consider the numerical solution of the problem (4.61)-(4.62) by using the exponential finite difference method. We will use the same mesh points with the Sec.3.2.3.

Since the domain of the problem (4.61) is unbounded in the underlying asset variable S, we consider the finite interval for the numerical solution and take $S \in [0, b]$ for the asset variable S for which b is chosen so that $0 < \underline{S} < \overline{S} < b$. Thus, the domain for the approximation becomes

$$(S, t) \in [0, b] \times [0, T].$$

We consider the equally spaced mesh points

$$S_j = jh,\ h = \Delta S,\ t_n = nk,\ k = \Delta t$$

with $0 \leq j \leq N, 0 \leq n \leq l, Nh = b$ and $lk = T$. The numerical approximation of the solution $V(S_j, t_n)$ is denoted by V_j^n. We use the following approximations of the partial derivatives

$$\frac{\partial V}{\partial t}(S_j, t_n) = \frac{V_j^{n+1} - V_j^n}{k} + O(k) \tag{4.64}$$

$$\frac{\partial^2 V}{\partial S^2}(S_j, t_n) = \frac{V_{j-1}^n - 2V_j^n + V_{j+1}^n}{h^2} + O(h^2) = \Delta_j^n(V) + O(h^2) \tag{4.65}$$

$$\frac{\partial V}{\partial S}(S_j, t_n) = \frac{V_{j+1}^n - V_{j-1}^n}{2h} + O(h^2) = \nabla_j^n(V) + O(h^2). \tag{4.66}$$

At the external nodes $S_{-1} = -h$ and $S_{N+1} = (N+1)h$, the approximations V_{-1}^n and V_{N+1}^n are obtained by

$$V_{-1}^n = 2V_0^n - U_1^n,\ \ V_{N+1}^n = 2V_N^n - V_{N-1}^n \tag{4.67}$$

for $0 \leq n \leq l$ by using linear interpolation. Moreover, using the approximation (4.65) we get

$$\Delta_0^n = \Delta_N^n = 0. \tag{4.68}$$

We know from the finite difference equation (4.58) that, exponential finite difference scheme suffers from the zero starting value. In order to avoid this difficulty, as we have done in Section 4.4, we introduce a parameter α and substitute into (4.61)

$$V(S, t) = \alpha - \overline{V}(S, t) \tag{4.69}$$

and obtain

$$\overline{V}_t + \frac{\sigma^2 S^2}{2\left[1 + \lambda(S, t) S \overline{V}_{SS}\right]^2} \overline{V}_{SS} + rS\overline{V}_S - r\overline{V} + r\alpha = 0. \tag{4.70}$$

Then we multiply (4.70) by $\frac{\partial F(\overline{V})}{\partial \overline{V}}$ and get

$$\frac{\partial F(\overline{V})}{\partial \overline{V}}\frac{\partial \overline{V}}{\partial t} = -\frac{\partial F(\overline{V})}{\partial \overline{V}}\left\{\frac{\sigma^2 S^2}{2\left[1+\lambda(S,t)S\overline{V}_{SS}\right]^2}\overline{V}_{SS} + rS\overline{V}_S - r\overline{V} + r\alpha\right\}. \qquad (4.71)$$

To obtain the exponential finite difference method, we choose

$$F(\overline{V}) = \ln\left|\overline{V}\right|$$

for which

$$\frac{\partial F(\overline{V})}{\partial \overline{V}} = \frac{1}{\overline{V}}.$$

Substituting them into (4.71), and using the backward Euler method for time variable t and central difference for variable S, we get the finite difference method

$$\frac{F(\overline{V}_j^n) - F(\overline{V}_j^{n-1})}{k} = -\frac{1}{\overline{V}_j^n}\left[\frac{\sigma^2 S_j^2}{2\left[1+\lambda(S_j,t_n)S_j\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{h^2}\right]^2}\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{h^2} + rS_j\frac{\overline{V}_{j+1}^n - \overline{V}_{j-1}^n}{2h} + r\alpha - r\overline{V}_j^n\right]. \qquad (4.72)$$

