

WONG TYPE OSCILLATION CRITERIA FOR NONLINEAR IMPULSIVE DIFFERENTIAL EQUATIONS*

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ABSTRACT. We present Wong-type oscillation criteria for nonlinear impulsive differential equations having discontinuous solutions and involving both negative and positive coefficients. We use a technique that involves the use of a nonprincipal solution of the associated linear homogeneous equation. The existence of such a solution was recently obtained by the present authors in [J. Math. Anal. Appl. 503 (2021) 125311]. As special cases, we have superlinear and sublinear Emden-Fowler equations under impulse effects. It is shown that the oscillatory behavior changes due to impulses. An example is also given to illustrate the importance of the results.

1. INTRODUCTION.

We start with recalling one of the most celebrated classical oscillation theorems of Wong [1] obtained for the linear nonhomogeneous equation

$$(p(t)x')' + q(t)x = f(t). \tag{1}$$

It says that (1) is oscillatory if

$$\limsup_{t \rightarrow \infty} \mathcal{H}_0(t) = -\liminf_{t \rightarrow \infty} \mathcal{H}_0(t) = \infty,$$

where \mathcal{H}_0 is given by

$$\mathcal{H}_0(t) = \int_T^t \frac{1}{p(\tau)v^2(\tau)} \int_T^\tau f(s)v(s) \, ds \, d\tau \tag{2}$$

and v is a positive nonprincipal solution [2, 3] of the corresponding homogeneous equation

$$(p(t)x')' + q(t)x = 0. \tag{3}$$

Recently, the above oscillation theorem has been extended by the present authors[4] to equations of impulsive type of the form

$$\begin{cases} (p(t)x')' + q(t)x = f(t), & t \neq \theta_i, \\ \Delta x + a_i x = f_i, \quad \Delta(p(t)x') + b_i x + c_i x' = g_i, & t = \theta_i. \end{cases}$$

Indeed, the first extension was given even earlier[5] for the much simpler case $a_i = c_i = 0$ as an application of nonprincipal solutions.

A successful generalization of the results in [1] to nonlinear differential equations can be found in [6, 7]. In particular, related oscillation theorems were derived in [6] for nonlinear equations with mixed type exponents $\beta > 1$ and $\alpha < 1$ of the form

$$(p(t)x')' + q(t)|x|^{\beta-1}x - r(t)|x|^{\alpha-1}x = f(t), \tag{4}$$

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where q and r are non-negative functions. Due to the sign of the coefficients, the limiting case $\alpha \rightarrow 1^-$ and $\beta \rightarrow 1^+$ has led an improvement of the Wong oscillation theorem mentioned above. We note that various extensions of Wong type oscillation theorems to nonlinear ordinary differential equations can be found in [8–11]. However, there is not much work in the literature on their impulsive counterparts [7, 12]. On the other hand, there are many results involving different types of oscillation theorems for nonlinear differential equations under impulse effects; see for instance [13–16] and the references cited therein.

In this work, inspired by the above mentioned studies, we will establish Wong type oscillation criteria for a general class of nonlinear impulsive differential equations of form

$$\begin{cases} (p(t)x')' + q(t)F(x) - r(t)G(x) = f(t), & t \neq \theta_i, \\ \Delta x + a_i F(x) - b_i G(x) = f_i, & t = \theta_i, \\ \Delta(p(t)x') + c_i F(x) - d_i G(x) = g_i, & t = \theta_i, \end{cases} \quad (5)$$

where $p > 0, q, r, f$ are piece-wise left continuous functions on $[0, \infty)$; F and G are continuous on \mathbb{R} ; $\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}, \{f_i\}$, and $\{g_i\}$ are real sequences; $1 - a_i + b_i > 0$; the sequence of impulses $\{\theta_i\}$ is unbounded and increasing. As usual,

$$\Delta y|_{t=\theta_i} = y(\theta_i^+) - y(\theta_i^-)$$

measures the jump of y at $t = \theta_i$.

A function $y(t)$ satisfying (5) is said to be a solution of (5) if y and $p(t)y'$ are left continuous on $(\theta_i, \theta_{i+1}]$, $\lim_{t \rightarrow \theta_i^+} y(t)$ and $\lim_{t \rightarrow \theta_i^+} p(t)y'(t)$ exist for each $i \in \mathbb{N}$. As usual, we say that a solution is oscillatory if it is neither eventually positive nor eventually negative.

It turns out that the extension to nonlinear impulsive equations of the form (5) is possible if \mathcal{H}_0 in (2) is replaced by

$$\mathcal{H}(t) = \int_a^t \frac{1}{p(s)v^2(s)} \left(\int_a^s \mu(s, \tau) f(\tau) v(\tau) d\tau + \sum_{i=\underline{n}(a)}^{\bar{n}(s)} \mu(s, \theta_i) M_i \right) ds + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{f_i}{(1 - a_i + b_i)v(\theta_i)}, \quad (6)$$

where a is an arbitrarily large real number,

$$\underline{n}(t) = \inf\{i : \theta_i \geq t\}, \quad \bar{n}(t) := \sup\{i : \theta_i < t\},$$

$$\mu(t, s) = \prod_{i=\underline{n}(s)}^{\bar{n}(t)} (1 - a_i + b_i), \quad t \geq s \geq a,$$

$$M_i = g_i v(\theta_i) - \frac{f_i}{(1 - a_i + b_i)} \left(p(\theta_i) v'(\theta_i) - (c_i - d_i) v(\theta_i) \right),$$

and v is a positive nonprincipal solution of

$$\begin{cases} (p(t)x')' + [q(t) - r(t)]x = 0, & t \neq \theta_i, \\ \Delta x + [a_i - b_i]x = 0, \quad \Delta(p(t)x') + [c_i - d_i]x = 0, & t = \theta_i \end{cases} \quad (7)$$

for $t \geq a$.

