# A new extended $\delta$-shock model with the consideration of shock magnitude[*]

by

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## Abstract

In this paper, a new $\delta$-shock model that takes into account the magnitude of shocks is introduced and studied from reliability perspective. According to the new model, the system breaks down if either a shock after non-critical shock occurs in a time length less than $\delta_1$ or a shock after a critical shock occurs in a time length less than $\delta_2$, where $\delta_1 < \delta_2$. The distribution of the system's lifetime is studied for both discrete and continuous intershock time distributions. It is shown that a new model is useful to describe a certain cold standby repairable system.

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## 1 Introduction

The $\delta$-shock model and its extensions have attracted great deal of attention in applied probability. Under the classical construction of the model, the system failure occurs when the time between two successive shocks is less than a given threshold $\delta$. The distribution and hence the reliability properties corresponding to the system's lifetime have been widely studied under the $\delta$-shock model for different types of shock arrival processes (Li and Kong (2007), Eryilmaz (2013), Eryilmaz (2017), Chadjiconstantinidis and Eryilmaz (2021)). Various extensions of the model have also been introduced and studied in the literature (Lorvand et al. (2020), Poursaeed (2021), Lorvand and Nematollahi (2022), Goyal et al. (2022), Lyu et al. (2023), Eryilmaz and Unlu (2023), Chadjiconstantinidis (2024)).

Under the ordinary version of the $\delta$-shock model, the recovery time ($\delta$) of the system does not depend on the magnitudes of shocks. In most real life situations, the recovery time of a system may depend on the size of the damage produced by the shock. In the present paper, we introduce a new version of the $\delta$-shock model by taking into account the maginutes of shocks. In particular, for two positive thresholds $\delta_1$ and $\delta_2$ such that $\delta_1 < \delta_2$, the system breaks down if either a shock after non-critical shock occurs in a time length less than $\delta_1$ or a shock after a critical shock occurs in a time length less than $\delta_2$. Thus, the recovery time depends on the type of the shock which is classified as either critical or non-critical. As it will be described in the paper, this new extended model is potentially useful to describe a certain cold standby repairable system with two components. Under this new model, the distribution of the system's lifetime is studied for both discrete and continuously distributed intershock times. Our method is based on the derivation of the probability generating function (Laplace-Stieltjes transform) for the discrete (continuous) intershock time distribution. The case of continuous intershock times is more common in the literature as Poisson shock arrival process defines exponentially distributed intershock times. The discretely distributed intershock times appear when the shocks occur randomly over the periods, i.e. when for example the shock arrival process is Binomial.

In the following, we describe the model. Consider a system that is subject to shocks over time. Define two positive thresholds $\delta_1$ and $\delta_2$ such that $\delta_1 < \delta_2$. Assume that a shock is classified as a critical shock if its magnitude is at least $c$, a given fixed quantity. The system fails if either a shock after a non-critical shock occurs in a time length less than $\delta_1$ or a shock after a critical shock occurs in a time length less than $\delta_2$. Let $X_i$ denote the time between $i$th and $(i+1)$th shocks for $i = 1, 2, ...$ and $X_0$ be the time until the first shock. Denote $Y_i$ be the magnitude of the $i$th shock which occurs at $X_0 + X_1 + ... + X_{i-1}$ for $i = 1, 2, ....$ It is assumed that $X_0, X_1, ...$

are independent and identically distributed and they are independent of $Y_i$s. There are two failure modes for the system under concern. In the first case, the system fails after the $n$th shock such that $Y_n < c, X_n < \delta_1$. In the second case, the system failure occurs after the $n$th shock when $Y_n \geq c, X_n < \delta_2$.

If $N$ denotes the number of shocks until the failure of the system, then

$$\{N = n\} \text{ iff } \bigcup_{i=0}^{n-1} \{S_{n-1} = i,\ i \text{ of } X\text{s} \geq \delta_2, n-i-1 \text{ of } X\text{s} \geq \delta_1 \ , \\ (Y_n < c, X_n < \delta_1 \ \cup Y_n \geq c, X_n < \delta_2)\} \ ,$$

where $S_{n-1}$ is the total number of shocks of size at least $c$ in $n-1$ shocks and $X$s represent $X_1, ..., X_{n-1}$. The lifetime of the system is then defined by

$$T = \sum_{i=0}^{N} X_i. \tag{1}$$

It can be shown that

$$\begin{aligned} P\{N = n\} &= [pP\{X \geq \delta_2\} + (1-p)P\{X \geq \delta_1\}]^{n-1} \\ &\quad \times [(1-p)P\{X < \delta_1\} + pP\{X < \delta_2\}] \ , \end{aligned} \tag{2}$$

for $n = 1, 2, ...,$ where $p = P\{Y \geq c\}$. The distribution of the random variable $N$ can be obtained by conditioning on the number of shocks of size at least $c$ and considering the two failure modes of the system.