Using $F(\overline{V}) = \ln\left|(\overline{V})\right|$ we have

$$\ln(\overline{V}_j^n) - \ln(\overline{V}_j^{n-1}) = -\frac{k}{\overline{V}_j^n}\left[\frac{\sigma^2 S_j^2}{2\left[1+\lambda(S_j,t_n)S_j\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{h^2}\right]^2}\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{h^2} + rS_j\frac{\overline{V}_{j+1}^n - \overline{V}_{j-1}^n}{2h} + r\alpha - r\overline{V}_j^n\right]. \qquad (4.73)$$

The property of logarithm $\ln(x/y) = \ln x - \ln y$ leads to

$$\ln\left|\frac{\overline{V}_j^n}{\overline{V}_j^{n-1}}\right| = -\frac{k}{\overline{V}_j^n}\left[\frac{\sigma^2 S_j^2}{2\left[1+\lambda(S_j,t_n)S_j\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{h^2}\right]^2}\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{h^2} + rS_j\frac{\overline{V}_{j+1}^n - \overline{V}_{j-1}^n}{2h} + r\alpha - r\overline{V}_j^n\right]$$

or equivalently

$$\frac{\overline{V}_j^n}{\overline{V}_j^{n-1}} = \exp\left\{-\frac{k}{\overline{V}_j^n}\left[\frac{\sigma^2 S_j^2}{2\left[1+\lambda(S_j,t_n)S_j\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{h^2}\right]^2}\frac{\overline{V}_{j+1}^n - 2\overline{V}_j^n + \overline{V}_{j-1}^n}{h^2} + rS_j\frac{\overline{V}_{j+1}^n - \overline{V}_{j-1}^n}{2h} + r\alpha - r\overline{V}_j^n\right]\right\}.$$

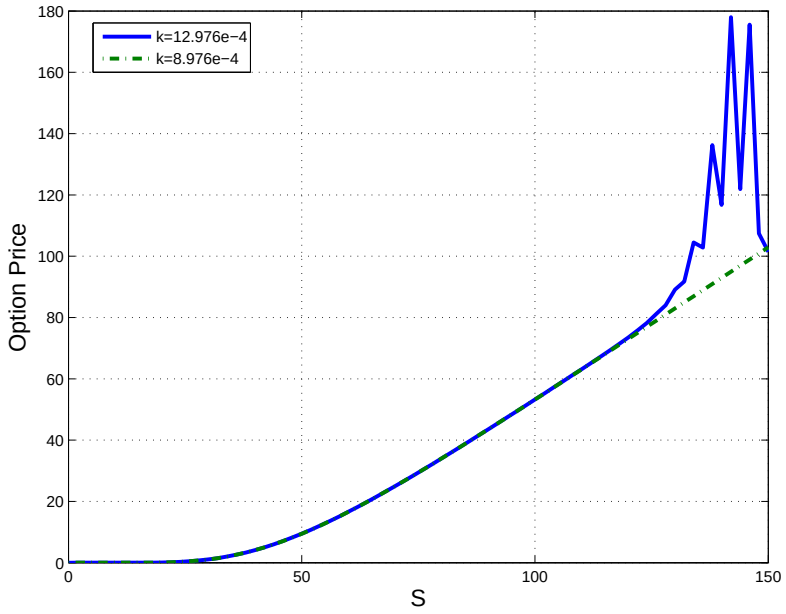

Figure 4.4: Numerical solutions for several temporal step size.

