

Complementability of isometric copies of ℓ_1 in transportation cost spaces*

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Abstract

This work aims to establish new results pertaining to the structure of transportation cost spaces. Due to the fact that those spaces were studied and applied in various contexts, they have also become known under different names such as Arens-Eells spaces, Lipschitz-free spaces, and Wasserstein spaces. The main outcome of this paper states that if a metric space X is such that the transportation cost space on X contains an isometric copy of ℓ_1 , then it contains a 1-complemented isometric copy of ℓ_1 .

Keywords. Arens-Eells space, Banach space, earth mover distance, Kantorovich-Rubinstein distance, Lipschitz-free space, transportation cost, Wasserstein distance

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In memory of all the people who have sacrificed their lives while fighting for Ukraine since 2014

1 Introduction

In this paper, we continue the study of Banach-space-theoretical properties of transportation cost spaces. The study of transportation cost spaces was launched by Kantorovich [20], see also [21]. As time passed, these spaces have proven to possess the high degree of importance within a variety of directions. This, in turn, have led to the diversity of names used for the spaces, the most popular names are mentioned in the Abstract. We stick to the term *transportation cost space* since, in our opinion, it immediately clarifies the circle of discussed problems and is consistent with the

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history of the subject. A detailed survey on the development of those notions along with relevant historical comments is presented in [25, Section 1.6].

Before we begin, let us recall some necessary definitions and facts. Let (X, d) be a metric space. If $f : X \rightarrow \mathbb{R}$ is a function possessing a finite support and satisfying the condition $\sum_{v \in \text{supp} f} f(v) = 0$, then, in a natural way, it can be viewed as a *transportation problem* (on X) of certain product from sites where it is available ($f(v) > 0$) to those where it is demanded ($f(v) < 0$).

Every transportation problem f admits a presentation of the form:

$$f = a_1(\mathbf{1}_{x_1} - \mathbf{1}_{y_1}) + a_2(\mathbf{1}_{x_2} - \mathbf{1}_{y_2}) + \cdots + a_n(\mathbf{1}_{x_n} - \mathbf{1}_{y_n}), \quad (1)$$

where $a_i \geq 0$, $x_i, y_i \in X$, and $\mathbf{1}_u(x)$, $u \in X$ stands for the *indicator function* of u . Since equality (1) can be regarded as a plan of carrying a_i units of the product from x_i to y_i , every representation of this form is said to be a *transportation plan* for f . In this interpretation, the sum $\sum_{i=1}^n a_i d(x_i, y_i)$ defines the *cost* of that plan.

In the sequel, $\text{TP}(X)$ denotes the real vector space of all transportation problems (on X). We endow $\text{TP}(X)$ with the *transportation cost norm* (or *transportation cost*, for short). Namely, for $f \in \text{TP}(X)$, the norm $\|f\|_{\text{TC}}$ is defined as the infimum of costs taken over all transportation plans given by (1).

For infinite X , the space $\text{TP}(X)$ with $\|\cdot\|_{\text{TC}}$ may not be complete, its completion is called the *transportation cost space* and denoted by $\text{TC}(X)$. When X is finite, the above spaces $\text{TC}(X)$ and $\text{TP}(X)$ are identical as sets. The notation $\text{TC}(X)$ is employed to highlight the normed vector space structure when we need it.

It can be effortlessly derived from the triangle inequality that, whenever $f \in \text{TC}(X)$ has a finite support, the infimum of costs of transportation plans is attained. Moreover, it is easy to notice that this occurs for a plan with $\{x_i\} = \{v : f(v) > 0\}$ and $\{y_i\} = \{v : f(v) < 0\}$. Notice, that a transportation plan that provides the infimum need not be unique. Any such a transportation plan for f , whose cost equals $\|f\|_{\text{TC}}$ is called an *optimal transportation plan* for $f \in \text{TP}(X)$ because this plan has the minimal possible cost. See [27] for a more detailed introduction.

Prior to presenting our results, it seems appropriate to outline the motivation for studying transportation cost spaces:

(1) The dual space of $\text{TC}(X)$ is the space of Lipschitz functions on X vanishing at a specified point and equipped with its natural norm. This makes the space $\text{TC}(X)$ an object of Classical Analysis, especially in cases where X is a classical metric space like \mathbb{R}^n .

(2) A metric space X admits a canonical isometric embedding into $\text{TC}(X)$ (Arens–Eells observation [4]). This fact makes $\text{TC}(X)$ a natural object of study in the theory of Metric Embeddings, see [28, Chapter 10].

(3) The norm in this space can be interpreted as a transportation cost.

(4) The transportation cost space $\text{TC}(X)$ can be regarded as a kind of a *linearization* of X , and can be used to generalize Banach-space-theoretical notions to

the case of metric spaces. This approach was suggested by Bill Johnson; later his idea was described in [7, p. 223], where its limitations were discovered. See also the discussion in [24].

(5) Transportation cost spaces were applied to solve some important problems of the Banach space theory, both of linear and non-linear theories. The respective program was initiated by Godefroy-Kalton in [17] and significantly developed by Kalton in [19].

In this paper, the study of Banach space geometry of $\text{TC}(X)$ is continued. More specifically, we focus at studying the relations between its structure and the structure of the space ℓ_1 . Available related results can be found in [1, 3, 9, 10, 11, 12, 16, 22, 25, 26].

