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ON PROPERTIES OF $q-$BERNSTEIN POLYNOMIALS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of **Master of Science in Mathematics Department, Atılım University**.

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ABSTRACT

On Properties of $q-$Bernstein Polynomials

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The aim of this thesis is to study the theory of the Bernstein polynomials and its recent extension to the q-calculus. The main focus of the present study is on the q-Bernstein polynomials which appeared twenty years ago and have been attracting many researchers afterward. This work exhibits a review of well-known results on the Bernstein polynomials along with the necessary preliminaries, introduction to the theory of the q-Bernstein polynomials, and some new developments. The latter include the result on strong operator limit for the sequence of the limit q-Bernstein operators and the proposition that the q-Bernstein operators are weakly Picard.

Keywords: Uniform convergence, Bernstein polynomials, q-calculus, q-Bernstein polynomials, limit q-Bernstein operators, iterates, weakly Picard operator.

ÖZ

$q-$Bernstein Polinomlarının Özellikleri Üzerine

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Bu tezin amacı Bernstein polinomları teorisini ve son genişletmesi olan q-kalkülüsü çalışmaktır. Bu çalışmanın temel odak noktası 20 yıl önce ortaya çıkan ve kısa sürede birçok araştırmacının dikkatini çeken q-Bernstein polinomlarıdır. Bu tez Bernstein polinomlarına dair bilinen bazı sonuçların derlemesinden, q-Bernstein polinomları teorisine kısa bir giriş ve bazı yeni gelişmelerden oluşmaktadır. Yeni gelişmeler kısmında; limit q-Bernstein operatör dizisinin kuvvetli operatör limiti ve q-Bernstein operatörlerinin zayıf Picard operatörler oldukları ifade edilmiştir.

Anahtar Kelimeler: Düzgün yakınsaklık, Bernstein polinomları, q-kalkülüs, q- Bernstein polinomları, limit q-Bernstein operatörleri, iterasyonlar, zayıf Picard operatör.

To Heba

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CHAPTER 1

INTRODUCTION

Modern Approximation Theory starts with the famous theorem attributed to K. Weierstrass:

Fundamental Theorem of Approximation Theory. Let $f \in C[a,b]$. Given $\varepsilon > 0$, there exists an algebraic polynomial P for which

$$|f(x) - P(x)| < \varepsilon$$

for all $x \in [a,b]$.

This remarkable result was proved in 1885, immediately triggering a flow of researches that offered new approaches and alternative proofs. For the history of the subject, we refer to [22]. One of the elegant and fruitful methods to prove this theorem was proposed by S. N. Bernstein in 1912. In article [4], he constructed in an explicit form a sequence of polynomials which converges uniformly to a given continuous function. This very short paper made a tremendous impact, both on mathematics and on its applications. Although in his work, Bernstein used the notions and ideas related to Probability Theory, the construction and proof can be re-established by purely analytical techniques. Such an approach is adopted in this thesis. Nowadays, the polynomials introduced in [4] are called the *Bernstein polynomials*. Chapter 3, contains their definition and some properties.

Later on, it was found that the Bernstein polynomials possess many remarkable properties, turning them into a field for intensive research. These polynomials have been studied intensively and their connections with various branches of analysis, numerical analysis, total positivity, the operator theory, and other disciplines have been investigated. The relation between the Bernstein polynomials and Probability Theory stipulated their significance in the study of probability distributions and statistical inference, where random Bernstein polynomials emerged. See [18] for details. Due to the fact that the Bernstein polynomials of a continuous function f on interval $[0, 1]$ form an approximating sequence of shape-preserving operators, these polynomials play an important role in computer-aided geometric design (CAGD). It is a noteworthy fact that the de Casteljeau algorithms, based primarily on the properties of the Bernstein polynomials, became a fundamental tool in the computer-aided geometric design. These algorithms are very effective in their applications in car and ship design, aircraft industry, as well as medical and geological sciences. The initial studies on the surface design using programmable computers were conducted by P. de Catseljau at Citroën Automobile SA company, P. Bézier at Regie Ranault, J. C. Ferguseon at the Boeing Aircraft Company, A. A. Ball at the British Aircraft Corporation, and at others. The significant role of the Bernstein polynomials in those researches was revealed by A. R. Forrest in 1972. For historical remark and overview of the subject see [5]. Therefore, the Bernstein polynomials are an important tool not only in mathematics, but also in engineering, industry, and other disciplines.

Currently, the bibliography on the Bernstein polynomials includes thousands of works, while new papers are constantly coming out and new applications and generalizations are being discovered. The aim of such emerging generalizations is to create appropriate tools for various problems of Analysis, Differential Equations, Numerical Analysis, and others. Due to the intensive development of q-Calculus, generalizations of Bernstein polynomials connected with q-Calculus have also appeared.

Alexandru Lupaş (1942 - 2007) was the person who pioneered the work on the q-versions of the Bernstein polynomials. In 1987, he introduced a q-analogue of the Bernstein operator and investigated its approximation and shape-preserving properties. This q-analogue, along with its convergence and shape-preserving properties, was studied in [13]. It has to be pointed out that the operators treated by A. Lupaş are given by rational functions rather than polynomials. Regrettably, the operators proposed by Lupaş remained unnoticed for a long while due to the very limited availability of his article published only in a domestic conference proceedings. Today, however, this situation has changed - see, for example [2] - thanks to the developments in communication technology.

Nevertheless, the most popular q-generalization which appeared 10 years after Lupaş's paper [13], belongs to G.M. Philips [19] who constructed new polynomials known today as the *q-Bernstein polynomials*. These polynomials were immediately brought into the spotlight and studied by a number of authors from different perspectives. The bibliography on the q-Bernstein polynomials comprises about 200 papers, and the research in the area is still in progress. While for $q = 1$, these polynomials coincide with the classical ones, for $q \neq 1$, we obtain new polynomials with quite different properties. Conventionally, the term *q-Bernstein polynomial* implies that $q \neq 1$. Comprehensive reviews of the available results, along with an extensive bibliography, are presented in [15, 20, 21] and references therein. The probabilistic aspects of the theory of q-Bernstein polynomials have been studied in [6] and [8].

It has to be pointed out that some properties of the classical Bernstein polynomials remain true for the q-Bernstein ones. This range of problems was investigated by G.M. Phillips and the related outcomes are summarized in [20, Chapter 7]. These results are discussed in Chapter 3.

In contrast, the results on the convergence properties of q-Bernstein polynomials are *not* similar to those of the classical ones. What is more, the convergence properties of the q-Bernstein polynomials for the cases $0 < q < 1$ and $q > 1$ are different from each other. In the case when $0 < q < 1$, an important role is played by the *limit q-Bernstein operator*. This operator $B_{\infty,q}$ was defined in [9] and its analytical and geometric properties have been studied in several papers. It should be mentioned that the same operator $B_{\infty,q}$ serves as a limit for a sequence of the q-Meyer-König and Zeller operators considered by T. Trif in [24]. In addition, similar limit operators were determined for other q-operators. An inclusive review of the results on the limit q-Bernstein operator is presented in [16].

As for the q-Bernstein polynomials themselves, their investigation establishes new connections between the relatively narrow class of operators and other areas, not only inside approximation theory, but also within functional analysis, algorithms, complex analysis, and others. The emergence of new phenomena adds further value to this field. Although many of the obtained results can be viewed as merely straightforward generalizations of those already known, studying them either requires different tools or leads to problems which do not even arise in the classical settings, such as, for example, the dependence of outcomes on parameter q. The present thesis investigates this type of problems for the limit q-Bernstein operator.

Since iterative methods are used widely in many areas such as the fixed point theory, matrix analysis, differential equations and numerical algorithms, the iterates of the Bernstein operators were studied in a number of papers using different techniques. The first results appeared in [11], and their alternative proofs are displayed in [1, 23]. In [23], it has also been established that the classical Bernstein operators are *weakly Picard operators*. The latter class of operators includes contraction operators. In this thesis, it will be shown that the q-Bernstein polynomials generate the weakly Picard operators for all $q > 0$. However, the approach adopted in [23] cannot be applied for all $q > 0$ since it is essentially based on the fact that classical Bernstein operators are *positive*.

The contents of this thesis has the following breakdown: Chapter 2 provides the necessary background with an overview of the main facts related to the notion of uniform continuity. In Chapter 3, the preliminary information on the Bernstein polynomials and their properties is given. The q-Bernstein polynomials are studied in Chapter 4, more precisely, Section 4.1 contains an overview of the needed facts related to the q-calculus, focusing on the properties of the q-integers and the q-binomial coefficients. Subsequently, the definition and simple properties of the q-Bernstein polynomials are presented. Finally, new results related to the q-Bernstein polynomials appear in Chapter 5. The first result is the theorem on the strong operator convergence of the sequence $\{B_{\infty,q_n}\}_{n\in\mathbb{N}_0}$ when $q_n \to a \in (0,1)$. Apart from that, the assertion that the q-Bernsten operator is weakly Picard for all $q > 0$ is proved. It is worth mentioning that the previously avaliable approaches can not be applied in the case $q > 1$, since the operator $B_{n,q}$ is not positive when $q > 1$.

CHAPTER 2

PRELIMINARIES

2.1 Uniform continuity

Definition 2.1.1 *A function* $f : A \to \mathbb{R}$ *is uniformly continuous on* A *if and only if, given* $\varepsilon > 0$, *there exist* $\delta = \delta(\varepsilon) > 0$ *such that for* $x, y \in A$,

$$\text{if} \qquad |x - y| < \delta, \quad \text{then} \quad |f(x) - f(y)| < \varepsilon.$$

Remark 2.1.2 *A uniformly continuous function is continuous, but the converse is not true.*

Example 2.1.3 *It is obvious that the function* $f(x) = 1/x$ *is continuous on* $A = (0, \infty)$, *since it is a rational function and its denominator does not vanish at any point in* A. *Now, to show that* f *is not uniformly continuous on* A, *choose* $\varepsilon = 1 > 0$ *and choose* $\{x_n\} = \frac{1}{n}$ *and* $\{y_n\} = \frac{1}{n+2}$. *Clearly,*

$$\lim_{n\to\infty}(x_n - y_n) = \lim_{n\to\infty}\left(\frac{1}{n} - \frac{1}{n+2}\right) = 0.$$

On the other hand,

$$|f(x_n) - f(y_n)| = |n - (n+2)| = 2 \geqslant 1 = \varepsilon$$

for all $n \in \mathbb{N}$. *Therefore,* f *is not uniformly continuous.*

However, if A is a bounded closed interval, the following theorem due to Cantor holds.

Theorem 2.1.4 *Let* $I = [a, b]$ *be a closed and bounded interval. If* $f : I \to \mathbb{R}$ *is continuous on* I, *then* f *is uniformly continuous on* I.

Proof. Assume that f is not uniformly continuous on I. Then, there exists $\varepsilon_0 > 0$ and two sequences $\{x_n\}$, $\{y_n\} \subseteq I$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| > \varepsilon_0$, for all $n \in \mathbb{N}$. Since I is bounded and $\{x_n\} \subseteq I$, $\{x_n\}$ is bounded. By the Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim\limits_{k\to\infty} x_{n_k} = z$. Since $\{x_{n_k}\} \subseteq I = [a, b]$, and the interval I is closed, we have $z \in [a, b]$. Now, we show that $\{y_{n_k}\} \to z$ as $k \to \infty$. Note that

$$0 \leqslant |y_{n_k} - z| = |y_{n_k} - x_{n_k} + x_{n_k} - z| \leqslant |y_{n_k} - x_{n_k}| + |x_{n_k} - z|$$

for all $k \in \mathbb{N}$. Since both sequences in the right hand side tend to 0, one has $\lim\limits_{k\to\infty} |y_{n_k} - z| = 0$, that is $\lim\limits_{k\to\infty} y_{n_k} = z$. Now, since $z \in I$ and f is continuous at z, one has $\lim\limits_{k\to\infty} f(x_{n_k}) = f(z)$, and $\lim\limits_{k\to\infty} f(y_{n_k}) = f(z)$.