This new model is potentially useful to describe and study a certain system. Consider a cold standby system that consists of two identical repairable components. At time $t = 0$, both of the two components are new, and one component starts working while the other component is in a cold standby state. The standby component is put into operation as soon as the active component fails, and a repair action is immediately taken for the failed component. A damage of random amount occurs upon the failure of a component. Assume that the size of the damage for the component who lives $X_i, i = 0, 1, ...$ is denoted by the random variable $Y_{i+1}$. If the size of the damage is at least $c$, then the repair time is $\delta_2$ while the time to repair is $\delta_1$ when the size of the damage is below $c$. That is, a longer repair time occurs when the damage magnitude exceeds the critical level. Thus, the lifetime of such a repairable system is represented by (1).

The rest of the paper is organized as follows. Section 2 involves the main results on the system's lifetime distribution. Sections 3 and 4 respectively contain numerical results and an application of the model.

## 2 The distribution of the system's lifetime

In this section, we study the distribution of the system's lifetime for both discrete and continuously distributed intershock times. In particular, we obtain the probability generating function (Laplace-Stieltjes transform) of the system's lifetime for the discrete (continuous) intershock time distribution.

### 2.1 The discrete case

Assume that the random variables $X_0, X_1, ...$ have common discrete distribution which has probability generating function (pgf) $\phi(z) = E(z^X)$. Define the following conditional probability generating functions

$$\phi_1(z) = E(z^X \mid X < \delta_1), \quad \bar{\phi}_1(z) = E(z^X \mid X \geq \delta_1),$$

and

$$\phi_2(z) = E(z^X \mid X < \delta_2), \quad \bar{\phi}_2(z) = E(z^X \mid X \geq \delta_2).$$

Manifestly, we have

$$\phi_i(z)P(X < \delta_i) + \bar{\phi}_i(z)P(X \geq \delta_i) = \phi(z),$$

for $i = 1, 2$.

**Theorem 1.** Assume that the random variables $X_0, X_1, ...$ have common discrete distribution which has pgf $\phi(z) = E(z^X)$. Then, the pgf of the system's lifetime is

$$\psi(z) = \frac{\phi(z)\left[(1-p)\phi_1(z)P(X < \delta_1) + p\phi_2(z)P(X < \delta_2)\right]}{1 - \left[p\bar{\phi}_2(z)P(X \geq \delta_2) + (1-p)\bar{\phi}_1(z)P(X \geq \delta_1)\right]}, \tag{3}$$

**Proof.** The pgf of the system's lifetime can be written as

$$\psi(z) = E(z^T) = E(z^{\sum_{i=0}^{N} X_i}) = E(z^{X_0})E(z^{\sum_{i=1}^{N} X_i}) = \phi(z)E(z^{\sum_{i=1}^{N} X_i})$$