As the most closely related predecessors of this research, the following results have to be cited:

Theorem 1.1 ([22]). *If a metric space X contains $2n$ elements, then $\text{TC}(X)$ contains a 1-complemented subspace isometric to ℓ_1^n .*

In what follows, some notions and facts of Graph Theory will be employed; we refer to [6] for the respective terminology. In this article, only finite graphs are considered. A graph is said to be *weighted* if each its edge e has *weight*, that is, is a real number $w(e)$ associated with e . A set M of edges in a graph G is called a *matching* if edges in M do not have common ends. A matching M in G is called *perfect* if each vertex in G is incident with an edge in M . The *weight* of a matching M is the sum of weights of all edges contained in M . A graph is called *complete* if any two vertices in it are joined by an edge. A *minimum weight perfect matching*, typically/mostly/most times studied for complete weighted graphs with the even number of vertices, is a perfect matching with the least possible weight.

To formulate the next theorem, we introduce, for a finite set $\{v_i\}_{i=1}^m$ in (X, d) , the complete weighted graph $K(\{v_i\}_{i=1}^m)$, whose vertex set is $\{v_i\}_{i=1}^m$, while the weight of an edge uv equals $d(u, v)$.

Theorem 1.2 ([26]). *The space $\text{TC}(X)$ contains ℓ_1 isometrically if and only if there exists a sequence of pairs $\{x_i, y_i\}_{i=1}^\infty$ in X , with all elements distinct, such that each set $\{x_i, y_i\}_{i=1}^n$ of edges is a minimum weight perfect matching in the graph $K(\{x_i, y_i\}_{i=1}^n)$.*

The assertion below is the main result of this paper.

Theorem 1.3. *If a metric space X is such that $\text{TC}(X)$ contains a subspace isometric to ℓ_1 , then $\text{TC}(X)$ contains a 1-complemented isometric copy of ℓ_1 .*

Remark 1.4. In general, a linear isometric copy of ℓ_1 in $\text{TC}(X)$ does not have to be complemented. This fact is a consequence of the following result proved in [17, Theorem 3.1]: There exists a metric space X_C such that $\text{TC}(X_C)$ contains a linear

isometric copy of $C[0, 1]$. To show that $\text{TC}(X_C)$ contains a linearly isometric copy of ℓ_1 which is not complemented, it suffices to combine this result with the two classical facts: (i) ℓ_1 admits a linear isometric embedding into $C[0, 1]$ (Banach-Mazur, see [5, Theorem 9, p. 185]), (ii) The image of this subspace is not complemented, for example, because the dual of $C[0, 1]$ is weakly sequentially complete, but the dual of ℓ_1 is not; see [14, Chapter IV].

Remark 1.4, which establishes the existence of non-complemented linear isometric copies of ℓ_1 in $\text{TC}(X)$, is based on important classical results. However, if we are interested in subspaces isometric to ℓ_1 which are only not 1-complemented, such example can be constructed in a more elementary way. We present such an example below.

Example 1.5. *There is a simple metric space X_K such that $\text{TC}(X_K)$ contains a linear isometric copy of ℓ_1 which is not 1-complemented.*

Recall that $K_{4,4}$ is a complete bipartite graph in which both parts have 4 vertices. The starting point of this example is the fact [13, Section 8] that $\text{TC}(K_{4,4})$ contains a linearly isometric copy of ℓ_∞^4 . It is a well-known observation of Grünbaum [18], that there exist a subspace in ℓ_∞^4 which is isometric to ℓ_1^3 and is not 1-complemented. We let X_K be the union of the vertex set $V(K_{4,4})$ and \mathbb{N} (with their usual metrics). Next, pick a vertex O in $V(K_{4,4})$ and introduce the metric on the union as follows: the distance between two points in $V(K_{4,4})$ or \mathbb{N} is equal to the original. The distance between $v \in V(K_{4,4})$ and $m \in \mathbb{N}$ is equal to $d(v, O) + m$. It is well-known (see [2, Section 3.1]) that for such metric one has $\text{TC}(X_K) = \text{TC}(K_{4,4}) \oplus_1 \ell_1$. By the aforementioned example of [13], this space contains a subspace isometric $\ell_\infty^4 \oplus_1 \ell_1$. Thus, by the observation of [18] stated above, the space contains a linear isometric copy of ℓ_1 which is not 1-complemented.

2 Proof of Theorem 1.3

Our proof of Theorem 1.3 is based on Theorem 1.2. The following result is the key lemma in the proof of Theorem 1.3.

Lemma 2.1. *Let (X, d) be a metric space containing a set $\{x_i, y_i\}_{i=1}^n$ of pairs forming a minimum weight matching in $K(\{x_1, \dots, x_n, y_1, \dots, y_n\})$. Then, there exists a surjective norm-1 projection $P_n : \text{TC}(X) \rightarrow L_n$, where L_n is the subspace of $\text{TC}(X)$ spanned by $\{\mathbf{1}_{x_i} - \mathbf{1}_{y_i}\}_{i=1}^n$.*

Proof. Denote the vector $\frac{\mathbf{1}_{x_i} - \mathbf{1}_{y_i}}{d(x_i, y_i)}$ by \mathfrak{s}_i . To prove this lemma, we are going to construct a sequence $\{t_{i,n}\}_{i=1}^n$ of 1-Lipschitz functions on X such that $\{\mathfrak{s}_i, t_{i,n}\}_{i=1}^n$ is a biorthogonal set, and also

$$P_n(f) := \sum_{i=1}^n t_{i,n}(f) \mathfrak{s}_i \tag{2}$$

is a surjective norm-1 projection $P_n : \text{TC}(X) \rightarrow L_n$.