But $|f(x_n) - f(y_n)| > \varepsilon_0$, by the assumption, we derive a contradiction. $\square$

Definition 2.1.5 *A function $f : A \to \mathbb{R}$ is Lipschitz on A, if there exists $k > 0$ such that*

$$|f(x) - f(y)| < k|x - y|, \quad \textit{for all} \quad x, y \in A.$$

Theorem 2.1.6 *If $f : A \to \mathbb{R}$ is Lipschitz on A, then it is uniformly continuous on A.*

Proof. Assume$f$ is Lipschitz on A. Then there exist $k > 0$ such that

$$|f(x) - f(y)| < k|x - y|,$$

for all $x, y \in A$. Let $\varepsilon > 0$ be given, and choose $\delta = \frac{\varepsilon}{k} > 0$. Then for $x, y \in A$ with $|x - y| < \delta$, one has $|f(x) - f(y)| \leqslant k|x - y| < k\delta = \varepsilon$. Since $\varepsilon > 0$, f is uniformly continuous on A. $\square$

Remark 2.1.7 *The converse of above theorem is false as it can be seen by the next example.*

Example 2.1.8 *Consider $f(x) = \sqrt{x}$ for $x \in I = [0, 2]$. Since $I = [0, 2]$ is closed and bounded, by Theorem 2.1.4, f is uniformly continuous on I.*

Now, we prove that f is not Lipschitz on I. Assume that f is Lipschitz on I, that is, there exists $k > 0$ such that $|f(x) - f(y)| < k|x - y|$ for all $x, y \in I = [0, 2]$. In particular, *this inequality should be true for* $y = 0$*. That is,* $|f(x)| < k|x|$ *for all* $x \in I = [0, 2]$*. Take* $x = \frac{1}{n^2} \in [0, 2]$*. The later inequality implies that*

$$\frac{1}{n} = f\left(\frac{1}{n^2}\right) < k\frac{1}{n^2}$$

or $n < k$ *for all* $n \in \mathbb{N}$*, which is a contradiction. Therefore,* f *is not Lipschitz on* I*.*

2.2 Modulus of continuity

Definition 2.2.1 *Let* f *be defined on an interval* I*. Set*

$$\omega(\delta; f) = \omega_f(\delta) = \sup |f(x) - f(y)|$$

where the sup *is taken over all pairs* $x, y \in I$ *for which* $|x - y| \leqslant \delta$*. The function* $\omega_f(\delta)$ *is called the modulus of continuity of* f *on* I*.*

Example 2.2.2 *The modulus of continuity of* $f(x) = x^2$ *on* $I = (0, 1)$ *is* $2\delta - \delta^2$*. To see this, choose* $y = x - \delta$*. Then*

$$\omega_f(\delta) = \sup_{\substack{x,y\in(0,1)\\ |x-y|\leqslant\delta}} |f(x) - f(y)| = \sup_{\substack{x,y\in(0,1)\\ |x-y|\leqslant\delta}} |x^2 - y^2| = \sup_{x,x-\delta\in(0,1)} |x^2 - (x - \delta)^2| = 2\delta - \delta^2.$$

2.3 Uniform convergence

Definition 2.3.1 *Let* D *be a subset of* $\mathbb{R}$ *and* $\{f_n\}$ *be a sequence of real valued functions defined on* D*. Then,* $\{f_n\}$ *converges uniformly to* f *if, given any* $\varepsilon > 0$*, there exists a natural number* $N = N(\varepsilon)$ *such that*

$$|f_n(x) - f(x)| < \varepsilon \textit{ for every } n > N \textit{ and for every } x \textit{ in } D.$$

It follows from the definition that if $f_n(x) \to f(x)$ uniformly on D, then $f_n(x) \to f(x)$ at every point $x \in D$. However, the converse is not true as the next example show.

Example 2.3.2 $f_n(x) = x^n - x^{2n}$ *converges to 0 on* $[0, 1]$ *but the convergence is not uniform.*

Note that $f_n(x) \to 0$ *as* $n \to \infty$ *for all* $x \in [0, 1]$. *So, if* f_n *converges uniformly, then it should converge to* 0. *Now,*

$$f_n'(x) = nx^{n-1} - 2nx^{2n-1} = nx^{n-1}(1 - 2x^n) = 0$$

when $x = 0$ *or* $x = 2^{-1/n}$. *Since*

$$f_n\left(2^{\frac{-1}{n}}\right) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \neq 0,$$

$f_n(x)$ *does not converge uniformly on* $[0, 1]$.

Example 2.3.3 $f_n(x) = x^n - x^{n+1}$ *converges to 0 uniformly on* $[0, 1]$.

Note that $f_n(x) \to 0$ *for all* $x \in [0, 1]$. *To see if the converge is uniform or not, lets find the maximum value of* $|f_n(x)|$ *for each n on* $[0, 1]$. *Now*

$$f_n'(x) = nx^{n-1} - (n + 1)x^n = x^{n-1}(n - (n + 1)x).$$

At the critical point $x = \frac{n}{n+1} \in [0, 1]$, *we have*

$$\begin{aligned} |f_n(x)| \leqslant \left|f_n\left(\frac{n}{n+1}\right)\right| &= \left(\frac{n}{n+1}\right)^n - \left(\frac{n}{n+1}\right)^{n+1} \\ &= \left(\frac{n}{n+1}\right)^n \left(1 - \frac{n}{n+1}\right) \\ &\leqslant \frac{1}{n+1} \to 0 \quad as \quad n \to \infty. \end{aligned}$$

Hence, $f_n(x) \to 0$ *uniformly on* $[0, 1]$.

CHAPTER 3

BERNSTEIN POLYNOMIALS

3.1 Introduction

In this section, we shall overview some properties of the Bernstein polynomials.

Definition 3.1.1 *Bernstein basis polynomials are defined by*

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \ldots, n, \quad n \in \mathbb{N}, \tag{3.1}$$

where $\binom{n}{k}$ *is the binomial coefficient:*

$$\binom{n}{k} = \prod_{k=1}^{n} \frac{n+1-k}{k} = \frac{n!}{k!(n-k)!}.$$

The Bernstein basis polynomials of degree n form a basis for the vector space of polynomials of degree $\leqslant n$.

Example 3.1.2 *One has*

$$\begin{aligned} p_{n,0}(x) &= \binom{n}{0} x^0 (1-x)^{n-0} = \frac{n!}{0!(n)!}\, 1(1-x)^n = (1-x)^n, \\ p_{n,1}(x) &= \binom{n}{1} x^1 (1-x)^{n-1} = \frac{n!}{1!(n-1)!}\, x(1-x)^{n-1} = nx(1-x)^{n-1}, \\ p_{n,2}(x) &= \binom{n}{2} x^2 (1-x)^{n-2} = \frac{n!}{2!(n-2)!}\, x^2(1-x)^{n-2} \\ &= \frac{n(n-1)(n-2)!}{2!(n-2)!}\, x^2(1-x)^{n-2} \\ &= \frac{n(n-1)}{2}\, x^2(1-x)^{n-2}, \end{aligned}$$

and

$$\begin{aligned}
p_{n,3}(x) &= \binom{n}{3} x^3(1-x)^{n-3} \\
&= \frac{n!}{3!(n-3)!}\, x^3(1-x)^{n-3} \\
&= \frac{n!}{3!(n-3)!}\, x^3(1-x)^{n-3} \\
&= \frac{n(n-1)(n-2)(n-3)!}{6(n-3)!}\, x^3(1-x)^{n-3} \\
&= \frac{n(n-1)(n-2)}{6}\, x^3(1-x)^{n-3}.
\end{aligned}$$

Definition 3.1.3 *Let $f : [0,1] \to \mathbb{R}$. The Bernstein polynomial of f of degree n is*

$$B_n(f;x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{k,n}(x), \quad n = 1, 2, \ldots \tag{3.2}$$

where $p_{k,n}$ are Bernstein basis polynomials given in (3.1)*. The Bernstein operator B_n on $C[0,1]$ is defined by $f \mapsto B_n f$.*

Using definition 3.1.3, one can immediately derive the following elementary results.

Proposition 3.1.4 *The Bernstein operators possess,*

(i) Linearity: For all $f, g : [0,1] \to \mathbb{R}$ and all $C_1, C_2 \in \mathbb{R}$ we have

$$B_n(C_1 f + C_2 g; x) = C_1 B_n(f;x) + C_2 B_n(g;x). \tag{3.3}$$

(ii) End-point interpolation:

$$B_n(f;0) = f(0), \quad B_n(f;1) = f(1). \tag{3.4}$$

(iii) Positivity: If $f(x) \geqslant 0$ for $x \in [0,1]$, then $B_n(f;x) \geqslant 0$ for all $x \in [0,1]$.

Example 3.1.5 *Let us write $B_1(f;x)$, $B_2(f;x)$, and $B_3(f;x)$.*

For $n = 1$ in (3.2)*, we get*

$$B_1(f;x) = \sum_{k=0}^{1} f\left(\frac{k}{1}\right)\binom{1}{k} x^k(1-x)^{1-k} = f(0)(1-x) + f(1)x.$$

For $n = 2$ in (3.2), *we get*

$$\begin{aligned} B_2(f;x) &= \sum_{k=0}^{2} f\left(\frac{k}{2}\right)\binom{2}{k}x^k(1-x)^{2-k} \\ &= f(0)(1-x)^2 + 2f\left(\frac{1}{2}\right)x(1-x) + f(1)x^2. \end{aligned}$$

For $n = 3$ in (3.2), *we get*

$$\begin{aligned} B_3(f;x) &= \sum_{k=0}^{3} f\left(\frac{k}{3}\right)\binom{3}{k}x^k(1-x)^{3-k} \\ &= f(0)(1-x)^3 + 3f\left(\frac{1}{3}\right)x(1-x)^2 + 3f\left(\frac{2}{3}\right)x^2(1-x) + f(1)x^3. \end{aligned}$$

Lemma 3.1.6 *For $f(x) \equiv 1$, one has $B_n(f;x) = 1$ for all $n \in \mathbb{N}$.*

Proof. Note that, from the Binomial Theorem,

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}.$$

When $f(x) \equiv 1$, one has

$$B_n(1;x) = \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = [x + (1-x)]^n = 1$$

for all $n = 1, 2, \ldots$ ☐

Lemma 3.1.7 *If $f(x) = x$, then $B_n(f;x) = x$ for all $n \in \mathbb{N}$. This means that*

$$\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k} = x \quad \text{for all } n = 1, 2, 3, \ldots$$

Proof. We know that

$$\frac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}.$$

Then

$$\begin{aligned} B_n(x;x) &= \sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=1}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k} \\ &= \sum_{k=1}^{n}\binom{n-1}{k-1}x^k(1-x)^{n-k} = \sum_{k=0}^{n-1}\binom{n-1}{k}x^{k+1}(1-x)^{n-k-1} \end{aligned}$$

$$= x\sum_{k=0}^{n-1}\binom{n-1}{k}x^k(1-x)^{n-1-k} = xB_{n-1}(1;x) = x$$

as stated. □

Using the preceding two Lemmas together with Proposition 3.1.4 (i), we conclude that the Bernstein operator leaves linear functions invariant, which is expressed by the next statement.

Corollary 3.1.8 *For any* $n \in \mathbb{N}$, $a, b \in \mathbb{R}$,

$$B_n(ax+b;x) = ax+b.$$

Lemma 3.1.9 *For all* $n = 2, 3, \dots$

$$B_n(x^2;x) = \frac{(n-1)x^2+x}{n} = x^2 + \frac{x(1-x)}{n}.$$

Proof.

$$\begin{aligned}
B_n(x^2;x) &= \sum_{k=0}^{n}\frac{k^2}{n^2}\binom{n}{k}x^k(1-x)^{n-k}\\
&= \sum_{k=1}^{n}\frac{k}{n}\binom{n-1}{k-1}x^k(1-x)^{n-k}\\
&= \sum_{k=0}^{n-1}\frac{k+1}{n}\binom{n-1}{k}x^{k+1}(1-x)^{n-k-1}\\
&= \frac{x}{n}\left[\sum_{k=0}^{n-1}k\binom{n-1}{k}x^k(1-x)^{n-k-1} + \sum_{k=0}^{n-1}\binom{n-1}{k}x^k(1-x)^{n-1-k}\right]\\
&= \frac{x}{n}\left[(n-1)\sum_{k=0}^{n-1}\binom{n-1}{k}x^k(1-x)^{n-k-1} + B_{n-1}(1;x)\right]\\
&= \frac{x}{n}\left[(n-1)B_{n-1}(x;x)+1\right]\\
&= \frac{x}{n}\left[(n-1)x+1\right] = \frac{(n-1)x^2+x}{n}.
\end{aligned}$$

□

The following theorem was proved by S. N. Bernstein [7].