Thus we obtain the exponential finite difference scheme

$$\overline{V}_j^{n-1} = \overline{V}_j^n \exp\left\{\frac{k}{\overline{V}_j^n}\left[\frac{\sigma^2 S_j^2}{2\left[1+\lambda(S_j,t_n)S_j\frac{\overline{V}_{j+1}^n-2\overline{V}_j^n+\overline{V}_{j-1}^n}{h^2}\right]^2}\frac{\overline{V}_{j+1}^n-2\overline{V}_j^n+\overline{V}_{j-1}^n}{h^2}\right.\right. \\ \left.\left. +rS_j\frac{\overline{V}_{j+1}^n-\overline{V}_{j-1}^n}{2h}+r\alpha-r\overline{V}_j^n\right]\right\} \tag{4.74}$$

for the generalized Black-Scholes pricing PDE (4.61) that for the case where interest rate ($r \geq 0$) and the reference volatility ($\sigma > 0$) are constant.

4.5.1 Numerical Results

We consider the European call option, (so that $f(S) = \max\{S - E, 0\}$) for an illiquid market with strike price $E = 50$. The asset variable S is chosen so that $\underline{S} = 20 \leq S \leq 80 = \overline{S}$. The following parameters are used in this test:

$$r = 0.06,\ \sigma = 0.4,\ T = 1,\ \alpha = -0.1,\ \beta = 100,\ \gamma = 1,\ h = \Delta S = 2,$$

Fig. 4.4 shows the stable and unstable cases for the option pricing value. The temporal step size $k = \Delta t = 8.976 \times 10^{-4}$ is a sufficient value for stability. However, for

$k = \Delta t = 12.976 \times 10^{-4}$ the stability condition is broken appearing spurious oscillations in the numerical solution. We note that the explicit method (3.14) was unstable for $k \geq 7.0698 \times 10^{-4}$. But exponential finite difference scheme (4.74) gives stable solution for the value of $k = 7.0698 \times 10^{-4}$. In the following tests, we consider the same stability step size requirement but with several price impact coefficient γ. Figure 4.5

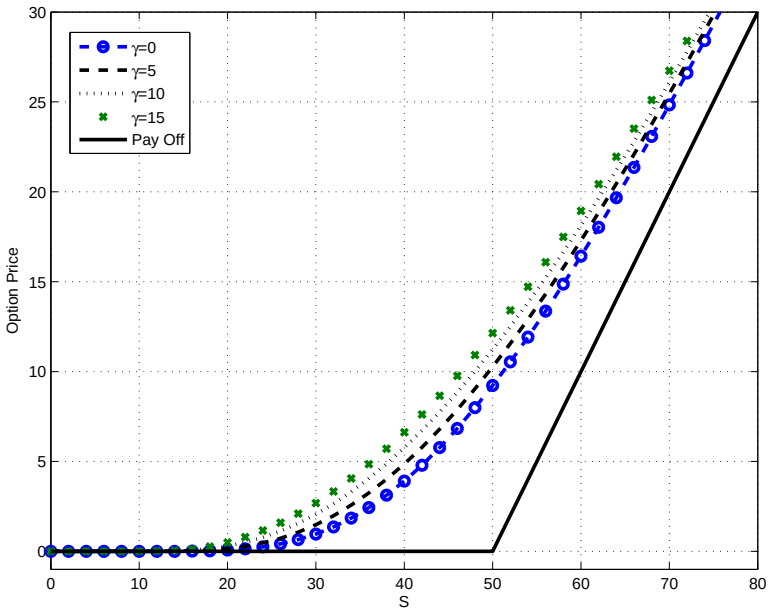

Figure 4.5: Variation of the option price with the parameter γ.

represents the effect of the price impact coefficient γ for option price $C(S,t)$ at time $t = 0$ with the temporal step size $\Delta t = 1.976 \times 10^{-4}$. From the figure we see that the grows in the value of γ yield the grows in option price.

Figure 4.6 shows that when the value of the price impact coefficient γ is increasing, the hedge ratio Δ is increasing for $S < E$. On the other hand when $S > E$ it is decreasing in γ. This indicates that lower liquidity spreads out the head ratio. We notice in Fig. 4.7 that Γ takes its maximum value for smaller values of S as illiquidity increases. Moreover, as γ grows maximum value of Gamma diminishes.