Remark 2.2. The dual of the space $\text{TC}(X)$ is identified as $\text{Lip}_0(X)$ - the space of Lipschitz functions on X having value 0 at a picked and fixed point O in X , called the *base point* (see [28, Chapter 10]). The action of $\text{Lip}_0(X)$ on $f \in \text{TP}(X)$ is given by $\sum_{v \in \text{supp} f} t(v)f(v)$, wherefrom it is extended to $\text{TC}(X)$ by continuity. Observe that any Lipschitz function t on X gives rise to a continuous linear functional on $\text{TC}(X)$, the functional is the same as the functional produced by $t - t(O) \in \text{Lip}_0(X)$. Because of this, in the selection of $t_{i,n}$ the condition $t_{i,n}(O) = 0$ may be dropped out.

At this point, we notice that, after establishing the biorthogonality, it suffices to prove that $\|P_n(f)\|_{\text{TC}} \leq \|f\|_{\text{TC}}$ for every $f \in \text{TC}(X)$ of the form $f = \mathbf{1}_w - \mathbf{1}_z$ for $w, z \in X$. This will be shown by using the reasoning of [22, p. 196]. For the convenience of the reader, the details are presented below. Indeed, by the definition of $\text{TC}(X)$, the space $\text{TP}(X)$ is dense in $\text{TC}(X)$, and for $g \in \text{TP}(X)$ the desired inequality can be derived from the case $f = \mathbf{1}_w - \mathbf{1}_z$ as follows. For each $g \in \text{TP}(X)$, there exists an optimal transportation plan (see Section 1 or [30, Proposition 3.16]). Hence, g can be represented as a sum $g = \sum_{i=1}^m g_i$, where all g_i are of the form $g_i = b_i(\mathbf{1}_{w_i} - \mathbf{1}_{z_i})$, $b_i \in \mathbb{R}$, and $\|g\|_{\text{TC}} = \sum_{i=1}^m \|g_i\|_{\text{TC}}$. Therefore, assuming that we proved the inequality $\|P_n(f)\|_{\text{TC}} \leq \|f\|_{\text{TC}}$ in the case $f = \mathbf{1}_w - \mathbf{1}_z$, we obtain:

$$\|P_n g\|_{\text{TC}} = \left\| P_n \left(\sum_{i=1}^m g_i \right) \right\|_{\text{TC}} \leq \sum_{i=1}^m \|P_n g_i\|_{\text{TC}} \leq \sum_{i=1}^m \|g_i\|_{\text{TC}} = \|g\|_{\text{TC}},$$

and, thus, $\|P_n\| \leq 1$.

To construct $\{t_{i,n}\}_{i=1}^n$, we need to restate the assumption that $\{x_i y_i\}_{i=1}^n$ is a minimum weight matching in $K(\{x_1, \dots, x_n, y_1, \dots, y_n\})$ in Linear Programming (LP) terms. Originally, this approach was suggested by Edmonds [15]. Below, we follow the presentation of this approach given in [23, Sections 7.3 and 9.2]. First, consider the minimum weight perfect matching problem on a complete weighted graph G with even number of vertices and weight $\mathbf{w} : E(G) \rightarrow \mathbb{R}$, $\mathbf{w} \geq 0$. By [23, Theorem 7.3.4], the minimum weight perfect matching problem can be reduced to the linear program **(LP1)** described below. Within the program, an *odd cut* designates the set of edges in G joining a subset of $V(G)$ of odd cardinality with its complement, while a *trivial odd cut* designates a set of edges joining one vertex with its complement. If \mathbf{x} is a real-valued function on $E(G)$ and A is a set of edges, we define $\mathbf{x}(A) := \sum_{e \in A} \mathbf{x}(e)$. The reduction means that the linear program has an integer solution corresponding to a minimum weight perfect matching.

Here comes the program.

- **(LP1)** minimize $\mathbf{w}^\top \cdot \mathbf{x}$ (where $\mathbf{x} : E(G) \rightarrow \mathbb{R}$)

- subject to

- (1) $x(e) \geq 0$ for each $e \in E(G)$
- (2) $x(C) = 1$ for each trivial odd cut C
- (3) $x(C) \geq 1$ for each non-trivial odd cut C .

Next, we introduce a variable y_C for each odd cut C .

The dual program of the program **(LP1)** is:

- **(LP2)** maximize $\sum_C y_C$
- subject to

- (D1)** $y_C \geq 0$ for each non-trivial odd cut C
- (D2)** $\sum_{C \text{ containing } e} y_C \leq w(e)$ for every $e \in E(G)$.

The Duality in Linear Programming [29, Section 7.4] - see also a summary in [23, Chapter 7] - states that the optima **(LP1)** and **(LP2)** are equal. Therefore, the total weight of the minimum weight perfect matching coincides with the sum of entries of the optimal solution of the dual problem.