The existence of principal and nonprincipal solutions of (7) and their applications to asymptotic integration of certain nonlinear impulsive differential equations can be found in [17]. For our purpose, we state a modified version as follows.

Lemma 1.1. *Assume that (7) has a positive solution. If $1 - a_i + b_i > 0$ for each $i \geq 1$, then there exist two solutions u and v of (7) defined on $[a, \infty)$ for some $a \geq 0$ that satisfy*

$$\lim_{t \rightarrow \infty} \frac{u(t)}{v(t)} = 0,$$

$$\int_a^\infty \frac{\mu(t, a)}{p(t)u^2(t)} dt = \infty, \quad \int_a^\infty \frac{\mu(t, a)}{p(t)v^2(t)} dt < \infty,$$

and

$$\frac{p(t)u'(t)}{u(t)} < \frac{p(t)v'(t)}{v(t)}, \quad t \geq a.$$

The functions u and v are called *principal and nonprincipal solutions, respectively*.

Obviously, equation (4) is a special case of (5), and hence the certain particular cases of equation (5) result in the Emden-Fowler type impulsive equations. The corresponding Wong type oscillation theorems will be stated in the present work. As a direction for further research we suggest investigating the corresponding results for arbitrary time scales; for some related results we refer the reader to [18–20].

2. LEMMAS

We give two useful lemmas that we will rely on in the proof of our main oscillation theorem. The first lemma is a construction of an integral equation that is equivalent to the given impulsive system (1), and the second one is a simple calculus application.

Let x be a real valued function defined on $[0, \infty)$. We define the operators J_1 , J_2 , and J_3 as follows:

$$(J_1x)(t) = \int_a^t \mu(t, \tau)v(\tau)Q(q(\tau), r(\tau), x(\tau)) d\tau,$$

$$(J_2x)(t) = \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \mu(t, \theta_i) \left\{ v(\theta_i)Q(c_i, d_i, x(\theta_i)) - \frac{1}{1 - a_i + b_i} [p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)]Q(a_i, b_i, x(\theta_i)) \right\},$$

and

$$(J_3x)(t) = c_1 + c_2 \int_a^t \frac{\mu(s, a)}{p(s)v^2(s)} ds + \int_a^t \frac{(J_1x)(s) + (J_2x)(s)}{p(s)v^2(s)} ds + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{Q(a_i, b_i, x(\theta_i))}{(1 - a_i + b_i)v(\theta_i)}, \quad (8)$$

where

$$Q(a, b, x) = (a - b)x - aF(x) + bG(x).$$

Lemma 2.1. *Let $v > 0$ be a nonprincipal solution of (7) given in Lemma 1.1. Then, x is a solution of (5) if and only if it satisfies the integral equation*

$$\frac{x}{v(t)} = J_3x + \mathcal{H}(t), \quad (9)$$

where \mathcal{H} and J_3 are as defined in (6) and (8), respectively.

Proof. Let $x(t)$ be a solution of (9) and $t \neq \theta_l$. It is not difficult to see that

$$(J_3x)' = \frac{1}{p(t)v^2(t)}(c_2\mu(t, a) + J_1x + J_2x). \quad (10)$$

Thus,

$$p(t)x' = p(t)v'(t)J_3x + \frac{1}{v(t)}(c_2\mu(t, a) + J_1x + J_2x) + p(t)z'$$

and so

$$\begin{aligned} (p(t)x')' &= (p(t)v'(t))'J_3x + p(t)v'(t)(J_3x)' - \frac{v'(t)}{v^2(t)}(c_2\mu(t, a) + J_1x + J_2x) \\ &\quad + \frac{1}{v(t)}((J_1x)' + (J_2x)') + (p(t)z')', \end{aligned} \quad (11)$$

where $z = v\mathcal{H}$. One can easily show that

$$(J_1x)' = \mu(t, t)v(t)Q(q(t), r(t), x), \quad (J_2x)'(t) = 0, \quad \mu(t, t) = 1.$$

Also, as shown in [4, Theorem 2.2], $z = v\mathcal{H}$ is a solution of

$$\begin{cases} (p(t)z')' + [q(t) - r(t)]z = f(t), & t \neq \theta_i, \\ \Delta z + [a_i - b_i]z = f_i, \quad \Delta(p(t)z') + [c_i - d_i]z = g_i, & t = \theta_i. \end{cases} \quad (12)$$

Taking the above considerations into account, it follows from (11) that

$$(p(t)x')' = -[q(t) - r(t)]v(t)J_3x + Q(q(t), r(t), x) - [q(t) - r(t)]z + f(t),$$

or

$$(p(t)x')' + q(t)F(x) - r(t)G(x) = f(t). \quad (13)$$

Now, we look at $t = \theta_i$. Clearly,

$$(J_3x)(\theta_i+) = (J_3x)(\theta_i) + \frac{Q(a_i, b_i, x(\theta_i))}{(1 - a_i + b_i)v(\theta_i)}. \quad (14)$$

Then,

$$\begin{aligned} \Delta x|_{t=\theta_i} &= \Delta[v(\theta_i)J_3x(\theta_i)] + \Delta z(\theta_i) \\ &= (1 - a_i + b_i)v(\theta_i)J_3x(\theta_i+) - v(\theta_i)J_3x(\theta_i) - (a_i - b_i)z(\theta_i) + f_i \\ &= -(a_i - b_i)x(\theta_i) + Q(a_i, b_i, x(\theta_i)) + f_i \end{aligned}$$

which implies that

$$\Delta x|_{t=\theta_i} + a_iF(x(\theta_i)) - b_iG(x(\theta_i)) = f_i. \quad (15)$$

Now, it is not difficult to see that $J_1x(\theta_i+) = (1 - a_i + b_i)J_1x(\theta_i)$ and

$$\begin{aligned} J_2x(\theta_i+) &= (1 - a_i + b_i)J_2x(\theta_i) + (1 - a_i + b_i)v(\theta_i)Q(c_i, d_i, x(\theta_i)) \\ &\quad - [p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)]Q(a_i, b_i, x(\theta_i)). \end{aligned}$$