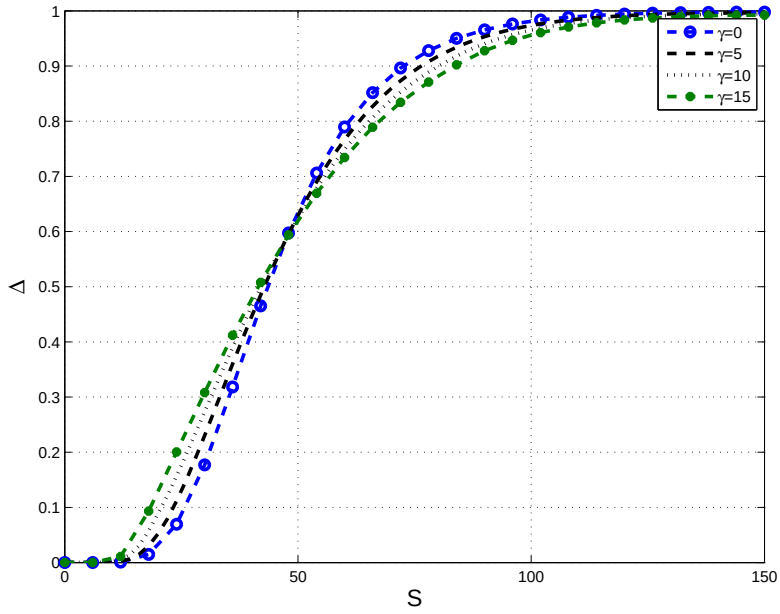

Figure 4.6: Variation of Δ with the parameter γ.

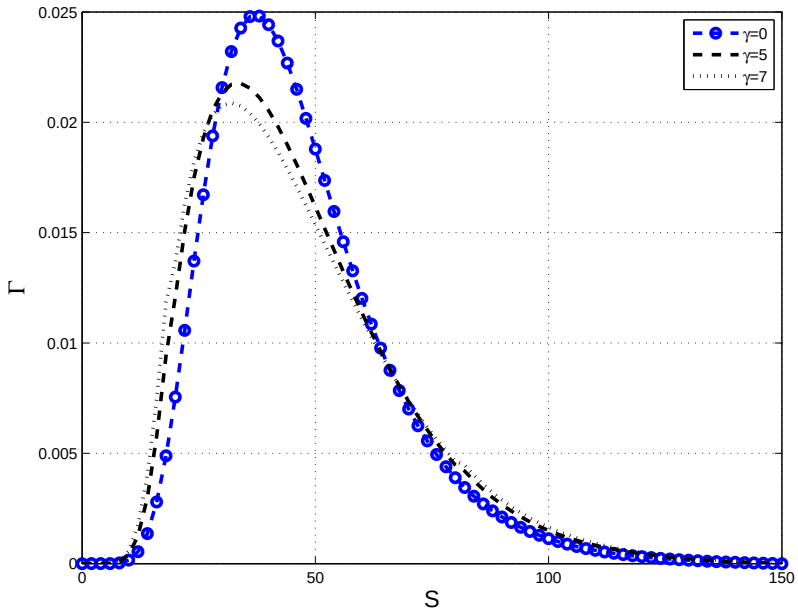

Figure 4.7: Variation of Γ with the parameter γ.

CHAPTER 5

CONCLUSION

In this thesis we investigated finite difference methods for the valuation of European options. We first reviewed the derivation of the original Black-Scholes partial differential equation and analytical solutions of the corresponding terminal-boundary value problems for the European call and put problems. Finite difference methods for the original Black-Scholes partial differential equation which is a linear second order parabolic equation are presented briefly.

Next, we studied an explicit finite difference method for a nonlinear Black-Scholes partial differential equation arising in an illiquid market given in the article [17] in detail. Monotonicity, stability and consistency properties of the explicit scheme is analyzed in a complete manner. Numerical experiments are performed confirming the stability conditions and behavior of the solution in terms of the change in the price impact coefficient γ. Variations of sensitivity of the option price and with respect to the change of the current price of the underlying stock, Δ, and Greek Γ are also investigated. It is seen that, the price of the option increases as γ increases and Δ grows for the price of asset smaller than the exercise price and diminishes otherwise, as γ increases. We also see that, maximum value of Γ is occurred for smaller values of the asset price as the parameter γ becomes larger.