In order to proceed, it is beneficial to recall some of the properties of optimal solutions $\{y_C\}$. **The following chain of relations/statements holds for all perfect matchings M in G and all $\{y_C\}$ satisfying **(D1)** and **(D2)**.**

$$w(M) = \sum_{e \in M} w(e) \stackrel{\text{(D2)}}{\geq} \sum_{e \in M} \sum_{C \text{ containing } e} y_C = \sum_C |M \cap C| y_C \stackrel{\text{(3)}}{\geq} \sum_C y_C, \quad (3)$$

where we use the fact that $|M \cap C| \geq 1$ for each perfect matching M and each odd cut C . See [23, p. 371].

If, in addition/on top of that/beyond that, M is a minimum weight perfect matching and y_C is an optimal dual solution, then the leftmost and the rightmost sides in (3) coincide, implying

$$w(e) = \sum_{C \text{ containing } e} y_C \quad (4)$$

for each $e \in M$ and

$$|M \cap C| = 1 \text{ for each non-trivial odd cut } C \text{ satisfying } y_C > 0. \quad (5)$$

Analysis in [23, p. 372–374] shows that we may assume that there exists a family \mathcal{H} of subsets of $V(G)$ which satisfies the conditions:

(P-1) \mathcal{H} is *nested* in the sense that for any $D, T \in \mathcal{H}$ either $D \subseteq T$ or $T \subseteq D$ or $D \cap T = \emptyset$.

(P-2) \mathcal{H} contains all singletons of $V(G)$.

(P-3) if C is a non-trivial odd cut, then $y_C > 0$ if and only if $C = \partial D$ for some $D \in \mathcal{H}$, where ∂D stands for the *edge boundary of D* , that is, the set of edges connecting D and $V(G) \setminus D$.

Furthermore, [8, Theorem 5.20] and [22, Lemma 14.11] established that if the weight function w satisfies $w(uv) = d(u, v)$ for some metric d on $V(G)$ and all $u, v \in V(G)$, then there is an optimal dual solution satisfying also $y_C \geq 0$ for all odd cuts, including trivial ones.

These results will be applied to the weighted graph $G = K(\{x_1, \dots, x_n, y_1, \dots, y_n\})$ and the matching $\{x_i y_i\}_{i=1}^n$. We denote the matching $\{x_i y_i\}_{i=1}^n$ by M_n , the graph $K(\{x_1, \dots, x_n, y_1, \dots, y_n\})$ by $K(M_n)$, and its vertex set by V_n .

Keeping the notation \mathcal{H} for the obtained nested family of subsets of V_n satisfying (P-1)–(P-3), we may and shall assume that \mathcal{H} is a *minimal family of subsets of $V(G)$ satisfying the conditions (P-1)–(P-3)*. To satisfy the condition (P-3), we can pick only one out of each complementary pair of subsets of $V(G)$. Therefore it may be assumed that *cardinalities of elements of \mathcal{H} do not exceed n , and there is at most one, if any, set in \mathcal{H} has cardinality n* . With these assumptions the correspondence between the edge boundaries of sets in \mathcal{H} and the cuts C which are either trivial or satisfy $y_C > 0$ is bijective.

With this in mind, it is only a slight abuse of notation to denote, for a uniquely determined $D \in \mathcal{H}$, the weight of ∂D by y_D , in particular, $y_{\{v\}}$ for a vertex v denotes the weight of the trivial cut separating vertex v from the rest of V_n (in $K(M_n)$).

Our next goal is to construct 1-Lipschitz functions $\{t_{i,n}\}_{i=1}^n$ satisfying the conditions $t_{i,n}(y_i) - t_{i,n}(x_i) = d(x_i, y_i)$ and $t_{i,n}(y_j) - t_{i,n}(x_j) = 0$, $i, j \in \{1, \dots, n\}$, $j \neq i$. Some features of this construction will be used to prove the inequality $\|P_n(f)\|_{\text{TC}} \leq \|f\|_{\text{TC}}$.

Using the notation $B_X(v, \mathfrak{r}) = \{x \in X : d(x, v) \leq \mathfrak{r}\}$ for $\mathfrak{r} > 0$, we define, for each $F \in \mathcal{H}$, the set

$$U_F = \bigcup_{v \in F} B_X \left(v, \sum_{D \subseteq F, v \in D \in \mathcal{H}} y_D \right). \quad (6)$$

Note that for a 1-element set $F = \{v\}$, $v \in V_n$, one has $U_F = B_X(v, y_F)$.

As a next step, we introduce three collections of 1-Lipschitz functions: $r_{\lambda, \theta, H} : X \rightarrow \mathbb{R}$ and $s_{\lambda, \theta, H} : X \rightarrow \mathbb{R}$ parameterized by $\lambda \in \mathbb{R}$, $\theta = \pm 1$, and $H \in \mathcal{H}$, and the collection $t_{D, F}$ parameterized by $D, F \in \mathcal{H}$. Here is the definition of $r_{\lambda, \theta, H}$:

$$r_{\lambda,\theta,H}(x) = \lambda + \theta \min_{v \in H} \{ \max \{ (d(x, v) - \sum_{v \in D \subsetneq H, D \in \mathcal{H}} y_D), 0 \} \}. \quad (7)$$

In the case where $H = \{v\}$, using the convention $\sum_{\emptyset} = 0$, one has:

$$r_{\lambda,\theta,\{v\}}(x) = \lambda + \theta d(x, v).$$

Note that $r_{\lambda,\theta,H}$ is equal to λ on $\bigcup_{D \subsetneq H, D \in \mathcal{H}} U_D$. The function $r_{\lambda,\theta,H}$ is 1-Lipschitz because $d(x, v)$ is 1-Lipschitz and all operations which we apply to it, namely, maximum, minimum, multiplication with ± 1 and addition of a constant preserve this property.