In view of (10) we can write

$$\begin{aligned} \Delta[p(t)v(t)(J_3x)'(t)]|_{t=\theta_i} &= \left[\frac{1 - a_i + b_i}{v(\theta_i+)} - \frac{1}{v(\theta_i)} \right] (c_2\mu(\theta_i, a) + J_1x(\theta_i) + J_2x(\theta_i)) \\ &\quad + \frac{1}{v(\theta_i+)} \left((1 - a_i + b_i)v(\theta_i)Q(c_i, d_i, x(\theta_i)) - [p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)]Q(a_i, b_i, x(\theta_i)) \right) \\ &= Q(c_i, d_i, x(\theta_i)) - \frac{p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)}{(1 - a_i + b_i)v(\theta_i)}Q(a_i, b_i, x(\theta_i)). \end{aligned} \quad (16)$$

On the other hand, using (14) we have

$$\begin{aligned} \Delta[p(t)v'(t)J_3x(t)]|_{t=\theta_i} &= [p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)]J_3x(\theta_i+) - p(\theta_i)v'(\theta_i)J_3x(\theta_i) \\ &= -(c_i - d_i)v(\theta_i)J_3x(\theta_i) + \frac{p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)}{(1 - a_i + b_i)v(\theta_i)}Q(a_i, b_i, x(\theta_i)). \end{aligned} \quad (17)$$

In view of (16), (17), and the fact that $z(t)$ is a solution of (12) it follows that

$$\begin{aligned}\Delta(p(t)x')|_{t=\theta_l} &= \Delta[p(\theta_l)v(\theta_l)(J_3x)'(\theta_l)] + \Delta[p(\theta_l)v'(\theta_l)J_3x(\theta_l)] + \Delta[p(\theta_l)z'(\theta_l)] \\ &= Q(c_l, d_l, x(\theta_l)) - (c_l - d_l)v(\theta_l)J_3x(\theta_l) - (c_l - d_l)z(\theta_l) + g_l,\end{aligned}$$

or

$$\Delta(p(t)x')|_{t=\theta_l} + c_lF(x(\theta_l)) - d_lG(x(\theta_l)) = g_l. \quad (18)$$

From (13), (15), and (18), we deduce that $x(t)$ is a solution of (5).

To prove the converse, let x be a solution of (5). Letting $x_1 = x$ and $x_2 = p(t)x'_1$, we may write (5) as a system of first order impulsive differential equations

$$\begin{cases} X' + A(t)X &= \varphi(t, X), & t \neq \theta_i \\ \Delta X + A_iX &= \varphi_i(X), & t = \theta_i, \end{cases} \quad (19)$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad A(t) = \begin{bmatrix} 0 & -1/p(t) \\ q(t) - r(t) & 0 \end{bmatrix}, \quad A_i = \begin{bmatrix} a_i - b_i & 0 \\ c_i - d_i & 0 \end{bmatrix},$$

and

$$\varphi(t, X) = \begin{bmatrix} 0 \\ Q(q(t), r(t), x_1) + f(t) \end{bmatrix}, \quad \varphi_i(X) = \begin{bmatrix} Q(a_i, b_i, x_1) + f_i \\ Q(c_i, d_i, x_1) + g_i \end{bmatrix}.$$

Let u and v be principal and nonprincipal solutions of (7), respectively. Clearly, one can write that [17],

$$u(t) = v(t) \int_a^t \frac{\mu(s, a)}{p(s)v^2(s)} ds. \quad (20)$$

We may then put $\Phi(t, s) = \Phi(t)\Phi^{-1}(s)$, where

$$\Phi(t) = \begin{bmatrix} u(t) & v(t) \\ p(t)u'(t) & p(t)v'(t) \end{bmatrix},$$

as a state transition matrix of the system

$$\begin{cases} X' + A(t)X &= 0, & t \neq \theta_i \\ \Delta X + A_iX &= 0, & t = \theta_i. \end{cases}$$

By using the relation (20) one can easily see that the entries a_{11} and a_{12} of $\Phi(t, s)$ are

$$a_{11} = v(t) \left[-p(s)v'(s) \int_s^t \frac{\mu(\tau, s)}{p(\tau)v^2(\tau)} d\tau + \frac{1}{v(s)} \right], \quad a_{12} = v(t)v(s) \int_s^t \frac{\mu(\tau, s)}{p(\tau)v^2(\tau)} d\tau.$$

Using the variation of parameters formula, we may write from (19) that

$$X(t) = \Phi(t, a)X(a) + \int_a^t \Phi(t, s)\varphi(s, X) ds + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \Phi(t, \theta_i+) \varphi_i(X).$$

Therefore,

$$x_1(t) = c_1 v(t) + c_2 v(t) \int_a^t \frac{\mu(s, a)}{p(s)v^2(s)} ds + v(t) \left\{ \int_a^t [Q(q(s), r(s), x_1(s)) + f(s)] v(s) \int_s^t \frac{\mu(\tau, s)}{p(\tau)v^2(\tau)} d\tau ds \right. \quad (21)$$

$$- \sum_{i=\underline{n}(a)}^{\bar{n}(t)} [Q(a_i, b_i, x_1(\theta_i)) + f_i] \left[p(\theta_i+) v'(\theta_i+) \int_{\theta_i}^t \frac{\mu(\tau, \theta_i+)}{p(\tau)v^2(\tau)} d\tau - \frac{1}{v(\theta_i+)} \right] \quad (22)$$

$$\left. + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} [Q(c_i, d_i, x_1(\theta_i)) + g_i] v(\theta_i+) \int_{\theta_i}^t \frac{\mu(\tau, \theta_i+)}{p(\tau)v^2(\tau)} d\tau \right\}. \quad (23)$$

By changing the order of integration in (21), the order of integration and summation both in (22) and in (23), and using the fact that $\mu(t, \theta_i+) = \mu(t, \theta_i)/(1 - a_i + b_i)$, we finally obtain that x_1 is a solution of (9). \square

We will also need the following simple lemma. The proof is elementary, so we omit it.