Theorem 3.1.10 *Let f be bounded on* $[0,1]$. *Then*

$$\lim_{n\to\infty} B_n(f;x) = f(x) \tag{3.5}$$

at any point $x \in [0,1]$ *where* f *is continuous. If* $f \in C[0,1]$, *then the convergence is uniform.*

Proof. Using the identities obtained in Lemmas 3.1.6, 3.1.7, and 3.1.9, we get

$$\begin{aligned}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} &= \sum_{k=0}^{n}\frac{k^2}{n^2}\binom{n}{k}x^k(1-x)^{n-k} - 2x\sum_{k=0}^{n}\frac{k}{n}\binom{n}{k}x^k(1-x)^{n-k}\\ &\quad + x^2\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\\ &= B_n(x^2;x) - 2xB_n(x;x) + x^2B_n(1;x)\\ &= \frac{(n-1)x^2+x}{n} - x^2 = \frac{x(1-x)}{n}.\end{aligned}$$

Before we proceed, let us note that for any $\delta > 0$ and $0 \leqslant x \leqslant 1$, we have

$$\sum_{\left|\frac{k}{n}-x\right|\geqslant\delta}\binom{n}{k}x^k(1-x)^{n-k} \leqslant \frac{1}{4n\delta^2}. \tag{3.6}$$

This notation means that the sum is taken over those values of $k = 0, 1, \ldots, n$ for which $\left|\frac{k}{n}-x\right| \geqslant \delta$. To prove (3.6), note that $\left|\frac{k}{n}-x\right| \geqslant \delta$ implies $\frac{1}{\delta^2}(\frac{k}{n}-x)^2 \geqslant 1$. Hence

$$\begin{aligned}\sum_{\left|\frac{k}{n}-x\right|\geqslant\delta}\binom{n}{k}x^k(1-x)^{n-k} &\leqslant \frac{1}{\delta^2}\sum_{\left|\frac{k}{n}-x\right|\geq\delta}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}\\ &\leqslant \frac{1}{\delta^2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}\\ &= \frac{x(1-x)}{n\delta^2} \leqslant \frac{1}{4n\delta^2}.\end{aligned}$$

The last inequality follows since $x(1-x) \leqslant \frac{1}{4}$ for all x. Using the result of Lemma 3.1.6, one can write

$$f(x) = \sum_{k=0}^{n} f(x)\binom{n}{k}x^k(1-x)^{n-k},$$

so that

$$\begin{aligned}|f(x) - B_n(f;x)| &= \sum_{k=0}^{n}\left|f(x) - f\left(\frac{k}{n}\right)\right| x^k(1-x)^{n-k}\\ &= \sum_{\left|\frac{k}{n}-x\right|<\delta}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}\\ &\quad + \sum_{\left|\frac{k}{n}-x\right|\geqslant\delta}\left|f(x) - f\left(\frac{k}{n}\right)\right|\binom{n}{k}x^k(1-x)^{n-k}.\end{aligned}$$

The function f is assumed to be bounded in $[0,1]$. Hence, there is $M > 0$ such that $|f(x)| \leqslant M$ and $|f(x) - f(y)| \leqslant 2M$ for any $x, y \in [0,1]$.

Let x be a point of continuity of f. Then for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Thus, using these estimates and using this δ in (3.6), we have

$$\begin{aligned}
|f(x) - B_n(f;x)| &\leqslant \sum_{|\frac{k}{n}-x|<\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} \\
&\qquad + \sum_{|\frac{k}{n}-x|\geqslant\delta} \left| f(x) - f\left(\frac{k}{n}\right) \right| \binom{n}{k} x^k (1-x)^{n-k} \\
&\leqslant \varepsilon \sum_{|\frac{k}{n}-x|<\delta} \binom{n}{k} x^k (1-x)^{n-k} + 2M \sum_{|\frac{k}{n}-x|\geqslant\delta} \binom{n}{k} x^k (1-x)^{n-k} \\
&\leqslant \varepsilon B_n(1;x) + \frac{2M}{4n\delta^2} \\
&= \varepsilon + \frac{M}{2n\delta^2}
\end{aligned}$$

From this inequality, we see that $|f(x) - B_n(f;x)| \leqslant 2\varepsilon$ for sufficiently large n. Since ε is arbitrary, (3.5) follows.

Suppose now that $f \in C[0,1]$, then f is uniformly continuous there. Given an $\varepsilon > 0$, we can find a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ for all x, y in $[0,1]$ satisfying $|x - y| < \delta$. The above inequality holds independently of the x selected and the convergence to $f(x)$ is uniform in $[0,1]$. □

Corollary 3.1.11 *If $f \in C[0,1]$, then given an $\varepsilon > 0$, we have for all n large enough*

$$|f(x) - B_n(f;x)| \leqslant \varepsilon, \quad x \in [0,1].$$

Corollary 3.1.12 *Let $f \in C[a,b]$. Then given $\varepsilon > 0$ we can find a polynomial $p(x)$ such that*

$$|f(x) - p(x)| \leqslant \varepsilon \quad \textit{for} \quad x \in [a,b].$$

Proof. Let $g(y) = f(a + (b-a)y)$. Clearly, $g \in C[0,1]$. Then, by Corollary 3.1.11, there is a polynomial $r(y)$ such that

$$|g(y) - r(y)| \leqslant \varepsilon \quad \text{for} \quad y \in [0,1].$$

Set

$$p(x) := r\left(\frac{x-a}{b-a}\right)$$

which is a polynomial in x and the required inequality follows. □

Lemma 3.1.13 *There is a constant c independent of n such that for $x \in [0,1]$,*

$$\sum_{\left|\frac{k}{n}-x\right| \geqslant n^{-1/4}} \binom{n}{k} x^k (1-x)^{n-k} \leqslant \frac{c}{n^{3/2}}$$

Proof. Set

$$S_m(x) = \sum_{k=0}^{n} (k-nx)^m \binom{n}{k} x^k (1-x)^{n-k}. \tag{3.7}$$

We have already established by Lemmas 3.1.6, 3.1.7 and 3.1.9, that $S_0(x) = 1$, $S_1(x) = 0$, and $S_2(x) = nx(1-x)$, respectively. Differentiating (3.7) with respect to x, we have

$$\begin{aligned} S'_m(x) &= \sum_{k=0}^{n} -mn(k-nx)^{m-1}\binom{n}{k} x^k(1-x)^{n-k} \\ &\quad + \sum_{k=0}^{n}(k-nx)^m \binom{n}{k}\left[kx^{k-1}(1-x)^{n-k} - (n-k)x^k(1-x)^{n-k-1}\right] \\ &= -mn\sum_{k=0}^{n}(k-nx)^{m-1}\binom{n}{k}x^k(1-x)^{n-k} \\ &\quad + \sum_{k=0}^{n}(k-nx)^m\binom{n}{k}x^k(1-x)^{n-k}\left(\frac{k}{x} - \frac{n-k}{1-x}\right) \\ &= -mnS_{m-1}(x) + \frac{1}{x(1-x)}\sum_{k=0}^{n}(k-nx)^{m+1}\binom{n}{k}x^k(1-x)^{n-k} \\ &= -mnS_{m-1}(x) + \frac{S_{m+1}(x)}{x(1-x)}, \end{aligned}$$

from which one gets

$$S_{m+1}(x) = x(1-x)\left[S'_m(x) + mnS_{m-1}(x)\right]. \tag{3.8}$$

Letting $m = 5$ in (3.8) one can obtain that

$$S_6(x) = P(x)n^3 + Q(x)n^2 + R(x)n,$$

where P, Q and R are polynomials in x. Hence, there exists a constant c, independent of n, such that

$$|S_6(x)| \leqslant cn^3.$$

Consider $\left|\frac{k}{n}-x\right| \geqslant \frac{1}{n^{1/4}}$. Then $\left(\frac{k-nx}{n}\right)^6 \geqslant \frac{1}{n^{3/2}}$, and $\frac{(k-nx)^6}{n^6} \cdot n^{3/2} \geqslant 1$. Hence, $1 \leqslant \frac{(k-nx)^6}{n^{9/2}}$.

Now,

$$\begin{aligned}
\sum_{\left|\frac{k}{n}-x\right| \geqslant n^{-1/4}} \binom{n}{k} x^k(1-x)^{n-k} &\leqslant \sum_{\left|\frac{k}{n}-x\right| \geqslant n^{-1/4}} \frac{(k-nx)^6}{n^{9/2}} \binom{n}{k} x^k(1-x)^{n-k} \\
&= \frac{1}{n^{9/2}} \sum_{\left|\frac{k}{n}-x\right| \geqslant n^{-1/4}} (k-nx)^6 \binom{n}{k} x^k(1-x)^{n-k} \\
&\leqslant \frac{1}{n^{9/2}} \sum_{k=0}^{n} (k-nx)^6 \binom{n}{k} x^k(1-x)^{n-k} \\
&= \frac{1}{n^{9/2}} S_6(x) \leqslant \frac{c}{n^{3/2}},
\end{aligned}$$

as claimed. ☐

Theorem 3.1.14 (Voronovskaya) *If f is bounded and twice differentiable on* $[0,1]$*, then for all* $x \in [0,1]$,

$$\lim_{n\to\infty} n[B_n(f;x) - f(x)] = \frac{1}{2}x(1-x)f''(x).$$

Proof. We have

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(x)}{2}(y-x)^2 + S(y)(y-x)^2,$$

where $\lim_{y\to x} S(y) = 0$. Set $y = \frac{k}{n}$

$$f\left(\frac{k}{n}\right) = f(x) + f'(x)\left(\frac{k}{n}-x\right) + \frac{f''(x)}{2}\left(\frac{k}{n}-x\right)^2 + S\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right)^2$$

Multiplying both sides by $\binom{n}{k}x^k(1-x)^{n-k}$ and summing from $k=0$ to n, we obtain:

$$\begin{aligned}
\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} = f(x)\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} \\
+ f'(x)\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)\binom{n}{k}x^k(1-x)^{n-k} \\
+ \frac{f''(x)}{2}\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k} \\
+ \sum_{k=0}^{n} S\left(\frac{k}{n}\right)\left(\frac{k}{n}-x\right)^2\binom{n}{k}x^k(1-x)^{n-k}
\end{aligned}$$

$$= f(x)S_0(x) + \frac{f'(x)}{n} S_1(x) + \frac{f''(x)}{2n^2} S_2(x) + \sum_{k=0}^{n} S\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k},$$

where S_m is defined by (3.7). Therefore,

$$B_n(f;x) = f(x) + \frac{1}{2n} f''(x)x(1-x) + \sum_{k=0}^{n} S\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k}.$$

Let us denote the third term in the right hand side by $\tilde{S}$. Now, since $\lim_{y\to x} S(y) = 0$, for any $\varepsilon > 0$, we can find n large enough so that $|y - x| < \frac{1}{n^{1/4}}$ implies $|S(y)| < \varepsilon$. Then

$$\begin{aligned} \left|\tilde{S}\right| &\leqslant \sum_{\left|\frac{k}{n}-x\right|<n^{-1/4}} \left|S\left(\frac{k}{n}\right)\right| \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \\ &\quad + \sum_{\left|\frac{k}{n}-x\right|\geqslant n^{-1/4}} \left|S\left(\frac{k}{n}\right)\right| \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} \\ &\leqslant \varepsilon \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1-x)^{n-k} + M \sum_{\left|\frac{k}{n}-x\right|\geqslant n^{-1/4}} \binom{n}{k} x^k (1-x)^{n-k}, \end{aligned}$$

where $M = \sup_{0\leqslant y\leqslant 1} S(y)(y-x)^2$. Using (3.7) and Lemma 3.1.13, one gets

$$\left|\tilde{S}\right| \leqslant \frac{\varepsilon x(1-x)}{n} + \frac{Mc}{n^{3/2}}.$$

Therefore,

$$\left| n\left[B_n(f;x) - f(x)\right] - \frac{x(1-x)}{2} f''(x) \right| = \left|n\tilde{S}\right| \leqslant \varepsilon x(1-x) + \frac{Mc}{n^{1/2}}.$$

Since ε is arbitrary, the conclusion follows. ☐

3.2 Iterates of Bernstein operators

In this section, some known results on the behavior of the iterates B_n^k as $k \to \infty$ are presented.