Now we define 1-Lipschitz functions $s_{\lambda,\theta,H} : X \rightarrow \mathbb{R}$ parameterized by $\lambda \in \mathbb{R}$, $\theta = \pm 1$, and $H \in \mathcal{H}$:

$$s_{\lambda,\theta,H}(x) = \lambda + \theta (\min_{v \in H} \{ \min \{ \max \{ (d(x, v) - \sum_{v \in D \subsetneq H, D \in \mathcal{H}} y_D), 0 \} \}, y_H \}). \quad (8)$$

In the case where $H = \{v\}$, the definition means the following:

$$s_{\lambda,\theta,\{v\}}(x) = \lambda + \theta \min \{ d(x, v), y_{\{v\}} \}.$$

Note that $s_{\lambda,\theta,H}$ is equal to λ on $\bigcup_{D \subsetneq H, D \in \mathcal{H}} U_D$ and to $\lambda + \theta y_H$ outside U_H . The function $s_{\lambda,\theta,H}$ is 1-Lipschitz for the same reason as $r_{\lambda,\theta,H}$.

Now we start constructing functions $t_{D,F}$ for different subsets $D, F \in \mathcal{H}$. As an initial point, let $D = D_1 = \{x_i\}$, $F = F_1 = \{y_i\}$, and $x_i y_i$ be an edge in the matching M_n . In this case, we denote $t_{D,F}$ also $t_{i,n}$ because these 1-Lipschitz functions will be the desired biorthogonal functionals for $\{\mathfrak{s}_i\}_{i=1}^n$.

Let $D_1 = \{x_i\} \subsetneq D_2 \subsetneq D_3 \subsetneq \dots \subsetneq D_{\tau_i}$ be elements of \mathcal{H} , where D_{τ_i} is the largest set in \mathcal{H} containing x_i but not containing y_i . Assume also that this increasing sequence is maximal in the sense that there is no $J \in \mathcal{H}$ satisfying $D_k \subsetneq J \subsetneq D_{k+1}$.

Similarly, let $F_1 = \{y_i\} \subsetneq F_2 \subsetneq F_3 \subsetneq \dots \subsetneq F_{\sigma_i}$ be a maximal increasing sequence of sets in \mathcal{H} with $x_i \notin F_{\sigma_i}$.

We define:

$$t_{i,n}(x) = \begin{cases} l_{i,n}(x) & \text{if } l_{i,n}(x) < y_{D_1} + \dots + y_{D_{\tau_i}} \\ h_{i,n}(x) & \text{if } h_{i,n}(x) > y_{D_1} + \dots + y_{D_{\tau_i}} \\ y_{D_1} + \dots + y_{D_{\tau_i}} & \text{otherwise} \end{cases} \quad (9)$$

where

$$l_{i,n}(x) = \min \{ r_{0,1,D_1}(x), r_{y_{D_1},1,D_2}(x), r_{y_{D_1}+y_{D_2},1,D_3}(x), \dots, r_{y_{D_1}+\dots+y_{D_{\tau_i-1}},1,D_{\tau_i}}(x) \}, \quad (10)$$

and

$$h_{i,n}(x) = \max\{r_{y_{D_1} + \dots + y_{D_{\tau_i}} + y_{F_{\sigma_i}, -1, F_{\sigma_i}}}(x), \dots, r_{y_{D_1} + \dots + y_{D_{\tau_i}} + y_{F_{\sigma_i}, \dots + y_{F_2, -1, F_2}}(x), r_{y_{D_1} + \dots + y_{D_{\tau_i}} + y_{F_{\sigma_i}, \dots + y_{F_1, -1, F_1}}(x)}\}. \quad (11)$$

It is not obvious that $t_{i,n}$ is well-defined, but it follows from the next argument demonstrating that $t_{i,n}$ is 1-Lipschitz.

Indeed, the functions $l_{i,n}$ and $h_{i,n}$, are 1-Lipschitz because they have been obtained from 1-Lipschitz functions using the maximum and minimum operations. To justify that $t_{i,n}$ is well-defined, it suffices to verify the 1-Lipschitz condition only for x and y satisfying $l_{i,n}(x) < y_{D_1} + \dots + y_{D_{\tau_i}}$ and $h_{i,n}(y) > y_{D_1} + \dots + y_{D_{\tau_i}}$. The 1-Lipschitz condition for such x and y guarantees that for some x inequalities $l_{i,n}(x) < y_{D_1} + \dots + y_{D_{\tau_i}}$ and $h_{i,n}(x) > y_{D_1} + \dots + y_{D_{\tau_i}}$ cannot hold simultaneously, and, therefore, $t_{i,n}$ is well-defined.