Lemma 2.2. *Suppose that $xF(x) > 0$ and $xG(x) > 0$ for $x \neq 0$. If*

$$\lim_{|x| \rightarrow \infty} \frac{F(x)}{x} > 1, \quad \lim_{|x| \rightarrow 0} \frac{F(x)}{x} < 1, \quad \lim_{|x| \rightarrow \infty} \frac{G(x)}{x} < 1, \quad \lim_{|x| \rightarrow 0} \frac{G(x)}{x} > 1, \quad (24)$$

then

$$F_m = -\min_{x \leq 0} [x - F(x)], \quad F_M = \max_{x \geq 0} [x - F(x)], \quad G_m = -\min_{x \geq 0} [x - G(x)], \quad G_M = \max_{x \leq 0} [x - G(x)]$$

exist as positive real numbers.

3. WONG'S OSCILLATION THEOREM

In addition to \mathcal{H} given by (6), we define

$$\begin{aligned} \bar{N}(t) := & \int_a^t \frac{1}{p(s)v^2(s)} \left\{ \int_a^s \mu(s, \tau)v(\tau) [q(\tau)F_m + r(\tau)G_m] d\tau + \sum_{i=\underline{n}(a)}^{\bar{n}(s)} \mu(s, \theta_i) [C_i F_m + D_i G_m] \right. \\ & \left. + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{1}{(1 - a_i + b_i)v(\theta_i)} [a_i F_m + b_i G_m] \right\} \end{aligned}$$

and

$$\begin{aligned} \underline{N}(t) := & \int_a^t \frac{1}{p(s)v^2(s)} \left\{ \int_a^s \mu(s, \tau)v(\tau) [q(\tau)F_m + r(\tau)G_m] d\tau + \sum_{i=\underline{n}(a)}^{\bar{n}(s)} \mu(s, \theta_i) [C_i F_m + D_i G_m] \right\} ds \\ & + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{1}{(1 - a_i + b_i)v(\theta_i)} [a_i F_m + b_i G_m], \end{aligned}$$

where $F_m, F_M, G_m,$ and G_M are as in Lemma 2.2, and

$$C_i := c_i v(\theta_i) - \frac{a_i}{1 - a_i + b_i} [p(\theta_i)v'(\theta_i) - (c_i - d_i)v(\theta_i)],$$

$$D_i := d_i v(\theta_i) - \frac{b_i}{1 - a_i + b_i} \left[p(\theta_i) v'(\theta_i) - (c_i - d_i) v(\theta_i) \right].$$

We may now state and prove our main result for the oscillation of (5).

Theorem 3.1. *Let (7) be nonoscillatory and $v(t) > 0$ be its nonprincipal solution. Suppose that (24) holds, and $q(t)$, $r(t)$, a_i , b_i , C_i , and D_i are non-negative. If*

$$\limsup_{t \rightarrow \infty} [\mathcal{H}(t) - \underline{N}(t)] = -\liminf_{t \rightarrow \infty} [\mathcal{H}(t) + \overline{N}(t)] = \infty, \quad (25)$$

then the impulsive equation (5) is oscillatory.

Proof. Let $x(t)$ be a solution of the nonlinear impulsive equation (5). By Lemma 2.1, $x(t)$ satisfies (9). We need to estimate $J_3 x$ given in (8). To do so, we need to estimate $J_1 x$ and $J_2 x$. Since $q(t)$ and $r(t)$ are non-negative, in view of Lemma 2.2, we can write from

$$(J_1 x)(t) = \int_a^t \mu(t, \tau) v(\tau) \left(q(\tau) [x - F(x)] - r(\tau) [x - G(x)] \right) d\tau$$

that

$$(J_1 x)(t) \begin{cases} \leq \int_a^t \mu(t, \tau) v(\tau) \left(q(\tau) F_M + r(\tau) G_m \right) d\tau, & x > 0, \\ \geq \int_a^t \mu(t, \tau) v(\tau) \left(q(\tau) F_m + r(\tau) G_M \right) d\tau, & x < 0. \end{cases} \quad (26)$$

Next, we rewrite $(J_2 x)(t)$ as

$$\begin{aligned} (J_2 x)(t) &= \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \mu(t, \theta_i) \left\{ v(\theta_i) \left((c_i - d_i) x(\theta_i) \right) - c_i F(x(\theta_i)) + d_i G(x(\theta_i)) \right\} \\ &\quad - \frac{1}{1 - a_i + b_i} \left[p(\theta_i) v'(\theta_i) - (c_i - d_i) v(\theta_i) \right] \left\{ (a_i - b_i) x(\theta_i) - a_i F(x(\theta_i)) + b_i G(x(\theta_i)) \right\} \\ &= \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \mu(t, \theta_i) \left\{ \left(C_i [x - F(x)] - D_i [x - G(x)] \right) \right\}. \end{aligned}$$

Since a_i , b_i , C_i and D_i are non-negative, by Lemma 2.2 we easily obtain

$$(J_2 x)(t) \begin{cases} \leq \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \mu(t, \theta_i) \left(C_i F_M + D_i G_m \right), & x > 0, \\ \geq \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \mu(t, \theta_i) \left(C_i F_m + D_i G_M \right), & x < 0. \end{cases} \quad (27)$$

On the other hand, it is not difficult to see that

$$\sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{Q(a_i, b_i, x(\theta_i))}{(1 - a_i + b_i) v(\theta_i)} \begin{cases} \leq \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{1}{(1 - a_i + b_i) v(\theta_i)} \left(a_i F_M + b_i G_m \right), & x > 0, \\ \geq \sum_{i=\underline{n}(a)}^{\overline{n}(t)} \frac{1}{(1 - a_i + b_i) v(\theta_i)} \left(a_i F_m + b_i G_M \right), & x < 0. \end{cases} \quad (28)$$