Definition 3.2.1 *Let B_n be Bernstein operator and $f : [0,1] \to \mathbb{R}$. The k-th iterate of B_n is defined by*

$$B_n^k(f;x) = B_n\left(B_n^{k-1}(f;x)\right), \quad k = 2, 3, \ldots,$$

where $B_n^1(f;x) = B_n(f;x)$.

The iterates of the Bernstein operators were studied in several papers, starting with [11]. Several results using different methods have been obtained by a number of authors. Below, two of their proofs are presented. The first proof uses the Banach fixed point theorem for contraction mappings. This approach was suggested by A. Rus in [23]. The second proof, attributed to Abel and Ivan, see [1], is based on some elementary properties of the Bernstein polynomials and positive operators.

Theorem 3.2.2 (Kelisky-Rivlin) *Given* $f \in C[0,1]$, *denote* $L_f(x) = f(0) + [f(1) - f(0)]x$. *If* $n \in \mathbb{N}$ *is fixed, then, for all* $f \in C[0,1]$, *we have*

$$\lim_{k\to\infty} B_n^k(f;x) = L_f(x),$$

and the convergence is uniform on $[0,1]$.

Proof. **1.** Consider the Banach space $C[0,1]$, with $\|f(x)\| = \max\limits_{x\in[0,1]} |f(x)|$, and, for $\alpha,\beta \in \mathbb{R}$, set $X_{\alpha,\beta} = \{f \in C[0,1] : f(0) = \alpha, f(1) = \beta\}$.

First, let us show that $X_{\alpha,\beta}$ is a closed subset of $C[0,1]$ and, as such, it is complete with respect to max norm. Indeed, if we take $\{f_n\} \in X_{\alpha,\beta}$ such that $f_n(x) \to f(x)$ uniformly on $[0,1]$, then

$$f(0) = \lim_{n\to\infty} f_n(0) = \alpha, \quad \text{and} \quad f(1) = \lim_{n\to\infty} f_n(1) = \beta,$$

that is, $f \in X_{\alpha,\beta}$.

Next, we observe that $X_{\alpha,\beta}$ is invariant under B_n whatever $\alpha,\beta \in \mathbb{R}$ and $n \in \mathbb{N}$. This is an immediate consequence of the end-point interpolation property given in (3.4).

Finally, let us prove that B_n is a contraction on $X_{\alpha,\beta}$ for all $\alpha,\beta \in \mathbb{R}$. Let $f, g \in X_{\alpha,\beta}$. Then

$$\begin{aligned}|B_n(f;x) - B_n(g;x)| &= |B_n(f-g)(x)| \\ &= \left|(f-g)(0)(1-x)^n + (f-g)\left(\frac{1}{n}\right)x(1-x)^{n-1}\right. \\ &\qquad \left. + \cdots + (f-g)\left(\frac{n-1}{n}\right)x^{n-1}(1-x) + (f-g)(1)x^n\right| \\ &= \left|(f-g)\left(\frac{1}{n}\right)\binom{n}{1}x(1-x)^{n-1}\right.\end{aligned}$$

$$\begin{aligned}
&+\cdots+(f-g)\left(\frac{n-1}{n}\right)\binom{n}{n-1}x^{n-1}(1-x)\Bigg| \\
&\leqslant \|f-g\|\left|\binom{n}{1}x(1-x)^{n-1}+\cdots+\binom{n}{n-1}x^{n-1}(1-x)\right| \\
&= \|f-g\|\cdot\left|\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}-(1-x)^n-x^n\right| \\
&= \|f-g\|\cdot|1-x^n-(1=x)^n| \\
&\leqslant \|f-g\|\cdot\left|1-\frac{1}{2^{n-1}}\right|.
\end{aligned}$$

So, B_n is a contraction, and thence, by the Banach fixed point theorem it has a unique fixed point, say $f^* \in X_{\alpha,\beta}$ so that $B_n^k(f;x) \to f^*$ for all $f \in X_{\alpha,\beta}$.

On the other hand, $f \in X_{\alpha,\beta}$ implies $L_f(x) = \alpha+(\beta-\alpha)x$. Since L_f is linear, $B_n(L_f(x)) = L_f(x)$, whence $f^* = L_f(x)$, that is

$$\lim_{k\to\infty} B_n^k(f;x) = f(0) + (f(1)-f(0))x,$$

which completes the proof. ☐

Proof. **2.** Notice that, for $n = 1$, the statement is obvious. Let $n \geqslant 2$. To begin with, consider the k-th iterate of B_n applied to $\psi(x) := x(1-x)$. Using Lemmas 3.1.7,3.1.9 and the linearity property, one has

$$B_n(\psi;x) = B_n(x;x) - B_n(x^2;x) = x - \left(x^2 + \frac{x(1-x)}{n}\right) = (1-1/n)\,\psi(x).$$

Applying B_n once more, we have:

$$B_n^2(\psi;x) = B_n(B_n(\psi;x)) = B_n\left(\psi(x)\left(1-\frac{1}{n}\right)\right) = \left(1-\frac{1}{n}\right)^2 \psi(x).$$

So, in general,

$$B_n^k(\psi;x) = \left(1-\frac{1}{n}\right)^k \psi(x).$$

Now, for any $f \in C[0,1]$, it can be readily seen that $B_n(f;x)-B_1(f;x)$ is a polynomial of degree at most n, which vanishes at $x = 0$ and $x = 1$ by the end-point interpolation property. Therefore, $B_n(f;x)-B_1(f;x) = x(1-x)R_f(x) = \psi(x)R_f$ for some polynomial R_f. Thus,

$$|B_n(f;x) - B_1(f;x)| \leqslant M_f\psi(x), \tag{3.9}$$

where $M_f = \max_{x\in[0,1]} |R_f(x)|$. By repeatedly applying the operator B_n to inequality (3.9) and taking advantage of the positivity of the operators B_n (see Proposition 3.1.4 (iii)), we obtain

$$|B_n^k(f;x) - B_1(f;x)| = |B_n^{k-1}(B_n(f;x) - B_1(f;x))| \leqslant \left(1 - \frac{1}{n}\right)^{k-1} M_f \psi(x).$$

Taking the limit as $k \to \infty$, one obtains $\lim_{k\to\infty} B_n^k(f;x) = B_1(f;x) = L_f(x)$, as stated. $\square$

Definition 3.2.3 *Let (X, d) be a metric space and $A : X \to X$ an operator. The operator A is called a weakly Picard operator (WPO) if the sequence $\{A^m(x)\}_{m\in\mathbb{N}}$ converges for all $x \in X$, and the limit (which may depend on x) is a fixed point of A.*

In terms of WPOs, we can reformulate the Theorem 3.2.2 as follows:

Theorem 3.2.4 *The Bernstein operator B_n is WPO and*

$$\lim_{k\to\infty} B_n^{\infty}(f;x) = f(0) + [f(1) - f(0)]x.$$

CHAPTER 4

q-BERNSTEIN POLYNOMIALS

4.1 Preliminaries

In this chapter, the notation and definitions from [3, Chapter 10] will be used.

Definition 4.1.1 *Let* $q > 0$. *For any* $n \in \mathbb{N}_0$, *the q-integer* $[n]_q$ *is defined by*

$$[n]_q := 1 + q + \cdots + q^{n-1}, \quad (n \in \mathbb{N}), \qquad [0]_q := 0$$

and the q-factorial $[n]_q!$ *by*

$$[n]_q! = [1]_q[2]_q \cdots [n]_q, \quad (n \in \mathbb{N}), \qquad [0]_q! := 1.$$

For integers $0 \leqslant k \leqslant n$, the $q-$binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Also, we employ the following standard notation of the q-calculus:

$$(a;q)_0 := 1, \quad (a;q)_k = \prod_{s=0}^{k-1}(1 - aq^s), \quad (a;q)_\infty := \prod_{s=0}^{\infty}(1 - aq^s).$$

Clearly, when $q = 1$, one has

$$[n]_1 = n, \quad [n]_1! = n!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}.$$

It is known that the binomial coefficients satisfy the well-known Pascal's rule

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k},$$

for inters k and n such that $1 \leqslant k \leqslant n$. Is there a similar rule for the q-Binomial coefficients? The answer is yes, as seen in the next result.

Lemma 4.1.2 *For integers k and n,*

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q, \qquad 1 \leqslant k \leqslant n. \tag{4.1}$$

Proof.

$$\begin{aligned} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q &= \frac{[n-1]_q!}{[k-1]_q![n-k]_q!} + q^k \frac{[n-1]_q!}{[k]_q[n-1-k]_q!} \\ &= \frac{[k]_q[n-1]_q! + q^k[n-k]_q[n-1]_q!}{[k]_q![n-k]_q!} \\ &= \frac{[n-1]_q!([k]_q + q^k[n-k]_q)}{[k]_q![n-k]_q!} \\ &= \frac{[n]_q!}{[k]_q![n-k]_q!} = \begin{bmatrix} n \\ k \end{bmatrix}_q. \end{aligned}$$

□

Remark 4.1.3 *Since* $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$, *the equality* (4.1) *can also be written as*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q,$$

upon replacing k by n − k.

Lemma 4.1.4 *For integers k and n*

$$\frac{[k]_q}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q, \qquad 1 \leqslant k \leqslant n. \tag{4.2}$$

Proof.

$$\frac{[k]_q}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[k]_q}{[n]_q} \frac{[n]_q!}{[k]_q[n-k]_q!} = \frac{[n-1]_q!}{[k-1]_q![n-k]_q!} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

□

Theorem 4.1.5 *If* $\lim\limits_{n\to\infty} q_n = 1$, *then* $\lim\limits_{n\to\infty} [n]_{q_n} = +\infty$. *In addition the following estimate holds:*

$$\max\left\{1 - q_n, \frac{1}{n}\right\} < \frac{1}{[n]_{q_n}} \leqslant 2 \max\left\{1 - q_n, \frac{1}{n}\right\}.$$

Proof. Case 1: If $q_n \geqslant 1$, then $[n]_{q_n} \geqslant n$, and the statement is obvious .

Case 2 : Let $0 < q_n < 1$. Since $[n]_{q_n} = 1 + q_n + \ldots + q_n^{n-1}$, we have

$$(1 - q_n)[n]_{q_n} = 1 - q_n^n. \tag{4.3}$$

Clearly, $[n]_{q_n} > nq^n$, that is, $q_n^n < \frac{[n]_{q_n}}{n}$. Together with (4.3), this yields

$$(1 - q_n)[n]_{q_n} > 1 - \frac{[n]_{q_n}}{n}, \quad \text{or} \quad \left(1 - q_n + \frac{1}{n}\right)[n]_{q_n} > 1.$$

Hence,

$$[n]_{q_n} > \frac{1}{1 - q_n + \frac{1}{n}} \to +\infty \quad \text{as} \quad n \to \infty.$$

What is more, we have the estimate:

$$\frac{1}{[n]_{q_n}} < 1 - q_n + \frac{1}{n} \leqslant 2 \max\left\{1 - q_n, \frac{1}{n}\right\}. \tag{4.4}$$

Since $[n]_{q_n} < n$ and $(1 - q_n)[n]_{q_n} = 1 - q_n^n < 1$, we have

$$\frac{1}{[n]_{q_n}} > \frac{1}{n} \quad \text{and} \quad \frac{1}{[n]_{q_n}} > 1 - q_n,$$

that is,

$$\frac{1}{[n]_{q_n}} > \max\left\{1 - q_n, \frac{1}{n}\right\}. \tag{4.5}$$

Combining (4.4) and (4.5), we get the required result. ☐

4.2 Definition and basic properties of the q-Bernstein polynomials

Definition 4.2.1 *Let $f : [0, 1] \to \mathbb{R}$, and $q > 0$. The q-Bernstein polynomial of f is*

$$B_{n,q}(f; x) := \sum_{k=0}^{n} f\left(\frac{[k]_q}{[n]_q}\right) p_{n,k}(q; x), \quad n = 1, 2, \ldots \tag{4.6}$$

where

$$p_{n,k}(q; x) = \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1 - q^s x), \quad n = 1, 2, \ldots$$

are the q-Bernstein basis polynomials.