Since all minima in the definitions above are over finite sets, we may, without loss of generality, assume that there exist $k \in \{0, \dots, \tau_i\}$ and $u \in D_{k+1}$ such that

$$\begin{aligned} l_{i,n}(x) &= r_{y_{D_1} + \dots + y_{D_k}, 1, D_{k+1}}(x) \\ &= y_{D_1} + \dots + y_{D_k} + \max\{(d(x, u) - \sum_{u \in D \subsetneq D_{k+1}, D \in \mathcal{H}} y_D), 0\}. \end{aligned} \quad (12)$$

Similarly, without loss of generality, we may assume that there exist $m \in \{0, \dots, \sigma_i\}$ and $w \in F_{m+1}$ such that

$$\begin{aligned} h_{i,n}(y) &= r_{y_{D_1} + \dots + y_{D_{\tau_i}} + y_{F_{\sigma_i}, \dots + y_{F_{m+1}, -1, F_{m+1}}}(x) \\ &= y_{D_1} + \dots + y_{D_{\tau_i}} + y_{F_{\sigma_i}} + \dots + y_{F_{m+1}} \\ &\quad - \max\{(d(y, w) - \sum_{w \in F \subsetneq F_{m+1}, F \in \mathcal{H}} y_F), 0\}. \end{aligned} \quad (13)$$

If both maxima in (12) and (13) are achieved at the first term, we get:

$$\begin{aligned} h_{i,n}(y) - l_{i,n}(x) &= y_{D_{k+1}} + \dots + y_{D_{\tau_i}} + y_{F_{\sigma_i}} + \dots + y_{F_{m+1}} \\ &\quad + \sum_{w \in F \subsetneq F_{m+1}, F \in \mathcal{H}} y_F + \sum_{u \in D \subsetneq D_{k+1}, D \in \mathcal{H}} y_D - d(u, x) - d(w, y) \\ &\leq d(u, w) - d(u, x) - d(w, y) \leq d(y, x), \end{aligned}$$

where the first inequality in the last row uses inequality **(D2)** for the edge joining u and w .

If the maxima are equal to 0, we obtain:

$$h_{i,n}(y) - l_{i,n}(x) = y_{D_{k+1}} + \dots + y_{D_{\tau_i}} + y_{F_{\sigma_i}} + \dots + y_{F_{m+1}} \leq d(y, x), \quad (14)$$

where the ultimate inequality follows from the fact that

$$d(x, u) - \sum_{u \in D \subsetneq D_{k+1}, D \in \mathcal{H}} y_D < 0$$

implies that x is inside U_D for some proper subset $D \subset D_{k+1}, D \in \mathcal{H}$. Similarly,

$$d(y, w) - \sum_{w \in F \subsetneq F_{m+1}, F \in \mathcal{H}} y_F < 0$$

implies that y is inside U_F for some proper subset $F \subset F_{m+1}, F \in \mathcal{H}$. This implies the inequality (14). “Mixed” cases can be treated in a “mixed” way.

Equation (9) allows to evaluate

$$t_{i,n}(y_i) - t_{i,n}(x_i) = y_{D_1} + \cdots + y_{D_{\tau_i}} + y_{F_{\sigma_i}} + \cdots + y_{F_1}.$$

On the other hand, (4) implies that this sum is equal to $d(x_i, y_i)$.

For the sequel, representation of functions $t_{i,n}$ as sums of functions $s_{\lambda,\theta,H}$ are needed. **Next we claim that the following identity is true:**

$$\begin{aligned} t_{i,n}(x) = & s_{0,1,D_1}(x) + s_{0,1,D_2}(x) + s_{0,1,D_3}(x) + \cdots + s_{0,1,D_{\tau_i}}(x) \\ & + s_{y_{F_{\sigma_i}},-1,F_{\sigma_i}}(x) + \cdots + s_{y_{F_2},-1,F_2}(x) + s_{y_{F_1},-1,F_1}(x). \end{aligned} \quad (15)$$

Its proof consists in checking that formulas (9) and (15) lead to the same values on different pieces of the space:

- $d(x, x_i)$ on $U_{D_1} = B_X(x_i, y_{D_1})$
- $\min\{r_{0,1,D_1}(x), r_{y_{D_1},1,D_2}(x)\}$ on U_{D_2}
- $\min\{r_{0,1,D_1}(x), r_{y_{D_1},1,D_2}(x), r_{y_{D_1}+y_{D_2},1,D_2}(x)\}$ on U_{D_3} ,
- \dots ,
- $l_{i,n}$ on $U_{D_{\tau_i}}$,
- $y_{D_1} + \cdots + y_{D_{\tau_i}}$ on the complement of the set $U_{D_{\tau_i}} \cup U_{F_{\sigma_i}}$.
- $h_{i,n}$ on $U_{F_{\sigma_i}}$,
- \dots ,
- $\max\{r_{y_{D_1}+\cdots+y_{D_{\tau_i}}+y_{F_{\sigma_i}}+\cdots+y_{F_2},-1,F_2}(x), r_{y_{D_1}+\cdots+y_{D_{\tau_i}}+y_{F_{\sigma_i}}+\cdots+y_{F_1},-1,F_1}(x)\}$ on U_{F_2} ,
- $r_{y_{D_1}+\cdots+y_{D_{\tau_i}}+y_{F_{\sigma_i}}+\cdots+y_{F_1},-1,F_1}(x)$ on $U_{F_1} = B_X(y_i, y_{F_1})$.