Employing (26), (27), and (28) in (8) leads to

$$(J_3x)(t) \begin{cases} \leq c_1 + c_2 \int_a^t \frac{\mu(s, a)}{p(s)v^2(s)} ds + \overline{N}(t), & x > 0, \\ \geq c_1 + c_2 \int_a^t \frac{\mu(s, a)}{p(s)v^2(s)} ds - \underline{N}(t), & x < 0. \end{cases} \quad (29)$$

By using the estimate (29) in (9), we easily obtain

$$\frac{x(t)}{v(t)} \leq c_1 + c_2 \int_a^t \frac{\mu(s, a)}{p(s)v^2(s)} ds + \mathcal{H}(t) + \overline{N}(t), \quad x > 0, \quad (30)$$

$$\frac{x(t)}{v(t)} \geq c_1 + c_2 \int_a^t \frac{\mu(s, a)}{p(s)v^2(s)} ds + \mathcal{H}(t) - \underline{N}(t), \quad x < 0. \quad (31)$$

In view of (25) and the fact that v is a nonprincipal solution, i.e.,

$$\int_a^t \frac{\mu(s, a)}{p(s)v^2(s)} ds < \infty,$$

it follows from (30) and (31) that

$$\limsup_{t \rightarrow \infty} \frac{x(t)}{v(t)} = -\liminf_{t \rightarrow \infty} \frac{x(t)}{v(t)} = \infty.$$

Since $v(t) > 0$, we conclude that $x(t)$ must be oscillatory. \square

In the next section we illustrate some important special cases of Theorem 3.1.

4. EMDEN-FOWLER TYPE IMPULSIVE EQUATIONS

Let $F(x) = |x|^{\beta-1}x$ and $G(x) = |x|^{\alpha-1}x$, where $0 < \alpha < 1 < \beta$, then (5) turns into the Emden-Fowler type impulsive equation with superlinear and sublinear terms

$$\begin{cases} (p(t)x')' + q(t)|x|^{\beta-1}x - r(t)|x|^{\alpha-1}x = f(t), & t \neq \theta_i, \\ \Delta x + a_i|x|^{\beta-1}x - b_i|x|^{\alpha-1}x = f_i, & t = \theta_i, \\ \Delta(p(t)x') + c_i|x|^{\beta-1}x - d_i|x|^{\alpha-1}x = g_i, & t = \theta_i. \end{cases} \quad (32)$$

By a simple calculation we have

$$F_m = F_M = \beta^{\beta/(1-\beta)}(\beta - 1), \quad G_m = G_M = \alpha^{\alpha/(1-\alpha)}(1 - \alpha).$$

Therefore, we obtain the following oscillation theorem.

Theorem 4.1. *Let (7) be nonoscillatory and $v(t) > 0$ be its nonprincipal solution. Suppose that $q(t)$, $r(t)$, a_i , b_i , C_i and D_i are non-negative. Then, the half-linear impulsive Emden-Fowler equation (32) is oscillatory provided that*

$$\limsup_{t \rightarrow \infty} [\mathcal{H}(t) - N(t)] = -\liminf_{t \rightarrow \infty} [\mathcal{H}(t) + N(t)] = \infty, \quad (33)$$

where $\mathcal{H}(t)$ is as given in (6) and

$$\begin{aligned} N(t) := & \int_a^t \frac{1}{p(s)v^2(s)} \left\{ \int_a^s \mu(s, \tau)v(\tau) [q(\tau)\beta^{\beta/(1-\beta)}(\beta-1) + r(\tau)\alpha^{\alpha/(1-\alpha)}(1-\alpha)] d\tau \right. \\ & + \left. \sum_{i=\underline{n}(a)}^{\bar{n}(s)} \mu(s, \theta_i) \left[C_i\beta^{\beta/(1-\beta)}(\beta-1) + D_i\alpha^{\alpha/(1-\alpha)}(1-\alpha) \right] \right\} ds \\ & + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{1}{(1-a_i+b_i)v(\theta_i)} [a_i\beta^{\beta/(1-\beta)}(\beta-1) + b_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)]. \end{aligned}$$

Taking $\alpha \rightarrow 1^-$ and $\beta \rightarrow 1^+$ in (32) results in a linear impulsive equation

$$\begin{cases} (p(t)x')' + [q(t) - r(t)]x = f(t), & t \neq \theta_i, \\ \Delta x + [a_i - b_i]x = f_i, & t = \theta_i, \\ \Delta(p(t)x') + [c_i - d_i]x = g_i, & t = \theta_i. \end{cases} \quad (34)$$

In this case, $\alpha^{\alpha/(1-\alpha)} \rightarrow 1/e$ and $\beta^{\beta/(1-\beta)} \rightarrow 1/e$, and so $F_m = F_M = G_m = G_M = 0$. Thus, we easily obtain the following corollary.

Corollary 4.1. *Let (7) be nonoscillatory and $v(t) > 0$ be its nonprincipal solution. Then, the linear impulsive Emden-Fowler equation (34) is oscillatory provided that*

$$\limsup_{t \rightarrow \infty} \mathcal{H}(t) = -\liminf_{t \rightarrow \infty} \mathcal{H}(t) = \infty,$$

where $\mathcal{H}(t)$ is as given in (6).

Finally, we state Wong type oscillation theorems for the superlinear impulsive Emden-Fowler type equation

$$\begin{cases} (p(t)x')' + q(t)|x|^{\beta-1}x = f(t), & t \neq \theta_i, \\ \Delta x + a_i|x|^{\beta-1}x = f_i, & t = \theta_i, \\ \Delta(p(t)x') + c_i|x|^{\beta-1}x = g_i, & t = \theta_i \end{cases} \quad (35)$$

and the sublinear impulsive Emden-Fowler type equation

$$\begin{cases} (p(t)x')' - r(t)|x|^{\alpha-1}x = f(t), & t \neq \theta_i, \\ \Delta x - b_i|x|^{\alpha-1}x = f_i, & t = \theta_i, \\ \Delta(p(t)x') - d_i|x|^{\alpha-1}x = g_i, & t = \theta_i. \end{cases} \quad (36)$$

The following results are readily available from Theorem 4.1.