By conditioning on $N$,

$$
\begin{aligned}
& E(z^{\sum_{i=1}^{N} X_i}) = \sum_{n=1}^{\infty} E(z^{\sum_{i=1}^{N} X_i} | N = n) P(N = n) = \sum_{n=1}^{\infty} E(z^{\sum_{i=1}^{n} X_i} I(N = n)) \\
= \; & \sum_{n=1}^{\infty} E(z^{\sum_{i=1}^{n} X_i} I(\bigcup_{i=0}^{n-1} \{S_{n-1} = i, \; i \text{ of } X\text{s} \geq \delta_2, n-i-1 \text{ of } X\text{s} \geq \delta_1 \; , \\
& (Y_n < c, X_n < \delta_1 \; \cup Y_n \geq c, X_n < \delta_2)\}) \\
= \; & \sum_{n=1}^{\infty} E(z^{X_n} I \left(Y_n < c, X_n < \delta_1 \; \cup Y_n \geq c, X_n < \delta_2\right)) \times \\
& E(z^{\sum_{i=1}^{n-1} X_i} I(\bigcup_{i=0}^{n-1} \{S_{n-1} = i, \; i \text{ of } X\text{s} \geq \delta_2, n-i-1 \text{ of } X\text{s} \geq \delta_1\}) \\
= \; & \sum_{n=1}^{\infty} \left(E(z^{X_n} I \left(Y_n < c, X_n < \delta_1\right) + E(z^{X_n} I(Y_n \geq c, X_n < \delta_2)\right)) \times \\
& E(z^{\sum_{i=1}^{n-1} X_i} I(\bigcup_{i=0}^{n-1} \{S_{n-1} = i, \; i \text{ of } X\text{s} \geq \delta_2, n-i-1 \text{ of } X\text{s} \geq \delta_1\}) \\
= \; & \sum_{n=1}^{\infty} (E(z^{X_n} | Y_n < c, X_n < \delta_1) P(Y_n < c, X_n < \delta_1) \\
+ \; & E(z^{X_n} | Y_n \geq c, X_n < \delta_2) P(Y_n \geq c, X_n < \delta_2)) \times \\
& E(z^{\sum_{i=1}^{n-1} X_i} I(\bigcup_{i=0}^{n-1} \{S_{n-1} = i, \; i \text{ of } X\text{s} \geq \delta_2, n-i-1 \text{ of } X\text{s} \geq \delta_1\}) \\
= \; & [(1-p)\phi_1(z) P(X < \delta_1) + p\phi_2(z) P(X < \delta_2)] \times \\
& \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} \binom{n-1}{j} (E(z^X I(X \geq \delta_1, Y < c)))^{n-j-1} (E(z^X I(X \geq \delta_2, Y > c)))^{j} \\
= \; & [(1-p)\phi_1(z) P(X < \delta_1) + p\phi_2(z) P(X < \delta_2)] \times \\
& \sum_{n=1}^{\infty} (E(z^X I(X \geq \delta_1, Y < c)) + E(z^X I(X \geq \delta_2, Y > c)))^{n-1} \\
= \; & [(1-p)\phi_1(z) P(X < \delta_1) + p\phi_2(z) P(X < \delta_2)] \times \\
& \sum_{n=1}^{\infty} (E(z^X | X \geq \delta_1, Y < c) P(X \geq \delta_1, Y < c) \\
+ \; & E(z^X | X \geq \delta_2, Y > c) P(X \geq \delta_2, Y > c))^{n-1} \\
= \; & \frac{[(1-p)\phi_1(z) P(X < \delta_1) + p\phi_2(z) P(X < \delta_2)]}{1 - \left[p\bar{\phi}_2(z) P(X \geq \delta_2) + (1-p)\bar{\phi}_1(z) P(X \geq \delta_1)\right]}
\end{aligned}
$$

Thus the proof is complete.■

**Corollary 1.** Note that in the special case when $c = 0$, i.e. $p = 1$ we have

$$\psi(z) = \frac{P\left\{X < \delta_2\right\}\phi(z)\phi_2(z)}{1 - P\left\{X \geq \delta_2\right\}\bar{\phi}_2(z)}$$

which is the pgf of the system's lifetime under classical $\delta$-shock model when $\delta = \delta_2$ (see, e.g. Theorem 1 of Eryilmaz and Kan (2021)).■

Consider a special family of discrete distributions known as discrete phase type distributions. Let $X_i$s have discrete phase-type distribution of order $m$ with representation $X_i \sim PH_m(\mathbf{a}, \mathbf{Q}), i = 0, 1, 2, ...$ Then,

$$P\left\{X_i = x\right\} = \mathbf{a}\mathbf{Q}^{x-1}\mathbf{u}',$$

for $x \in \mathbb{N}$, for suitably defined $\mathbf{a}$, $\mathbf{Q}$ and $\mathbf{u}'$=$(\mathbf{I} - \mathbf{Q})\mathbf{e}'$, where $\mathbf{e} = (1, ..., 1)_{1\times m}$ (see, e.g. He (2014)). In this case, $\phi(z) = E(z^X) = \mathbf{a}z(\mathbf{I} - z\mathbf{Q})^{-1}\mathbf{u}'$, and

$$\bar{\phi}_i(z) = \frac{1}{P\left\{X \geq \delta_i\right\}}\mathbf{a}z(\mathbf{I} - \mathbf{Q})(\mathbf{I} - z\mathbf{Q})^{-1}(z\mathbf{Q})^{\delta_i - 1}\mathbf{e}',$$

and

$$\phi_i(z) = \frac{1}{P\left\{X < \delta_i\right\}}\left[\phi(z) - \bar{\phi}_i(z)P\left\{X_i \geq \delta\right\}\right],$$

for $i = 1, 2$ (Eryilmaz and Kan (2021)).