To finalize the proof of biorthogonality of $\{t_{i,n}, \mathfrak{s}_i\}_{i=1}^n$ (see the paragraph preceding (2)), let $j \neq i, j \in \{1, \dots, n\}$. By condition (5), $x_j \in D_k$ implies $y_j \in D_k$, the same for F_k . Consequently, x_j and y_j satisfy exactly one of the following conditions: (i) There is a unique k such that $x_j, y_j \in D_k$, but not in D_{k-1} , (ii) There is a unique k such that $x_j, y_j \in F_k$, but not in F_{k-1} , (iii) $x_j, y_j \notin D_{\tau_i} \cup F_{\sigma_i}$. The biorthogonality condition $t_{i,n}(x_j - y_j) = 0$ is clear in the case (iii). As the cases (i) and (ii) are similar, only case (i) will be proven. Observe that $U_{\{x_j\}}$ and $U_{\{y_j\}}$ are contained in U_{D_k} . Also, the condition (4) implies that the sum of radii of the balls $U_{\{x_j\}}$ and $U_{\{y_j\}}$ does not exceed the distance between their centers, thus $U_{\{x_j\}} \neq U_{\{y_j\}}$. Combining definitions (8) and (6) of $s_{\lambda, \theta, H}$ and U_F , respectively, one derives that $s_{\lambda, \theta, H}(x) = s_{\lambda, \theta, H}(y) = \lambda$ for every $x, y \in \bigcup_{F \subseteq H} U_F$. Also, combining the definitions (8) and (6), one obtains that $s_{\lambda, \theta, H}(x) = s_{\lambda, \theta, H}(y) = \lambda + \theta y_H$ for $x, y \notin U_H$. Therefore, the equality $s_{\lambda, \theta, H}(x_j) = s_{\lambda, \theta, H}(y_j)$ holds for all $\lambda \in \mathbb{R}$, $\theta = \pm 1$, and $H \in \{D_1, \dots, D_{\tau_i}, F_1, \dots, F_{\sigma_i}\}$. Hence $t_{i,n}(x_j) - t_{i,n}(y_j) = 0$, this completes our proof of biorthogonality of $\{t_{i,n}, \mathfrak{s}_i\}_{i=1}^n$.

Now, necessary preparations will be made for the proof of $\|P_n(f)\|_{\text{TC}} \leq \|f\|_{\text{TC}}$ for every $f \in \text{TC}(X)$ of the form $f = \mathbf{1}_w - \mathbf{1}_z$ for $w, z \in X$.

We start by picking one of the smallest $D \in \mathcal{H}$ satisfying $w \in U_D$ (ties may be resolved arbitrarily) and one of the smallest $F \in \mathcal{H}$ satisfying $z \in U_F$. Note that it is possible that w or z are not in U_D for any $D \in \mathcal{H}$. Without loss of generality, the possible options for D and F can be listed as:

- (a) $D \cap F = \emptyset$
- (b) $D \subsetneq F$
- (c) $D = F$
- (d) F is undefined; meaning that z is not contained in U_H for any $H \in \mathcal{H}$.
- (e) D and F are undefined, that is, w and z are not contained in U_H for any $H \in \mathcal{H}$.

At this stage, it has to be proven that

$$\sum_{i=1}^n |t_{i,n}(z) - t_{i,n}(w)| \leq d(z, w). \quad (16)$$

For the sequel, the following notation is needed: $O_X(v, \mathfrak{r}) = \{x \in X : d(x, v) \leq \mathfrak{r}\}$ for $\mathfrak{r} > 0$, and

$$I_F = \bigcup_{v \in F} O_X \left(v, \sum_{D \subseteq F, v \in D \in \mathcal{H}} y_D \right). \quad (17)$$

To prove (16) in Case (a), we use the following argument. Let $D_{\beta(z,w)}$ be the largest set in \mathcal{H} satisfying $D_{\beta(z,w)} \supseteq D$ and $z \notin I_{D_{\beta(z,w)}}$. Note that z cannot belong

to I_D because in such a case the set U_F intersects I_D . Juxtaposing this observation/fact/remark with the definition of U_D , the condition $D \cap F = \emptyset$ and the condition **(D2)**, one obtains a contradiction. **The next sentence is totally unclear to me. Maybe, it can be changed as follows:** The case when $F_{\zeta(w,z)} \supseteq F$ is the largest set satisfying $w \notin I_{F_{\zeta(w,z)}}$ can be analyzed/considered/proved likewise/similarly. Next, also, let $D = D_1 \subsetneq D_2 \subsetneq \dots \subsetneq D_{\beta(z,w)}$ and $F = F_1 \subsetneq F_2 \subsetneq \dots \subsetneq F_{\zeta(w,z)}$ be the maximal chains of subsets in \mathcal{H} . **Then what?? Is it clear?**

The corresponding functions $t_{D,F}$ are constructed in the same manner as $t_{i,n}$:

$$t_{D,F}(x) = s_{0,1,D_1}(x) + s_{0,1,D_2}(x) + s_{0,1,D_3}(x) + \dots + s_{0,1,D_{\beta(z,w)}}(x) + s_{y_{F_{\zeta(w,z)}}, -1, F_{\zeta(w,z)}}(x) + \dots + s_{y_{F_2}, -1, F_2}(x) + s_{y_{F_1}, -1, F_1}(x). \quad (18)$$

The fact that $t_{D,F}$ is 1-Lipschitz can be checked in the same way as for $t_{i,n}$.

Note that each of the summands in the right-hand side of (18) appears in exactly one of the sums in (15). It cannot be present in several because the vertex leading to oddness of the cut related to a set $H \in \mathcal{H}$ should be in the corresponding pair $\{x_i, y_i\}$.