Remark 4.2.2 *Empty products and sums are taken as 1 and 0, respectively.*

Proposition 4.2.3 *The q-Bernstein operators enjoy the following properties:*

(i) Linearity: For all $f, g : [0,1] \to \mathbb{R}$ *and all* $C_1, C_2 \in \mathbb{R}$ *we have*

$$B_{n,q}(C_1 f + C_2 g; x) = C_1 B_{n,q}(f; x) + C_2 B_{n,q}(g; x). \tag{4.7}$$

(ii) End-point interpolation:

$$B_{n,q}(f; 0) = f(0), \quad B_{n,q}(f; 1) = f(1). \tag{4.8}$$

(iii) Positivity:
If $q \in (0,1)$ *and* $f(x) \geqslant 0$ *for* $x \in [0,1]$, *then* $B_{n,q}(f; x) \geqslant 0$ *for all* $x \in [0,1]$.

Example 4.2.4 *The q-Bernstein polynomials* $B_{1,q}(f; x)$, $B_{2,q}(f; x)$ *and* $B_{3,q}(f; x)$:

For $n = 1$ *in* (4.6) *one writes*

$$\begin{aligned}
B_{1,q}(f; x) &= \sum_{k=0}^{1} f\left(\frac{[k]_q}{[1]_q}\right)\begin{bmatrix}1\\k\end{bmatrix}_q x^k \prod_{s=0}^{-k}(1 - q^s x)\\
&= f\left(\frac{[0]_q}{[1]_q}\right)\begin{bmatrix}1\\0\end{bmatrix}_q x^0 \prod_{s=0}^{0}(1 - q^s x) + f\left(\frac{[1]_q}{[1]_q}\right)\begin{bmatrix}1\\1\end{bmatrix}_q x \prod_{s=0}^{-1}(1 - q^s x)\\
&= f(0)(1 - x) + f(1)x,
\end{aligned}$$

and, for $n = 2$,

$$\begin{aligned}
B_{2,q}(f; x) &= \sum_{k=0}^{2} f\left(\frac{[k]_q}{[2]_q}\right)\begin{bmatrix}2\\k\end{bmatrix}_q x^k \prod_{s=0}^{1-k}(1 - q^s x)\\
&= f\left(\frac{[0]_q}{[2]_q}\right)\begin{bmatrix}2\\0\end{bmatrix}_q x^0 \prod_{s=0}^{1}(1 - q^s x)(x) + f\left(\frac{[1]_q}{[2]_q}\right)\begin{bmatrix}2\\1\end{bmatrix}_q x \prod_{s=0}^{0}(1 - q^s x)\\
&\quad + f\left(\frac{[2]_q}{[2]_q}\right)\begin{bmatrix}2\\2\end{bmatrix}_q x^2 \prod_{s=0}^{-1}(1 - q^s x)\\
&= f(0)\frac{[2]_q!}{[0]_q![2]_q!}(1 - x)(1 - qx) + f\left(\frac{1}{1+q}\right)\frac{[2]_q!}{[1]_!}x(1 - x)\\
&\quad + f(1)\frac{[2]_q!}{[1]_q![1]_q!}x^2\\
&= f(0)(1 - x)(1 - qx) + (1 + q)f\left(\frac{1}{1+q}\right)x(1 - x) + f(1)x^2,
\end{aligned}$$

and, for $n = 3$,

$$B_{3,q}(f; x) = \sum_{k=0}^{3} f\left(\frac{[k]_q}{[3]_q}\right)\begin{bmatrix}3\\k\end{bmatrix}_q x^k \prod_{s=0}^{2-k}(1 - q^s x)$$

$$\begin{aligned}&= f(0)(1-x)(1-qx)(1-q^2x)\\ &\qquad + (1+q+q^2)f\left(\frac{1}{1+q+q^2}\right)x(1-x)(1-qx)\\ &\qquad + (1+q+q^2)f\left(\frac{1+q}{1+q+q^2}\right)x^2(1-x)+f(1)x^3.\end{aligned}$$

Example 4.2.5 *For $f(x) \equiv 1$ we get*

$$\begin{aligned}B_{1,q}(1;x) &= (1-x)+x = 1,\\ B_{2,q}(1;x) &= (1-x)(1-qx)+(1+q)x(1-x)+x^2\\ &= 1-qx-x+qx^2+x-x^2+qx-qx^2+x^2 = 1,\end{aligned}$$

and,

$$\begin{aligned}B_{3,q}(1;x) = (1-x)(1-qx)(1-q^2x)&+(1+q+q^2)x(1-x)(1-qx)\\ &+(1+q+q^2)x^2(1-x)+x^3 = 1.\end{aligned}$$

Example 4.2.6 *For $f(x) = x$ we get*

$$\begin{aligned}B_{1,q}(x;x) &= x,\\ B_{2,q}(x;x) &= x(1-x)+x^2 = x,\end{aligned}$$

and,

$$B_{3,q}(x;x) = x(1-x)(1-qx)+x^2(1-x)(1+q)+x^3 = x.$$

Lemma 4.2.7 *$B_{n,q}(f;x) = 1$ for all $n \geqslant 1$ when $f(x) = 1$.*

Proof. We will use mathematical induction on n. For $n = 1$, one has the result as obtained in Example 4.2.5. Assume that $B_{m,q}(1;x) = 1$ for some $m \geqslant 1$. Then, for $n = m+1$, one has

$$\begin{aligned}B_{m+1,q}(1;x) &= \sum_{k=0}^{m+1}\begin{bmatrix}m+1\\ k\end{bmatrix}_q x^k\prod_{s=0}^{m-k}(1-q^sx)\\ &= \sum_{k=0}^{m}\begin{bmatrix}m+1\\ k\end{bmatrix}_q x^k\prod_{s=0}^{m-k}(1-q^sx)+\begin{bmatrix}m+1\\ m+1\end{bmatrix}_q x^m+1\prod_{s=0}^{m-m-1}(1-q^sx)\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_q x^k \prod_{s=0}^{m-k} (1-q^s x) \\
&= \begin{bmatrix} m+1 \\ 0 \end{bmatrix}_q x^0 \prod_{s=0}^{m} (1-q^s x) + \sum_{k=1}^{m+1} \begin{bmatrix} m+1 \\ k \end{bmatrix}_q x^k \prod_{s=0}^{(m-k)} (1-q^s x) \\
&= \prod_{s=0}^{m} (1-q^s x) + \sum_{k=0}^{m} \begin{bmatrix} m+1 \\ k+1 \end{bmatrix}_q x^{k+1} \prod_{s=0}^{m-k-1} (1-q^s x).
\end{aligned}$$

Using (4.1), one obtains

$$\begin{aligned}
B_{m+1,q}(1;x) &= \prod_{s=0}^{m} (1-q^s x) + \sum_{k=0}^{m} \left(\begin{bmatrix} m \\ k \end{bmatrix}_q + q^{k+1} \begin{bmatrix} m \\ k+1 \end{bmatrix}_q \right) x^{k+1} \prod_{s=0}^{m-k-1} (1-q^s x) \\
&= \prod_{s=0}^{m} (1-q^s x) + \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q x^{k+1} \prod_{s=0}^{m-k-1} (1-q^s x) \\
&\qquad + \sum_{k=0}^{m} \begin{bmatrix} m \\ k+1 \end{bmatrix}_q q^{k+1} x^{k+1} \prod_{s=0}^{m-k-1} (1-q^s x) \\
&= \prod_{s=0}^{m} (1-q^s x) + x + \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q q^k x^k \prod_{s=0}^{m-k} (1-q^s x) - \begin{bmatrix} m \\ 0 \end{bmatrix}_q \prod_{s=0}^{m} (1-q^s x) \\
&\qquad + \begin{bmatrix} m \\ m+1 \end{bmatrix}_q q^{m+1} x^{m+1} \prod_{s=0}^{-1} (1-q^s x) \\
&= \prod_{s=0}^{m} (1-q^s x) + x + \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q q^k x^k \prod_{s=0}^{m-k} (1-q^s x) - \prod_{s=0}^{m} (1-q^s x) + 0 \\
&= x + \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q t^k \prod_{s=0}^{m-k} (1-q^{s-1} t) \qquad (\text{ where } \quad t = xq) \\
&= x + \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q t^k \prod_{s=-1}^{m-1-k} (1-q^s t) \\
&= x + \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q t^k \left(1-q^{-1} t\right) \prod_{s=0}^{m-1-k} (1-q^s t) \\
&= x + \left(1-q^{-1} t\right) \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q t^k \prod_{s=0}^{m-1-k} (1-q^s t) \\
&= x + \left(1-q^{-1} t\right) B_{m,q}(1;t) \\
&= x + 1 - q^{-1} t = 1
\end{aligned}$$

which completes the proof. □

Lemma 4.2.8 *$B_{n,q}(f;x) = x$ for all $n \geqslant 1$ if $f(x) = x$.*

Proof. Again we will use the induction on n. The result for $n = 1$ can be seen in Example 4.2.6. Assume that $B_{m,q}(x; x) = x$ for some $m \geqslant 1$. Then,

$$
\begin{aligned}
B_{m+1,q}(f; x) &= \sum_{k=0}^{m+1} \frac{[k]_q}{[m+1]_q} \begin{bmatrix} m+1 \\ k \end{bmatrix}_q x^k \prod_{s=0}^{m-k}(1 - q^s x) \\
&= \frac{[0]_q}{[m+1]_q} \begin{bmatrix} m+1 \\ 0 \end{bmatrix}_q x^0 \prod_{s=0}^{m}(1 - q^s x) \\
&\quad + \sum_{k=1}^{m+1} \frac{[k]_q}{[m+1]_q} \begin{bmatrix} m+1 \\ k \end{bmatrix}_q x^k \prod_{s=0}^{m-k}(1 - q^s x) \\
&= \sum_{k=1}^{m+1} \frac{[k]_q}{[m+1]_q} \begin{bmatrix} m+1 \\ k \end{bmatrix}_q x^k \prod_{s=0}^{m-k}(1 - q^s x) \\
&= \sum_{k=0}^{m} \frac{[k+1]_q}{[m+1]_q} \begin{bmatrix} m+1 \\ k+1 \end{bmatrix}_q x^{k+1} \prod_{s=0}^{m-k-1}(1 - q^s x).
\end{aligned}
$$

Using (4.2) in the previous equation, one gets

$$
\begin{aligned}
B_{m+1,q}(f; x) &= \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q x^{k+1} \prod_{s=0}^{m-k-1}(1 - q^s x) \\
&= x \sum_{k=1}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q x^{k} \prod_{s=0}^{m-k-1}(1 - q^s x) \\
&= x B_{m,q}(1; x) = x,
\end{aligned}
$$

which completes the proof. □

Similar to the case of Bernstein operators, using the preceding two lemmas together with the linearity property, one can get the following result.

Corollary 4.2.9 *For any* $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ *and* $q > 0$, *we have* $B_{n,q}(ax + b; x) = ax + b$.

Lemma 4.2.10 *If* $f(x) = x^2$, *then* $B_{n,q}(x^2; x) = x^2 + \frac{x(1-x)}{[n]_q}$.