Chapter 4 is the main part of the thesis. We introduced the exponential finite difference method for heat equation following [13] with some stability results. Then we applied the method for the linear and nonlinear Black-Scholes equations. We investigated order, consistency, linear stability domain and convergence of the method. For the linear and nonlinear problems discussed in Chapter 2 and Chapter 3, we found nu-

merical schemes as a result of the implementation of the exponential finite difference method. We analyzed the performence of the method by comparing its solution by the analytical and numerical solution obtained from explicit method for the linear BSE. We have shown that approximation to the current value of the European call option of the exponential finite difference scheme is always better than the explicit scheme. To show the convergence rate in space of the proposed numerical scheme, we have performed some numerical experiments for various values of ΔS and fixed value of Δt and verified the second order convergence in space of the scheme. Moreover, numerical experiments are performed to investigate the variation of option price in terms of the price impact coefficient γ. We have seen that, all results are similar to ones in the literature. Indeed, we have seen that, exponential finite difference method gives a stable solution for the choice of a time step value where the explicit method given in [17] gives an unstable solution.

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APPENDIX A

Appendix

In this part, we recall some definitions (see e.g. [46] and the references therein) in probability theory.

Definition A.0.1 *Let Ω be a nonempty set, and let $\mathcal{F}$ be a collection of subsets of Ω. We say that $\mathcal{F}$ is σ-algebra provided that:*

(i) the empty set $\varnothing$ belongs to $\mathcal{F}$,

(ii) whenever a set A belongs to $\mathcal{F}$, its complement A^c also belongs to $\mathcal{F}$, and

(iii) whenever a sequence of sets $A_1, A_2, \ldots$ belongs to $\mathcal{F}$, their union $\cup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{F}$.

Definition A.0.2 *Let Ω be a nonempty set, and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. A probability measure $\mathbb{P}$ is a function that, to every set $A \in \mathcal{F}$, assigns a number in $[0, 1]$, called the probability of A and written $\mathbb{P}(A)$. We require*

(i) $\mathbb{P}(\Omega) = 1$, and

(ii) (countable additivity) whenever $A_1, A_2, \ldots$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.

Definition A.0.3 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A real-valued function X de-*

fined on Ω is called random variable if the sets

$$\{X \leq x\} := \{\omega \in \Omega | X(\omega) \leq x\} = X^{-1}((-\infty, x])$$

are measurable for all $x \in \mathbb{R}$. That is, $\{X \leq x\} \in \mathcal{F}$.

Definition A.0.4 *For $x \in \mathbb{R}$ the (cumulative) distribution function $F(x)$ of a random variable X is defined by*

$$F(x) := \mathbb{P}(X \leq x).$$

satisfying

$$\lim_{x \to -\infty} F(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} F(x) = 1.$$

For all $x \in \mathbb{R}$ if $f(x) \geq 0$ and

$$F(x) = \int_{-\infty}^{x} f(t)dt,$$

then f is called a probability density function.

If X has the density f, then the expectation or mean of X is defined by

$$\mu := \mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)dx$$

and the variance is given as

$$\sigma^2 := Var[X] := \mathbb{E}[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)dx.$$

Definition A.0.5 (Normal Distribution) *X is normally distributed with expectation μ and variance σ^2 and denoted by $X \sim N(\mu, \sigma^2)$ if its density is*

$$\phi_{\mu,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2} \quad \text{for} \quad -\infty < x < \infty.$$

Definition A.0.6 (The Standard Normal Distribution) *The normal distribution with mean 0 and variance 1 is called the standard normal distribution.*

The probability density function of the standard normal distribution is

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad \text{for} \quad -\infty < x < \infty,$$

and the cumulative distribution function is

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \quad \text{for} \quad -\infty < x < \infty.$$