Corollary 4.2. *Let the homogeneous equation associated with (35) be nonoscillatory and $v(t) > 0$ be its nonprincipal solution. Suppose that $q(t)$, a_i and $C_{i1} := c_iv(\theta_i) - a_ip(\theta_i)v'(\theta_i)$ are non-negative. Then, the superlinear impulsive Emden-Fowler equation (35) under impulse effects is oscillatory provided that*

$$\limsup_{t \rightarrow \infty} [\mathcal{H}_1(t) - N_1(t)] = -\liminf_{t \rightarrow \infty} [\mathcal{H}_1(t) + N_1(t)] = \infty,$$

where

$$\mathcal{H}_1(t) := \int_a^t \frac{1}{p(s)v^2(s)} \left(\int_a^s \prod_{i=\underline{n}(\tau)}^{\bar{n}(s)} (1-a_i) f(\tau)v(\tau) d\tau + \sum_{i=\underline{n}(a)}^{\bar{n}(s)} \prod_{k=i}^{\bar{n}(s)} (1-a_k) M_{i1} \right) ds + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{f_i}{(1-a_i)v(\theta_i)},$$

with

$$M_{i1} := g_i v(\theta_i) - \frac{f_i}{(1 - a_i)} \left(p(\theta_i) v'(\theta_i) - c_i v(\theta_i) \right),$$

and

$$\begin{aligned} N_1(t) := & \int_a^t \frac{1}{p(s)v^2(s)} \left\{ \int_a^s \prod_{i=\underline{n}(\tau)}^{\bar{n}(s)} (1 - a_i) v(\tau) q(\tau) \beta^{\beta/(1-\beta)} (\beta - 1) d\tau \right. \\ & \left. + \sum_{i=\underline{n}(a)}^{\bar{n}(s)} \left[\prod_{k=i+1}^{\bar{n}(s)} (1 - a_k) C_{i1} \beta^{\beta/(1-\beta)} (\beta - 1) \right] \right\} ds + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{a_i}{(1 - a_i) v(\theta_i)} \beta^{\beta/(1-\beta)} (\beta - 1). \end{aligned}$$

Corollary 4.3. *Let the homogeneous equation associated with (36) be nonoscillatory and $v(t) > 0$ be its nonprincipal solution. Suppose that $r(t)$, b_i and $D_{i2} := d_i v(\theta_i) - b_i p(\theta_i) v'(\theta_i)$ are non-negative. Then, the sublinear Emden-Fowler equation (36) under impulse effects is oscillatory provided that*

$$\limsup_{t \rightarrow \infty} [\mathcal{H}_2(t) - N_2(t)] = -\liminf_{t \rightarrow \infty} [\mathcal{H}_2(t) + N_2(t)] = \infty,$$

where

$$\mathcal{H}_2(t) := \int_a^t \frac{1}{p(s)v^2(s)} \left(\int_a^s \prod_{i=\underline{n}(\tau)}^{\bar{n}(s)} (1 + b_i) f(\tau) v(\tau) d\tau + \sum_{i=\underline{n}(a)}^{\bar{n}(s)} \prod_{k=i}^{\bar{n}(s)} (1 + b_k) M_{i2} \right) ds + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{f_i}{(1 + b_i) v(\theta_i)},$$

with

$$M_{i2} := g_i v(\theta_i) - \frac{f_i}{(1 + b_i)} \left(p(\theta_i) v'(\theta_i) + d_i v(\theta_i) \right),$$

and

$$\begin{aligned} N_2(t) := & \int_a^t \frac{1}{p(s)v^2(s)} \left\{ \int_a^s \prod_{i=\underline{n}(\tau)}^{\bar{n}(s)} (1 + b_i) v(\tau) r(\tau) \alpha^{\alpha/(1-\alpha)} (1 - \alpha) d\tau \right. \\ & \left. + \sum_{i=\underline{n}(a)}^{\bar{n}(s)} \left[\prod_{k=i+1}^{\bar{n}(s)} (1 + b_k) D_{i2} \alpha^{\alpha/(1-\alpha)} (1 - \alpha) \right] \right\} ds + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{b_i}{(1 + b_i) v(\theta_i)} \alpha^{\alpha/(1-\alpha)} (1 - \alpha). \end{aligned}$$

5. EXAMPLE

In this section, we provide an example to illustrate the efficiency of Theorem 4.1. Consider an Emden-Fowler type impulsive equation with superlinear and sublinear terms

$$\begin{cases} (t^2(t^2 + 1)x')' + 4|x|^{\beta-1}x - 10(t^2 + 1)|x|^{\alpha-1}x = \sin t, & t \neq i, \quad t \geq 1, \\ \Delta x + i|x|^{\beta-1}x - \frac{i^2 + 1}{i}|x|^{\alpha-1}x = (-1)^i i^3(i + 1), & t = i, \\ \Delta(t^2(t^2 + 1)x') + 2i^2(i^2 + 1)|x|^{\beta-1}x - 2(i^2 + 1)^2|x|^{\alpha-1}x = 2i, & t = i, \quad i > 1. \end{cases} \quad (37)$$