Proof.

$$
\begin{aligned}
B_{n,q}(x^2; x) &= \sum_{k=0}^{n} \frac{[k]_q^2}{[n]_q^2} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1 - q^s x) \\
&= \sum_{k=1}^{n} \frac{[k]_q}{[n]_q} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1 - q^s x)
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{k=0}^{n-1} \frac{[k+1]_q}{[n]_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^{k+1} \prod_{s=0}^{n-2-k} (1-q^s x) \\
&= \frac{x}{[n]_q} \left[\sum_{k=0}^{n-1} [k]_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-2-k} (1-q^s x) + \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-2-k} (1-q^s x) \right] \\
&= \frac{x}{[n]_q} \left\{ [n-1]_q \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-2-k} (1-q^s x) + B_{n-1,q}(1;x) \right\} \\
&= \frac{x}{[n]_q} \left[(n-1) B_{n-1,q}(x;x) + x(n-1) + 1 \right] \\
&= \frac{x}{[n]_q} \left[[n-1]_q x + 1 \right] \\
&= \frac{[n-1]_q x^2 + x}{[n]_q} \\
&= x^2 + \frac{x(1-x)}{[n]_q}.
\end{aligned}$$

□

4.3 Convergence of the q-Bernstein polynomials in the case $q \in (0,1)$

Theorem 4.3.1 *Let $\{q_n\}$ denote a sequence such that $0 < q_n < 1$ and $q_n \to 1$ as $n \to \infty$. Then, for any $f \in C[0,1]$, $B_{n,q_n}(f;x)$ converges uniformly to $f(x)$ on $[0,1]$.*

Proof. Using the identities obtained in Lemmas 4.2.7, 4.2.8 and 4.2.10, we get

$$\begin{aligned}
\sum_{k=0}^{n} \left(\frac{[k]_q}{[n]_q} - x \right)^2 \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1-q^s x) &= \sum_{k=0}^{n} \frac{[k]_q^2}{[n]_q^2} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1-q^s x) \\
&\quad - 2x \sum_{k=0}^{n} \frac{[k]_q}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1-q^s x) \\
&\quad + x^2 \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k} (1-q^s x) \\
&= B_{n,q}(x^2;x) - 2x B_{n,q}(x;x) + x^2 B_{n,q}(1;x) \\
&= \frac{x - x^2}{[n]_q}.
\end{aligned}$$

Note that $\left|\frac{[k]_q}{[n]_q} - x\right| \geqslant \delta$ implies $\frac{1}{\delta^2}(\frac{[k]_q}{[n]_q} - x)^2 \geqslant 1$. Hence

$$\begin{aligned}
\sum_{\left|\frac{[k]_q}{[n]_q}-x\right|\geqslant\delta} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x) &\leqslant \frac{1}{\delta^2} \sum_{\left|\frac{[k]_q}{[n]_q}-x\right|\geq\delta} \left(\frac{[k]_q}{[n]_q} - x\right)^2 \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x) \\
&\leqslant \frac{1}{\delta^2} \sum_{k=0}^{n} \left(\frac{[k]_q}{[n]_q} - x\right)^2 \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x) \\
&= \frac{x(1-x)}{[n]_q\delta^2} \leqslant \frac{1}{4[n]_q\delta^2}.
\end{aligned}$$

Since $x(1-x) \leqslant \frac{1}{4}$ for all x, the last inequality results in

$$\sum_{\left|\frac{[k]_q}{[n]_q}-x\right|\geqslant\delta} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x) \leqslant \frac{1}{4[n]_q\delta^2}. \tag{4.9}$$

Using Lemma 4.2.7, one writes

$$f(x) = \sum_{k=0}^{n} f(x) \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x),$$

so that

$$\begin{aligned}
|f(x) - B_{n,q}(f;x)| &= \sum_{k=0}^{n} \left| f(x) - f\left(\frac{[k]_q}{[n]_q}\right)\right| \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x) \\
&= \sum_{\left|\frac{[k]_q}{[n]_q}-x\right|<\delta} \left| f(x) - f\left(\frac{[k]_q}{[n]_q}\right)\right| \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x) \\
&\quad + \sum_{\left|\frac{[k]_q}{[n]_q}-x\right|\geqslant\delta} \left| f(x) - f\left(\frac{[k]_q}{[n]_q}\right)\right| \begin{bmatrix} n \\ k \end{bmatrix}_q x^k \prod_{s=0}^{n-1-k}(1-q^s x).
\end{aligned}$$

Since the function f is assumed to be continuous on $[0,1]$, it is bounded and there exists $M > 0$ such that $|f(x)| \leqslant M$ for all $x \in [0,1]$ and $|f(\alpha) - f(\beta)| \leqslant 2M$ for any two values $\alpha, \beta \in [0,1]$.

Let $x \in [0,1]$. Then for any given $\varepsilon > 0$, we can find a δ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$. Thus, using these estimates and using this δ in (4.9) , we have

$$\begin{aligned}
|f(x) - B_{n,q}(f;x)| &\leqslant \sum_{\left|\frac{[k]_q}{[n]_q}-x\right|<\delta} \left| f(x) - f\left(\frac{[k]_q}{[n]_q}\right)\right| \begin{bmatrix} n \\ k \end{bmatrix}_x^k \prod_{s=0}^{n-1-k}(1-q^s x) \\
&\quad + \sum_{\left|\frac{[k]_q}{[n]_q}-x\right|\geqslant\delta} \left| f(x) - f\left(\frac{[k]_q}{[n]_q}\right)\right| \begin{bmatrix} n \\ k \end{bmatrix}_x^k \prod_{s=0}^{n-1-k}(1-q^s x)
\end{aligned}$$

$$\begin{aligned}&\leqslant \varepsilon \sum_{\left|\frac{[k]_q}{[n]_q}-x\right|<\delta} \begin{bmatrix} n \\ k \end{bmatrix}_x^k \prod_{s=0}^{n-1-k}(1-q^s x) + \frac{2M}{4\delta^2[n]_q}\\ &= \varepsilon + \frac{M}{\delta^2[n]_q}.\end{aligned}$$

From this inequality, since $\{q_n\} \to 1$ as $n \to \infty$, one has $|f(x) - B_n(f;x)| \leqslant 2\varepsilon$ when n is sufficiently large. As ε is arbitrary, (4.3.1) follows.

Being continuous on $[0,1]$, f is uniformly continuous there. Therefore, the above inequality holds independently from the choice of x and, as a result, the convergence to $f(x)$ is uniform in $[0,1]$. $\square$

Lemma 4.3.2 *Let $f \in C[0,1]$. Then the series $\sum_{k=0}^{\infty} \frac{f(1-q^k)}{(q;q)_k} x^k$ converges on $[0,1)$.*

Proof. First of all note that $\lim\limits_{k\to\infty}(q;q)_k = (q;q)_\infty$. Since f is continuous on $[0,1]$ and $\lim\limits_{k\to\infty}(1-q^k) = 1$, we have $\lim\limits_{k\to\infty} f(1-q^k) = f(1)$. Now, we consider the following two cases.

Case 1: Let $f(1) \neq 0$. Use the ratio test:

$$\begin{aligned}\rho &= \lim_{k\to\infty}\left|\frac{f(1-q^{k+1})x^{k+1}}{(q;q)_{k+1}} \cdot \frac{(q;q)_k}{f(1-q^k)x^k}\right|\\ &= |x| \lim_{k\to\infty}\left|\frac{f(1-q^{k+1})}{f(1-q^k)(1-q^{k+1})}\right| = |x|\frac{f(1)}{f(1)} = |x|.\end{aligned}$$

Thus, the series converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. For $x = 1$ the series becomes $\sum_{k=0}^{\infty} \frac{f(1-q^k)}{(q;q)_k}$. Since

$$\lim_{k\to\infty}\left|\frac{f(1-q^k)}{(q;q)_k}\right| = \frac{f(1)}{(q;q)_\infty} \neq 0,$$

the series $\sum_{k=0}^{\infty} \frac{f(1-q^k)}{(q;q)_k}$ diverges by the general term test. Thus, the series $\sum_{k=0}^{\infty} \frac{f(1-q^k)}{(q;q)_k} x^k$ converges for $x \in [0,1)$ when $f(1) \neq 0$.

Case 2: Let $f(1) = 0$. Since f is continuous on $[0,1]$, it is bounded there. That is, there exists a positive number M such that $|f(x)| \leqslant M$ for all $x \in [0,1]$. In particular $|f(1-q^k)| \leqslant M$ for all k. Moreover, $(q;q)_k \geqslant (q;q)_\infty$. Therefore,

$$\left|\frac{f(1-q^k)x^k}{(q;q)_k}\right| \leqslant \frac{M}{(q;q)_\infty}|x^k| = M_1|x^k|,$$

where $M_1 = M/(q;q)_\infty$. Since the geometric series $\sum_{k=0}^{\infty} M_1|x|^k$ converges when $|x| < 1$, using the Comparison Test for series, we conclude that the series $\sum_{k=0}^{\infty} \frac{f(1-q^k)}{(q;q)_k} x^k$ converges for $x \in [0,1)$. $\square$

For the sequel, we need the following definition.

Definition 4.3.3 *For all* $q \in (0,1)$, *and* $f \in C[0,1]$, *the limit q-Bernstein operator is defined by* $f \mapsto B_{\infty,q}f$, *where*

$$B_{\infty,q}(f;x) = \begin{cases} \sum_{k=0}^{\infty} f(1-q^k)p_{\infty,k}(q;x) & \text{if } x \in [0,1) \\ f(1) & \text{if } x = 1. \end{cases} \tag{4.10}$$

where

$$p_{\infty,k}(q;x) = \frac{x^k (x;q)_\infty}{(q;q)_k}. \tag{4.11}$$

Lemma 4.3.4 *For all* $k = 0, 1, \ldots,$

$$\lim_{n\to\infty} p_{n,k}(q;x) = p_{\infty,k}(q;x) \quad \textit{uniformly on} \quad [0,1].$$

Proof. First of all, note that

$$\begin{aligned} 0 &\leqslant \prod_{s=0}^{n-1-k}(1-q^s x) - \prod_{s=0}^{\infty}(1-q^s x) \\ &\leqslant 1 - \prod_{s=n-k}^{\infty}(1-q^s x) \leqslant 1 - \prod_{s=n-k}^{\infty}(1-q^s) \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

uniformly on $[0,1]$. Note also that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q x^k \to \frac{x^k}{(q;q)_k} \quad \text{as} \quad n \to \infty,$$

uniformly on $[0,1]$. Thus, $p_{n,k}(q;x) \to p_{\infty,k}(q;x)$ uniformly on $[0,1]$. $\square$

Theorem 4.3.5 *Let* $f \in C[0,1]$. *Then* $B_{n,q}(f;x) \to B_{\infty,q}(f;x)$, *as* $n \to \infty$ *uniformly for* $x \in [0,1]$.

Proof. It suffices to prove the uniform convergence for $x \in [0,1)$ due to the end-point interpolation property.

Since f is continuous, we have

$$\lim_{n\to\infty} f\left(\frac{[k]_q}{[n]_q}\right) = f\left(\lim_{n\to\infty}\frac{[k]_q}{[n]_q}\right) = f\left(\lim_{n\to\infty}\frac{1-q^k}{1-q^n}\right) = f(1-q^k). \tag{4.12}$$

In addition, by Euler's Identity (cf. [3, Corollary 10.2.2.(a)]) we have

$$\sum_{k=0}^{\infty} p_{\infty,k}(q;x) = 1, \qquad \text{for} \quad x \in [0,1). \tag{4.13}$$

To prove the statement, we choose $a \in (0,1)$ in such a way that $|f(t) - f(1)| < \varepsilon/3$ for $a \leqslant t \leqslant 1$. Let N be a positive integer satisfying the condition $1 - q^{N+1} \geqslant a$.

We estimate the difference

$$\Delta := |B_{n,q}(f;x) - B_{\infty,q}(f;x)|$$

for $n > N$ and $x \in [0,1)$. Using Lemma 4.2.7 and (4.13) we get

$$\begin{aligned}
\Delta &= \left|\sum_{k=0}^{n}\left(f\left(\frac{[k]_q}{[n]_q}\right) - f(1)\right) p_{n,k}(q;x) - \sum_{k=0}^{\infty}\left(f(1-q^k) - f(1)\right) p_{\infty,k}(q;x)\right| \\
&\leqslant \left|\sum_{k=0}^{N}\left(f\left(\frac{[k]_q}{[n]_q}\right) - f(1)\right) p_{n,k}(q;x) - \sum_{k=0}^{N}\left(f(1-q^k) - f(1)\right) p_{\infty,k}(q;x)\right| \\
&\quad + \sum_{k=N+1}^{n}\left|f\left(\frac{[k]_q}{[n]_q}\right) - f(1)\right| p_{n,k}(q;x) + \sum_{k=N+1}^{\infty}\left|f(1-q^k) - f(1)\right| p_{\infty,k}(q;x) \\
&=: S_1 + S_2 + S_3.
\end{aligned}$$

Using Lemma 4.3.4 and the fact $f\left(\frac{[k]_q}{[n]_q}\right) \to f(1-q^k)$ as $n \to \infty$ for all $k = 1, 2, \dots, N$, we conclude that $S_1 < \varepsilon/3$ for n sufficiently large. Using Lemma 4.2.7 and positivity of $p_{n,k}(q;x)$ we get

$$S_2 < \frac{\varepsilon}{3}\sum_{k=N+1}^{n} p_{n,k}(q;x) \leqslant \frac{\varepsilon}{3}.$$

Similarly,

$$S_3 < \frac{\varepsilon}{3}\sum_{k=N+1}^{\infty} p_{\infty,k}(q;x) \leqslant \frac{\varepsilon}{3}.$$

Thus, $\Delta < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the statement follows. □

4.4 Iterates of the q-Bernstein operator

Definition 4.4.1 *Let $B_{n,q}$ be Bernstein operator and $f : [0,1] \to \mathbb{R}$. The k-th iterate of $B_{n,q}$ is defined by*

$$B_{n,q}^{k}(f;x) = B_{n,q}\left(B_{n,q}^{k-1}(f;x)\right), \quad k = 2,3,\ldots,$$

where $B_{n,q}^{1}(f;x) = B_{n,q}(f;x)$.