**Corollary 2.** Let $X_i \sim PH_m(\mathbf{a}, \mathbf{Q}), i = 0, 1, 2, ....$ Then, the pgf of the system's lifetime is

$$\psi(z) = \frac{\phi(z)\left[\phi(z) - (1-p)\mathbf{a}z(\mathbf{I} - \mathbf{Q})(\mathbf{I} - z\mathbf{Q})^{-1}(z\mathbf{Q})^{\delta_1 - 1}\mathbf{e}' - p\mathbf{a}z(\mathbf{I} - \mathbf{Q})(\mathbf{I} - z\mathbf{Q})^{-1}(z\mathbf{Q})^{\delta_2 - 1}\mathbf{e}'\right]}{1 - \left[p\mathbf{a}z(\mathbf{I} - \mathbf{Q})(\mathbf{I} - z\mathbf{Q})^{-1}(z\mathbf{Q})^{\delta_2 - 1}\mathbf{e}' + (1-p)z(\mathbf{I} - \mathbf{Q})(\mathbf{I} - z\mathbf{Q})^{-1}(z\mathbf{Q})^{\delta_1 - 1}\mathbf{e}'\right]}$$

In the following Corollary, we obtain the pgf of the system's lifetime when the intershock times have geometric distribution with mean $\frac{1}{p_1}$. It can be obtained as a special case of Corollary 2 when $\mathbf{a} = 1, \mathbf{Q} = 1 - p_1$ and $\mathbf{u}'$=$(\mathbf{I} - \mathbf{Q})\mathbf{e}' = p_1$.

**Corollary 3.** Let $X_0, X_1, ...$ have the geometric distribution with probability mass function (pmf) $P(X_i = x) = p_1(1-p_1)^{x-1}, x = 0, 1, 2, ...,$ for $i = 1, 2, ....$ Then the pgf of system's lifetime is

$$\psi(z) = \frac{p_1^2 z^2 - (1-p)p_1^2 q_1^{\delta_1 - 1} z^{\delta_1 + 1} - p p_1^2 q_1^{\delta_2 - 1} z^{\delta_2 + 1}}{1 - 2q_1 z + q_1^2 z^2 - (1-p)p_1 q_1^{\delta_1 - 1} z^{\delta_1} + (1-p)p_1 q_1^{\delta_1} z^{\delta_1 + 1} - p p_1 q_1^{\delta_2 - 1} z^{\delta_2} + p p_1 q_1^{\delta_2} z^{\delta_2 + 1}},$$

where $q_1 = 1 - p_1$.■

**Corollary 4.** Let $X_0, X_1, ...$ have the geometric distribution with pmf $P(X_i = x) = p_1(1-p_1)^{x-1}, x = 0, 1, 2, ...,$ for $i = 1, 2, ....$ Then the MTTF of the system is

$$E(T) = \frac{2(1-p_1) - (1-p_1)^{\delta_1}(1-p) - p(1-p_1)^{\delta_2}}{p_1\left[1 - p_1 - (1-p_1)^{\delta_1}(1-p) - p(1-p_1)^{\delta_2}\right]}.■$$

For the case when the random variables $X_0, X_1, ...$ have geometric distribution, the pgf of the random variable $T$ has the following form:

$$\psi(z) = E(z^T) = \frac{c_1 z + ... + c_m z^m}{1 + d_1 z + ... + d_m z^m} \tag{4}$$

for some $m \geq 1$ and real constants $c_1, ..., c_m$ and $d_1, ..., d_m$. A random variable that has the pgf in the form of (4) is said to have matrix-geometric distribution. Using the coefficients in (4), the probability mass function and survival function of the random variable $T$ can be expressed respectively as

$$P\{T = t\} = \boldsymbol{\pi}\mathbf{Q}^{t-1}\mathbf{u}', \tag{5}$$

and

$$P\left\{T>t\right\}=\boldsymbol{\pi}\mathbf{Q}^{t}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{u}',$$

where $\boldsymbol{\pi}=(1,0,...,0)$,

$$\mathbf{Q}=\begin{bmatrix} -d_1 & 0 & 0 & \cdots & 0 & 1 \\ -d_m & 0 & 0 & \cdots & 0 & 0 \\ -d_{m-1} & 1 & 0 & \cdots & 0 & 0 \\ -d_{m-2} & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -d_2 & 0 & 0 & & 1 & 0 \end{bmatrix} \quad , \quad \mathbf{u}'=\begin{bmatrix} c_1 \\ c_m \\ c_{m-1} \\ c_{m-2} \\ \vdots \\ c_2 \end{bmatrix}$$

(see, e.g. Bladt and Nielsen (2017)).