Comparing with the general form (32), we observe that $\theta_i = i$, $p(t) = t^2(t^2 + 1)$, $q(t) = 4$, $r(t) = 10(t^2 + 1)$, $a_i = i$, $b_i = (i^2 + 1)/i$, $c_i = 2i^2(i^2 + 1)$, $d_i = 2(i^2 + 1)^2$, $f(t) = \sin t$, $f_i = (-1)^i i^3(i + 1)$ and $g_i = 2i$. Thus, $1 - a_i + b_i = (i + 1)/i > 0$,

$$C_i = c_i v(\theta_i) - \frac{a_i}{1 - a_i + b_i} \left[p(\theta_i) v'(\theta_i) - (c_i - d_i) v(\theta_i) \right] = 0$$

and

$$D_i = d_i v(\theta_i) - \frac{b_i}{1 - a_i + b_i} \left[p(\theta_i) v'(\theta_i) - (c_i - d_i) v(\theta_i) \right] = 0,$$

where $v(t) = it^2$, $t \in (i-1, i]$ is a nonprincipal solution of the associated homogeneous equation

$$\begin{cases} (t^2(t^2+1)x')' + 4x - 10(t^2+1)x = 0, & t \neq i, \quad t \geq 1, \\ \Delta x + ix - \frac{i^2+1}{i}x = 0, \quad \Delta(t^2(t^2+1)x') + 2i^2(i^2+1)x - 2(i^2+1)^2x = 0, & t = i, \quad i > 1. \end{cases}$$

For convenience, we write

$$\begin{aligned} N(t) &= \int_1^t \frac{1}{p(s)v^2(s)} (y_1(s) + y_2(s)) \, ds \\ &\quad + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{1}{(1-a_i+b_i)v(\theta_i)} [a_i\beta^{\beta/(1-\beta)}(\beta-1) + b_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)], \\ \mathcal{H}(t) &= y_3(t) + y_4(t), \end{aligned}$$

where

$$\begin{aligned} y_1(t) &:= \int_a^t \mu(t, \tau)v(\tau) [q(\tau)\beta^{\beta/(1-\beta)}(\beta-1) + r(\tau)\alpha^{\alpha/(1-\alpha)}(1-\alpha)] \, d\tau, \\ y_2(t) &:= \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \mu(t, \theta_i) [C_i\beta^{\beta/(1-\beta)}(\beta-1) + D_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)], \\ y_3(t) &:= \int_a^t \frac{1}{p(s)v^2(s)} \left(\int_a^s \mu(s, \tau)f(\tau)v(\tau) \, d\tau + \sum_{i=\underline{n}(a)}^{\bar{n}(s)} \mu(s, \theta_i)M_i \right) \, ds, \\ y_4(t) &:= \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{f_i}{(1-a_i+b_i)v(\theta_i)}. \end{aligned}$$

Let $a = 1$, $t \in (k-1, k]$, and $s \in (j-1, j]$ for $k > j > 2$. It follows that

$$\mu(s, \tau) = (\bar{n}(s) + 1)/\underline{n}(\tau) = (j+1)/\underline{n}(\tau),$$

and so $\mu(s, i) = (j+1)/i$, $\underline{n}(\tau) = i-1$ for $\tau \in (i-1, i]$, and $\underline{n}(\tau) = j-1$ for $\tau \in (j-1, s)$. Thus, we have

$$\begin{aligned} y_1(s) &= (j+1) \sum_{i=2}^{j-1} \frac{i}{i-1} \int_{i-1}^i \tau^2 [4\beta^{\beta/(1-\beta)}(\beta-1) + 10(\tau^2+1)\alpha^{\alpha/(1-\alpha)}(1-\alpha)] \, d\tau \\ &\quad + \frac{(j+1)j}{j-1} \int_{j-1}^s \tau^2 [4\beta^{\beta/(1-\beta)}(\beta-1) + 10(\tau^2+1)\alpha^{\alpha/(1-\alpha)}(1-\alpha)] \, d\tau \\ &= O(s^6), \quad s \rightarrow \infty, \end{aligned}$$

$$y_2(s) = \sum_{i=2}^j \frac{j+1}{i} [C_i\beta^{\beta/(1-\beta)}(\beta-1) + D_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)] = 0,$$

and

$$\begin{aligned}
& \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{1}{(1-a_i+b_i)v(\theta_i)} [a_i\beta^{\beta/(1-\beta)}(\beta-1) + b_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)] \\
&= \sum_{i=2}^{\bar{n}(t)} \frac{i}{(i+1)i^3} \left[i\beta^{\beta/(1-\beta)}(\beta-1) + \frac{i^2+1}{i}\alpha^{\alpha/(1-\alpha)}(1-\alpha) \right] \\
&= O(1), \quad t \rightarrow \infty.
\end{aligned}$$

Therefore,

$$\begin{aligned}
N(t) &= \sum_{i=2}^{k-1} \int_{i-1}^i \frac{1}{s^6(s^2+1)i^2} (y_1(s) + y_2(s)) ds + \int_{k-1}^t \frac{1}{s^6(s^2+1)k^2} (y_1(s) + y_2(s)) ds \\
&\quad + \sum_{i=\underline{n}(a)}^{\bar{n}(t)} \frac{1}{(1-a_i+b_i)v(i)} [a_i\beta^{\beta/(1-\beta)}(\beta-1) + b_i\alpha^{\alpha/(1-\alpha)}(1-\alpha)] \\
&= O(1), \quad t \rightarrow \infty.
\end{aligned} \tag{38}$$

On the other hand, we calculate that $M_i = 2i^4[1 - (-1)^i i^3(i+1)(i^2+1)]$,

$$\begin{aligned}
\int_a^s \mu(s, \tau) f(\tau) v(\tau) d\tau &= (j+1) \sum_{i=2}^{j-1} \frac{i}{i-1} \int_{i-1}^i \tau^2 \sin \tau d\tau + \frac{(j+1)j}{j-1} \int_{j-1}^s \tau^2 \sin \tau d\tau \\
&= O(s^4), \quad s \rightarrow \infty,
\end{aligned} \tag{39}$$