The next Theorem shows that behavior of iterates for $q > 0$ is similar to the case when $q = 1$, see Theorem 3.2.2. Namely, the iterates converge to the linear function $L_f(x) = f(0) + [f(1) - f(0)]x$.

Theorem 4.4.2 *Let $n \in \mathbb{N}$ is fixed, and $q > 0$. For $f \in C[0,1]$, one has*

$$\lim_{k\to\infty} B_{n,q}^{k}(f;x) = L_f(x)$$

and the convergence is uniform on $[0,1]$.

The proof of this theorem can be found in [14, 17]. In Chapter 5, a new proof is provided which is not similar to the proofs given in [1] and [23] for the classical Bernstein operators.

CHAPTER 5

MAIN RESULTS

5.1 Convergence of the sequence $\{B_{\infty,q_n}\}_{n=0}^{\infty}$

In this chapter, some new results on the limit q-Bernstein operator defined by (4.10) will be established. Throughout the sections 5.1-5.3, it is assumed that $q_n \in (0,1)$, so that the corresponding operators B_{∞,q_n} are well-defined on $C[0,1]$.

The following definition is taken from [12, Section 4.9, Definition 4.9-1].

Definition 5.1.1 ([12]) *A sequence $\{T_n\}$ of operators on a Banach space X is said to be* **strongly operator convergent** *if $\{T_n x\}$ converges in the norm of X for every $x \in X$.*

Theorem 5.1.2 *Let $q_n \to a \in (0,1)$. Then, for every $f \in C[0,1]$, one has*

$$B_{\infty,q_n}(f;x) \to B_{\infty,a}(f;x) \quad as \quad n \to \infty$$

uniformly on $[0,1]$. In other words, $B_{\infty,a}$ is the strong operator limit of B_{∞,q_n}.

Assumption: Clearly, it can be assumed without any loss of generality that $0 < q_n \leqslant a_1 < a_2 < 1$, where a_1 and a_2 are any numbers such that $a < a_1 < a_2 < 1$. From here on, this is taken to be satisfied. Before the proof is presented, we give some auxiliary results.

5.2 Auxiliary results

Lemma 5.2.1 *The following inequalities are true:*

(i) if $|u| \leqslant \frac{1}{2}$, *then* $|\ln(1+u)| \leqslant 2|u|$;

(ii) if $|\ln t| \leqslant 1$, *then* $|t-1| \leqslant 3|\ln(t)|$.

Proof. (i) Note that the equality is true for $u = 0$. Let $u \in (0, 1/2]$ and consider $f(x) = \ln(1+x)$ for $x \in [0, u]$. Then by the Mean Value Theorem there is a point $c \in [0, u]$ such that

$$\frac{\ln(1+u) - \ln(1+0)}{u} = \frac{1}{1+c}.$$

Since $c \in [0, 1/2]$, the above equality implies that

$$\frac{2}{3} \leqslant \frac{\ln(1+u)}{u} \leqslant 1,$$

which results in $|\ln(1+u)| \leqslant |u|$ for $u \in [0, 1/2]$.

For $f(x) = \ln(1+x)$ on $x \in [u, 0]$, when $u \in [-1/2, 0)$, we get

$$1 \leqslant \frac{\ln(1+u)}{u} \leqslant 2,$$

which yields $|\ln(1+u)| \leqslant 2|u|$ for $u \in [-1/2, 0]$. Therefore,

$$|\ln(1+u)| \leqslant 2|u| \quad \text{for} \quad |u| \leqslant \frac{1}{2}.$$

(ii) Let $|\ln t| \leqslant 1$ and apply the Mean Value Theorem to the function $f(x) = e^x$ for $x \in [0, \ln t]$ whenever $t \geqslant 1$ and $x \in [\ln t, 0]$ whenever $t < 1$. Then, there is a point c such that

$$\frac{e^{\ln t} - e^0}{\ln t - 0} = e^c.$$

Since $c \in [0, 1]$, we see that

$$\left|\frac{t-1}{\ln t}\right| \leqslant e^c \leqslant e \leqslant 3,$$

which gives us $|t-1| \leqslant 3\,|\ln t|$ as claimed. ☐

Lemma 5.2.2 *The following estimate holds:*

$$\left|q_n^j - a^j\right| \leqslant C_1(a) a_2^j |q_n - a|,$$

where $C_1(a)$ *is independent from* j *and* n.

Proof. Since $q_n^j - a^j = (q_n - a)(q_n^{j-1} + q_n^{j-2}a + \ldots + a^{j-1})$, we have

$$\begin{aligned}
\left|q_n^j - a^j\right| &= \left|(q_n - a)(q_n^{j-1} + q_n^{j-2}a + \ldots + a^{j-1})\right| \\
&\leqslant |q_n - a| \left|a_1^{j-1} + a_1^{j-2}a_1 + \ldots + a_1^{j-1}\right| \\
&= |q_n - a| \, (ja_1^{j-1}) \\
&\leqslant |q_n - a| \, a_2^j \frac{ja_1^{j-1}}{a_2^j} \\
&\leqslant C_1(a) a_2^j \, |q_n - a| \, .
\end{aligned}$$

This is because the sequence $\left\{\frac{ja_1^{j-1}}{a_2^j}\right\}$ is bounded, say by $C_1(a)$. $\square$

Lemma 5.2.3 *For all $j \in \mathbb{N}$, the following estimate hold :*

$$\left|\frac{q_n^j - a^j}{1 - a^j}\right| \leqslant C_2(a) a_2^j |q_n - a|,$$

where $C_2(a)$ is independent from j and n.

Proof. Clearly,

$$\left|\frac{q_n^j - a^j}{1 - a^j}\right| \leqslant \frac{\left|q_n^j - a^j\right|}{1 - a} \leqslant \frac{C_1(a) a_2^j |q_n - a|}{1 - a} = C_2(a) a_2^j |q_n - a|,$$

by virtue of Lemma 5.2.2. $\square$

Corollary 5.2.4 *For all $j \in \mathbb{N}$ and $x \in [0, 1]$, the following inequality is valid:*

$$\left|\frac{(q_n^j - a^j)x}{1 - a^j x}\right| \leqslant C_2(a) x a_2^j |q_n - a|.$$

Proof. Since $1 - a^j x \geqslant 1 - a^j \geqslant 1 - a$, one has

$$\left|\frac{(q_n^j - a^j)x}{1 - a^j x}\right| \leqslant \left|\frac{q_n^j - a^j}{1 - a^j}\right| x \leqslant C_2(a) x a_2^j |q_n - a|,$$

as claimed. $\square$

Lemma 5.2.5 *Let $\lim\limits_{n\to\infty} q_n = a < 1$. Then, for $n \in \mathbb{N}$, one has*

$$\left|\ln\left(1 + \frac{a^j - q_n^j}{1 - a^j}\right)\right| \leqslant C_3(a) a_2^j |q_n - a| \quad \textit{for all} \quad j \in \mathbb{N}.$$

Proof. Take $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|q_n - a| \leqslant \frac{1}{2C_2(a)a_2}$. By using Lemma 5.2.1, we derive that for $n > N_1$

$$\left|\ln\left(1 + \frac{a^j - q_n^j}{1 - a^j}\right)\right| \leqslant 2C_2(a)a_2^j|q_n - a| := C_3(a)a_2^j|q_n - a|$$

for all $j \in \mathbb{N}$. □

Lemma 5.2.6 *The following estimate holds for all $n, k \in \mathbb{N}$:*

$$\left|\frac{1}{(q_n; q_n)_k} - \frac{1}{(a; a)_k}\right| \leqslant C_4(a)|q_n - a|,$$

where $C_4(a)$ is independent from n and k.

Proof. First, notice that

$$(q_n; q_n)_k = (a_1; a_1)_k > (a_1; a_1)_\infty,$$

whence

$$0 < \frac{1}{(q_n; q_n)_k} < \frac{1}{(a_1; a_1)_\infty}.$$

Therefore,

$$\begin{aligned}\left|\frac{1}{(q_n; q_n)_k} - \frac{1}{(a_1; a_1)_k}\right| &= \frac{1}{(q_n; q_n)_k}\left|1 - \frac{(q_n; q_n)_k}{(a_1; a_1)_k}\right| \\ &\leqslant \frac{1}{(a_1; a_1)_\infty}\left|\prod_{j=1}^{k}\frac{1 - q_n^j}{1 - a^j} - 1\right| \\ &= \frac{1}{(a_1; a_1)_\infty}\left|\prod_{j=1}^{k}\left(1 + \frac{a^j - q_n^j}{1 - a^j}\right) - 1\right|.\end{aligned}$$

Set $t := \prod_{j=1}^{k}\left(1 + \frac{a^j - q_n^j}{1 - a^j}\right)$ and consider

$$|\ln t| = \left|\sum_{j=1}^{k}\ln\left(1 + \frac{a^j - q_n^j}{1 - a^j}\right)\right|.$$

By Lemma 5.2.5, one has, for n large enough,

$$\begin{aligned}|\ln t| &\leqslant \sum_{j=1}^{k}\left|\ln\left(1 + \frac{a^j - q_n^j}{1 - a^j}\right)\right| \leqslant \sum_{j=1}^{k} C_3(a)a_2^j|q_n - a| \\ &\leqslant C_3(a)|q_n - a|\sum_{j=1}^{\infty} a_2^j := \tilde{C}_3(a)\,|q_n - a|.\end{aligned}$$

Thence, $|\ln t| \leqslant 1$ for n large enough and Lemma 5.2.1 implies that for those n, $|t - 1| \leqslant 3\tilde{C}_3(a)|q_n - a|$. As a result, we conclude that for all $n \in \mathbb{N}$, the Lemma 5.2.6 holds with $C_4(a) = 3\tilde{C}_3(a)/(a_1; a_1)_\infty > 0$. □

Lemma 5.2.7 *For all $x \in [0,1]$, the following estimate holds:*

$$\left|\prod_{j=1}^{\infty}(1-q_n^j x)-\prod_{j=1}^{\infty}(1-a^j x)\right| \leqslant C_5(a)x|q_n-a|.$$

Proof. First, notice that $(1-q_n^j x) \geqslant (1-a_1^j x) \geqslant (1-a_1^j)$ for all n and all $x \in [0,1]$. Hence,

$$\left|\prod_{j=1}^{\infty}(1-q_n^j x)-\prod_{j=1}^{\infty}(1-a^j x)\right| \leqslant \frac{1}{(a_1;a_1)_\infty}\left|\prod_{j=1}^{\infty}\left(1+\frac{(a^j-q_n^j)x}{1-a^j}\right)-1\right|.$$

As in the proof of Lemma 5.2.6, set:

$$t := \prod_{j=1}^{\infty}\left(1+\frac{(a^j-q_n^j)x}{1-a^j}\right),$$

and use Corollary 5.2.4 along with Lemma 5.2.5 to obtain

$$|\ln t| \leqslant \sum_{j=1}^{\infty}\left|\ln\left(1+\frac{(a^j-q_n^j)x}{1-a^j}\right)\right| \leqslant C_3(a)x|q_n-a|$$

for n large enough. Finally, one obtains:

$$\left|\prod_{j=1}^{\infty}(1-q_n^j x)-\prod_{j=1}^{\infty}(1-a^j x)\right| \leqslant \frac{1}{(a_1;a_1)_\infty}C_3(a)x|q_n-a| := C_5(a)x|q_n-a|.$$

□

Corollary 5.2.8 *If $\lim_{n\to\infty} q_n = a$, then one has:*

$$\prod_{j=1}^{\infty}(1-q_n^j x) \to \prod_{j=1}^{\infty}(1-a^j x) \quad as \quad n\to\infty$$

uniformly on $[0,1]$.