Therefore, on one hand,

$$t_{D,F}(x) = \sum_{i=1}^n t_{D,F,i}(x), \quad (19)$$

where $t_{D,F,i}(x)$ is the sum of those summands in (18) which are present in the decomposition of $t_{i,n}$.

On the other hand, since $t_{D,F}$ is 1-Lipschitz, one gets:

$$|t_{D,F}(w) - t_{D,F}(z)| \leq d(w, z). \quad (20)$$

As a result,

$$\begin{aligned} t_{D,F}(z) - t_{D,F}(w) &= \sum_{i=1}^n (t_{D,F,i}(z) - t_{D,F,i}(w)) \\ &= \sum_{i=1}^n |t_{D,F,i}(z) - t_{D,F,i}(w)| \\ &= \sum_{i=1}^n |t_{i,n}(z) - t_{i,n}(w)|. \end{aligned}$$

The latter equalities have been established with the help of the following observations: (1) All summands in (18) have larger value at z than at w ; (2) All other functions of the form $s_{\lambda,\theta,H}$ have the same values at z and w .

This statement **which statement? Is it clear?** is obvious in the case where $I_H, H \in \mathcal{H}$, does not contain any of w and z since $s_{\lambda, \theta, H}$ is constant outside I_H .

The function $s_{\lambda, \theta, H}$ is also constant on the set $\bigcup_{D \subsetneq H, D \in \mathcal{H}} U_D$, implying that/yielding that $s_{\lambda, \theta, H}(w) \neq s_{\lambda, \theta, H}(z)$ is possible/may hold/holds only if either one w, z is in $\bigcup_{D \subsetneq H, D \in \mathcal{H}} U_D$ and the other is not, or at least one of them is in $I_H \setminus \bigcup_{D \subsetneq H, D \in \mathcal{H}} U_D$. In both cases, $s_{\lambda, \theta, H}$ has to be present in the corresponding sum introduced in cases (a)-(e).

Case (b): Consider the maximal increasing sequence of sets in \mathcal{H} of the form $D = D_1 \subsetneq D_2 \subsetneq \dots \subsetneq D_n = F$. Construct the function $t_{D, F}$ and complete the proof as will be described in Case (d).

Case (c): In this case, define $t_{D, F}$ as $s_{0, 1, D}$ and use a simpler version of the argument of Cases (a) and (b).

In Case (d), consider the maximal increasing sequence of sets in \mathcal{H} of the form $D = D_1 \subsetneq D_2 \subsetneq \dots \subsetneq D_{\beta(z, w)}$.

Form the function $t_{D, F}$ as in the first line of (18), and repeat the same argument as in Case (a).

In Case (e), $t_{i, n}(w) = t_{i, n}(z)$ for every i , whence the conclusion follows. This completes our proof of Lemma 2.1. \square

Now we shall use this result to prove Theorem 1.3.

Proof. Let a metric space X be such that $\text{TC}(X)$ contains a linear isometric copy of ℓ_1 . By Theorem 1.2, this implies that there exists a sequence of pairs $\{x_i, y_i\}_{i=1}^\infty$ in X , with all elements distinct, such that each set $\{x_i, y_i\}_{i=1}^n$ of edges is a minimum weight perfect matching in the graph $K(\{x_i, y_i\}_{i=1}^n)$.

For each $n \in \mathbb{N}$, find functionals $\{t_{i, n}\}_{i=1}^n$ by applying Lemma 2.1. The next step in the proof is to define 1-Lipschitz functions $\{t_i\}_{i=1}^\infty$ on X as weak* limits of the sequences $\{t_{i, n}\}_{n=i}^\infty$. More precisely, we pick a free ultrafilter \mathcal{U} on \mathbb{N} and let $t_i = w^* - \lim_{i, \mathcal{U}} t_{i, n}$. (We may understand the limit as pointwise after replacing the functions $t_{i, n}$ with $t_{i, n}(x) - t_{i, n}(O)$ for some base point O .) Then, we define a mapping $P : \text{TC}(X) \rightarrow \overline{\text{lin}(\{\mathfrak{s}_i\}_{i=1}^\infty)}$ by $P(f) = \sum_{i=1}^\infty t_i(f) \mathfrak{s}_i$, where $\overline{\text{lin}(\{\mathfrak{s}_i\}_{i=1}^\infty)}$ denotes the closure of the linear span of the sequence $\{\mathfrak{s}_i\}_{i=1}^\infty$.

The fact that the sequence $\{\mathfrak{s}_i\}_{i=1}^\infty$ is isometrically equivalent to the unit vector basis of ℓ_1 was observed in [22] (and is easy to check).

Therefore, to justify that the map P is well-defined and at the same time it is a projection of norm 1, it suffices to show that for any $m \in \mathbb{N}$, there holds:

$$\left\| \sum_{i=1}^m t_i(f) \mathfrak{s}_i \right\|_{\text{TC}} \leq \|f\|_{\text{TC}}.$$

However, this is true because, by Lemma 2.1,

$$\left\| \sum_{i=1}^m t_{i,n}(f) \mathfrak{s}_i \right\|_{\text{TC}} \leq \|f\|_{\text{TC}}$$

and, in addition, $\sum_{i=1}^m t_i(f) \mathfrak{s}_i$ is a (strong) limit of $\sum_{i=1}^m t_{i,n}(f) \mathfrak{s}_i$ as $n \rightarrow \infty$ through \mathcal{U} . \square

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