and

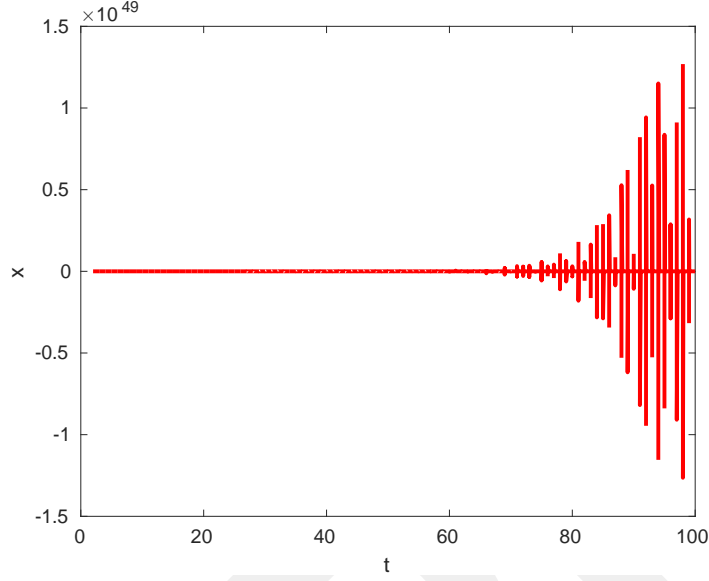
$$\begin{aligned}
\sum_{i=\underline{n}(a)}^{\bar{n}(s)} \mu(s, \theta_i) M_i &= \sum_{i=2}^j \frac{j+1}{i} 2i^4 [1 - (-1)^i i^3(i+1)(i^2+1)] \\
&= (-1)^{j+1} j^{10} + O(j^9), \quad j \rightarrow \infty.
\end{aligned} \tag{40}$$

Then, from (39) we have

$$\begin{aligned}
\int_a^t \frac{1}{p(s)v^2(s)} \int_a^s \mu(s, \tau) f(\tau) v(\tau) d\tau &= \sum_{j=2}^{k-1} \int_{j-1}^j \frac{1}{j^2 s^6 (s^2+1)} \int_1^s \mu(s, \tau) f(\tau) v(\tau) d\tau ds \\
&\quad + \int_{k-1}^t \frac{1}{k^2 s^6 (s^2+1)} \int_1^s \mu(s, \tau) f(\tau) v(\tau) d\tau ds \\
&= O(1), \quad t \rightarrow \infty.
\end{aligned} \tag{41}$$

In a similar way, (40) implies that

$$\int_a^t \frac{1}{p(s)v^2(s)} \sum_{i=\underline{n}(a)}^{\bar{n}(s)} \mu(s, \theta_i) M_i ds = (-1)^k \frac{k^8}{7t^7} + O(1), \quad t \rightarrow \infty. \tag{42}$$

FIGURE 1. Graph of solution of (37) for $t \in [1, 100]$.

Finally, for $t \in (k-1, k]$ we can write

$$y_4(t) = \sum_{i=2}^k (-1)^i i = (-1)^k \frac{k+2}{4} + \frac{3}{4}. \quad (43)$$

Properties (41), (42), and (43) imply $\limsup_{t \rightarrow \infty} [y_3(t) + y_4(t)] = -\liminf_{t \rightarrow \infty} [y_3(t) + y_4(t)] = \infty$, or

$$\limsup_{t \rightarrow \infty} \mathcal{H}(t) = -\liminf_{t \rightarrow \infty} \mathcal{H}(t) = \infty. \quad (44)$$

Combining (38) and (44), we see that (33) holds. Since all the conditions of Theorem 4.1 are satisfied, we conclude that (37) is oscillatory. The oscillatory behavior of the solution is shown in Figure 1 for $\alpha = 1/2$ and $\beta = 3/2$.

It is worth mentioning that the impulsive equation

$$\begin{cases} (t^2(t^2+1)x')' + 4|x|^{1/2}x - 10(t^2+1)|x|^{-1/2}x = \sin t, & t \neq i, \quad t \geq 1, \\ \Delta(t^2(t^2+1)x') + 2i^2(i^2+1)|x|^{1/2}x - 2(i^2+1)^2|x|^{-1/2}x = 2i, & t = i, \quad i > 1, \end{cases} \quad (45)$$

obtained from (37) by setting $\Delta x = 0$, and the nonimpulsive equation

$$(t^2(t^2+1)x')' + 4|x|^{1/2}x - 10(t^2+1)|x|^{-1/2}x = \sin t, \quad t \geq 1, \quad (46)$$

obtained by removing impulses in (37) are nonoscillatory. This is illustrated in Figure 2 and Figure 3, respectively.

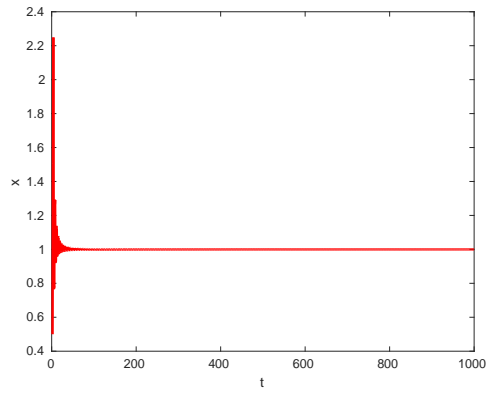


FIGURE 2. Graph of solution of (45)

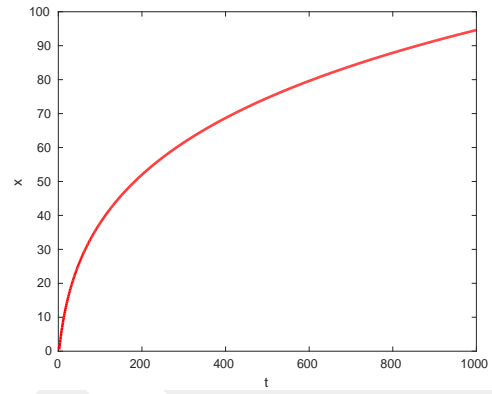


FIGURE 3. Graph of solution of (46)

Graphs of the solutions of equations (45) and (46) for $t \in [1, 1000]$.

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