Lemma 5.2.9 *For every $k \in \mathbb{N}$, the following estimate holds:*

$$|p_{\infty,k}(q_n;x)-p_{\infty,k}(a;x)| \leqslant C_6(a)|q_n-a|,$$

and therefore $p_{\infty,k}(q_n;x) \to p_{\infty,k}(a;x)$ as $n\to\infty$ uniformly on $[0,1]$.

Proof.

$$|p_{\infty,k}(q_n;x)-p_{\infty,k}(a;x)| = x^k(1-x)\left|\frac{1}{(q_n;q_n)_k}\prod_{j=1}^{\infty}\left(1-q_n^j x\right)-\frac{1}{(a;a)_k}\prod_{j=1}^{\infty}\left(1-a^j x\right)\right|$$

$$\leqslant \left|\frac{1}{(q_n;q_n)_k} - \frac{1}{(a;a)_k}\right| \prod_{j=1}^{\infty}(1-q_n^j x)$$
$$+ \frac{1}{(a;a)_k}\left|\prod_{j=1}^{\infty}(1-q_n^j x) - \prod_{j=1}^{\infty}(1-a^j x)\right|.$$

Using the results of Lemmas 5.2.6 and 5.2.7, we obtain

$$\begin{aligned}|p_{\infty,k}(q_n;x) - p_{\infty,k}(a;x)| &\leqslant C_4(a)|q_n - a| + \frac{1}{(a;a)_k} C_5(a)x|q_n - a| \\ &\leqslant \left(C_4(a) + \frac{C_5(a)}{(a;a)_\infty}\right)|q_n - a| := C_6(a)|q_n - a|.\end{aligned}$$

□

Lemma 5.2.10 *Let N be a fixed positive integer and set:*

$$S_N(q;x) = \sum_{k=0}^{N} f(1-q_n^k)p_{\infty,k}(q_n;x).$$

Then, for any $f \in C[0,1]$, $S_N(q_n;x) \to S_N(a;x)$ as $n \to \infty$ uniform on $[0,1]$. The uniform convergence follows from the fact that $f(1-q_n^k) \to f(1-a^k)$ for each $k \in \mathbb{N}_0$ and Lemma 5.2.9. What is more, the following estimate can be derived:

$$|S_N(q_n;x) - S_N(a;x)| \leqslant \omega_f(C_1(a)|q_n - a|) + (N+1)M_f C_5(a)|q_n - a|.$$

Proof. Consider

$$\begin{aligned}|S_N(q_n;x) - S_N(a;x)| &= \left|\sum_{k=0}^{N} f(1-q_n^k)p_{\infty,k}(q_n;x) - \sum_{k=0}^{N} f(1-a^k)p_{\infty,k}(a;x)\right| \\ &= \left|\sum_{k=0}^{N} f(1-q_n^k)p_{\infty,k}(q_n;x) - \sum_{k=0}^{N} f(1-q_n^k)p_{\infty,k}(a;x)\right. \\ &\quad \left. + \sum_{k=0}^{N} f(1-q_n^k)p_{\infty,k}(a;x) - \sum_{k=0}^{N} f(1-a^k)p_{\infty,k}(a;x)\right|.\end{aligned}$$

Using the triangle inequality, we derive

$$\begin{aligned}|S_N(q_n;x) - S_N(a;x)| &\leqslant \left|\sum_{k=0}^{N} f(1-q_n^k)p_{\infty,k}(q_n;x) - \sum_{k=0}^{N} f(1-q_n^k)p_{\infty,k}(a;x)\right| \\ &\quad + \left|\sum_{k=0}^{N} f(1-q_n^k)p_{\infty,k}(a;x) - \sum_{k=0}^{N} f(1-a^k)p_{\infty,k}(a;x)\right| \\ &\leqslant \sum_{k=0}^{N} f(1-q_n^k)\left|p_{\infty,k}(q_n,x) - p_{\infty,k}(a,x)\right|\end{aligned}$$

$$+\sum_{k=0}^{N} p_{\infty,k}(a;x)\left|f(1-q_n^k)-f(1-a^k)\right|.$$

With the help of Lemmas 5.2.2 and 5.2.9

$$\begin{aligned}|S_N(q_n;x)-S_N(a;x)| &\leqslant C_6(a)|q_n-a|\sum_{k=0}^{N}\left|f(1-q_n^k)\right| + \max_{0\leqslant k\leqslant N}\left|f(1-q_n^k)-f(1-a^k)\right| \\ &\leqslant C_6(a)|q_n-a|(N+1)M_f+\omega_f(C_1(a)|q_n-a|).\end{aligned}$$

□

Remark 5.2.11 *If f satisfies the Lipschitz condition on* $[0,1]$, *then*

$$|S_N(q_n;x)-S_N(a;x)| \leqslant C_7(a)|q_n-a|.$$

5.3 Proof of Theorem 5.1.2

Proof. Let $\varepsilon > 0$ be given. Since the limit q-Bernstein operator leaves linear functions invariant, it can be assumed without loss of generality that $f(0) = f(1) = 0$. As f is uniformly continuous on $[0,1]$, there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \varepsilon$ for all $x, y \in [0,1]$. Select $N \in \mathbb{N}$ in such a way that $a_1^k < \delta$ for $k > N$. Clearly, the selection of N depends only on the value of a_1 and, therefore, on a. Then, for $k \geqslant N+1$, $f(1-q_n^k) < \varepsilon$ and $f(1-a^k) < \varepsilon$. Consequently,

$$\begin{aligned}\left|B_{\infty,q_n}(f;x)-B_{\infty,a}(f;x)\right| &\leqslant |S_N(q_n;x)-S_N(a;x)| + \left|\sum_{k=N+1}^{\infty} f(1-q_n^k)p_{\infty,k}(q_n;x)\right| \\ &\quad + \left|\sum_{k=N+1}^{\infty} f(1-a^k)p_{\infty,k}(a;x)\right| \\ &:= J_1+J_2+J_3.\end{aligned}$$

By Lemma 5.2.10, we conclude that $J_1 < \varepsilon$ for n large enough. As for J_2, we have

$$J_2 \leqslant \sum_{k=N+1}^{\infty} |f(1-q_n^k)|p_{\infty,k}(q_n;x) \leqslant \varepsilon\cdot\sum_{k=N+1}^{\infty} p_{\infty,k}(q_n;x) < \varepsilon\cdot 1 = \varepsilon.$$

The term J_3 can be estimated in the same way. Finally, we obtain that, for n large enough,

$$\left|B_{\infty,q_n}(f;x)-B_{\infty,a}(f;x)\right| < 3\varepsilon.$$

The proof of the theorem is complete. □

5.4 Iterates of q-Bernstein operators

The iterates of q-Bernstein operators were given in Definition 4.4.1. Here, we present a new proof of Theorem 4.4.2. Additionally, a new observation stating that for all $q > 0$, q-Bernstein operators are weakly Picard operators is made. As a novelty of this section, the observation is proved to be true for all $q > 0$ while the previously known methods ([1, 23]) are not applicable to the case $q > 1$, since, in this case, the positivity of the operator $B_{n,q}$ fails.

Theorem 5.4.1 *Let $n \in \mathbb{N}$ is fixed, and $q > 0$. For $f \in C[0,1]$, one has*

$$\lim_{k\to\infty} B^k_{n,q}(f;x) = L_f(x)$$

and the convergence is uniform on $[0,1]$.

Proof. Since $B_{n,q}(f;x) = P_n(x)$ is a polynomial of degree $\leqslant n$, to find the limit of iterate, it suffices to consider the case when f is a polynomial of degree at most n.

If $n = 1$, then, by virtue of Example 4.2.4, we have

$$B_{1,q}(f;x) = L_f(x),$$

and there is nothing to prove. We take $n \geqslant 2$. By Corollary 4.2.9, the statement of the Theorem holds if f is a polynomial of degree $\leqslant 1$. Now, we want to apply the induction on the degree of a polynomial. By the linearity of $B_{n,q}$, it is enough to examine the monomials x^m of degree $m \leqslant n$. Assume that the statement is true for $f(x) = 1, x, \dots, x^{m-1}$. Consider $B_{n,q}(x^m;x)$. By [14, Lemma 3] one has:

$$\begin{aligned} B_{n,q}(x^m;x) &= \alpha x^m + P(x), \\ &= \alpha x^m + (P(x) - L_P(x)) + L_P(x) := \alpha x^m + \rho(x) + L_P(x), \end{aligned}$$

where $\alpha = 1 - \frac{1}{[n]_q} \in (0,1)$. Further,

$$\begin{aligned} B^2_{n,q}(x^m;x) &= \alpha B_{n,q}(x^m) + B_{n,q}\rho(x) + L_P(x) \\ &= \alpha^2 x^m + \alpha\rho(x) + \alpha L_P(x) + B_{n,q}\rho(x) + L_P(x) \\ &= \alpha^2 x^m + (\alpha + B_{n,q})\rho(x) + (1+\alpha)L_P(x), \end{aligned}$$

and, in general,

$$B^k_{n,q}(x^m;x) = \alpha^k x^m + \left[\alpha^{k-1} + \alpha^{k-2}B_{n,q} + \cdots + B^{k-1}_{n,q}\right]\rho(x) + (1+\alpha+\cdots+\alpha^{k-1})L_P(x).$$

Let $k \to \infty$. Then $\alpha^k x^m \to 0$ uniformly on $[0,1]$ since $\alpha \in (0,1)$. Also,

$$(1+\alpha+\cdots+\alpha^{k-1})L_P(x) \to \frac{L_P(x)}{1-\alpha}$$

uniformly on $[0,1]$. We are going to prove that the function in brackets tends to 0 uniformly on $[0,1]$. Choose $\varepsilon > 0$. By the induction assumption, there is $K \in \mathbb{N}$ such that

$$\|B^j\rho(x)\| < \frac{\varepsilon(1-\alpha)}{2} \quad \text{for} \quad j > K.$$

Hence,

$$\begin{aligned}\|\alpha^{k-1}\rho + \cdots + B^{k-1}_{n,q}(\rho;x)\| &\leqslant \|\alpha^{k-1}\rho(x) + \cdots + \alpha^{k-K}B^{K-1}_{n,q}(\rho(x);x)\| \\ &\quad + \frac{\varepsilon(1-\alpha)}{2}(1+\alpha+\cdots+\alpha^j) \\ &\leqslant \alpha^{k-K}\|\rho(x) + \cdots + B^{K-1}_{n,q}(\rho(x);x)\| + \frac{\varepsilon}{2}.\end{aligned}$$

Now, denote $\|\rho(x) + \cdots + B^{K-1}_{n,q}\rho(x)\| = M$ and select $k_0 > K$ in such a way that

$$\alpha^{k-K} \leqslant \frac{\varepsilon}{2M} \quad \text{for all} \quad k \geqslant k_0.$$

As a result, for $k \geqslant k_0$, we obtain:

$$\|\alpha^{k-1}\rho + \cdots + B^{k-1}_{n,q}(\rho;x)\| \leqslant \frac{\varepsilon}{2M}\cdot M + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore,

$$\lim_{k\to\infty} B^k_{n,q}(x^m;x) = \frac{L_P(x)}{1-\alpha},$$

and the convergence is uniform on $[0,1]$. Now, by the end-point interpolation, we conclude that $\lim\limits_{k\to\infty} B^k_{n,q}(x^m;x) = x$. Since

$$B_{n,q}(f;x) = A_n x^n + A_{n-1}x^{n-1} + \cdots + A_1 x + A_0$$

it follows that

$$\begin{aligned}\lim_{k\to\infty} B^k_{n,q}(f;x) &= \lim_{k\to\infty}\left[A_n B^k_{n,q}(x^n) + A_{n-1}B^k_{n,q}(x^{n-1}) + \cdots + A_1 B^k_{n,q}(x) + A_0\right] \\ &= (A_n + A_{n-1} + \cdots + A_1)x + A_0.\end{aligned}$$

Using again the end-point interpolation point property, one concludes that

$$\lim_{k\to\infty} B_{n,q}^{k}(f;x) = L_f(x)$$

and the convergence is uniform on $[0, 1]$ as claimed. □

It has been noticed in Corollary 4.2.9 that q-Bernstein operators preserve linear functions. That is, all linear functions are fixed points of the q-Bernstein operators. Therefore, taking the Definition 3.2.3 into account, the following conclusion can be derived.

Theorem 5.4.2 *For all $q > 0$, the q-Bernstein operator is a weakly Picard operator.*

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