For $\delta_1>3$ and $\delta_2>\delta_1+2$, if the intershock times have geometric distribution, then we have $c_1=0, c_2=p_1^2, c_3=0,...,c_{\delta_1}=0, c_{\delta_1+1}=-(1-p)p_1^2q_1^{\delta_1-1}, c_{\delta_1+2}=0,...,c_{\delta_2}=0, c_{\delta_2+1}=-pp_1^2q_1^{\delta_2-1}, d_1=-2q_1, d_2=q_1^2, d_3=0,...,d_{\delta_1-1}=0, d_{\delta_1}=-(1-p)p_1q_1^{\delta_1-1}$, $d_{\delta_1+1}=(1-p)p_1q_1^{\delta_1}, d_{\delta_1+2}=0,...,d_{\delta_2-1}=0, d_{\delta_2}=-pp_1q_1^{\delta_2-1}, d_{\delta_2+1}=pp_1q_1^{\delta_2}$ and hence the system's lifetime has matrix-geometric distribution of order $m=\delta_2+1$. Thus, for given values of $\delta_1$ and $\delta_2$, the exact distribution of $T$ can be obtained via

(5). If for example, $\delta_1 = 4$ and $\delta_2 = 7$, then

$$\mathbf{Q} = \begin{bmatrix} 2q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -pp_1q_1^{\delta_2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ pp_1q_1^{\delta_2-1} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -(1-p)p_1q_1^{\delta_1} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ (1-p)p_1q_1^{\delta_1-1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -q_1^2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{u}' = \begin{bmatrix} 0 \\ -pp_1^2q_1^{\delta_2-1} \\ 0 \\ 0 \\ -(1-p)p_1^2q_1^{\delta_1-1} \\ 0 \\ 0 \\ p_1^2 \end{bmatrix}$$

The hazard rate of the system can be computed from

$$r(t) = \frac{P\{T = t\}}{P\{T \geq t\}} = \frac{\boldsymbol{\pi}\mathbf{Q}^{t-1}\mathbf{u}'}{\boldsymbol{\pi}\mathbf{Q}^{t+1}(\mathbf{I}-\mathbf{Q})^{-1}\mathbf{u}'}.$$

For the mean residual life function $m(t) = E(T - t \mid T \geq t)$, we have

$$m(t+1) = \frac{m(t)}{1 - h(t)} - 1$$

(see, e.g. Salvia (1996)).

## 2.2 The continuous case

Analogously to the proof of Theorem 1, we can obtain the following result.

**Theorem 2.** Assume that the random variables $X_0, X_1, \ldots$ have common continuous distribution. Then, the Laplace-Stieltjes transform of the system's lifetime is

$$L(t) = E(e^{-tX_0}) \frac{L_1(t)(1-p)P(X<\delta_1) + L_2(t)pP(X<\delta_2)}{1-[p\bar{L}_2(t)P(X>\delta_2) + (1-p)\bar{L}_1(t)P(X>\delta_1)]},$$

where

$$\begin{aligned} L_1(t) &= E(e^{-tX}|X<\delta_1), \qquad L_2(t) = E(e^{-tX}|X<\delta_2) \\ \bar{L}_1(t) &= E(e^{-tX}|X>\delta_1), \qquad \bar{L}_2(t) = E(e^{-tX}|X>\delta_2). \end{aligned}$$

**Corollary 5.** Let $X_0, X_1, ...$ have the exponential distribution with the density function $f(x) = \lambda e^{-\lambda x}, x > 0$. Then, the LST of $T$ is:

$$L(t) = \frac{\lambda^2 e^{t\delta_2} - (1-p)\lambda^2 e^{-\lambda\delta_1} e^{t(\delta_2-\delta_1)} - p\lambda^2 e^{-\lambda\delta_2}}{\lambda^2 e^{t\delta_2} + 2t\lambda e^{t\delta_2} + t^2 e^{t\delta_2} - p\lambda^2 e^{-\lambda\delta_2} - p\lambda t e^{-\lambda\delta_2} - (1-p)\lambda e^{-\lambda\delta_1} e^{t(\delta_2-\delta_1)}(\lambda+t)}. \tag{6}$$

**Remark 1.** As it has been shown by Corollary 3, the system's lifetime has matrix-geometric distribution if the interarrival times between successive shocks follow geometric distribution. Thus, the distribution of the system's lifetime can be easily computed by the equation (5). However, if the interarrival times have exponential distribution, then the LST given by (6) does not have the following form

$$L(t) = E(e^{-tT}) = \frac{b_q t^{q-1} + ... + b_2 t + b_1}{t^q + a_q t^{q-1} ... + a_2 t + a_1} \tag{7}$$

which is the standard form of the LST of matrix-exponential distribution. For sufficiently large $q$, using the Taylor expansions for the terms $e^{t\delta_2}$ and $e^{(\delta_2-\delta_1)t}$ an approximate LST that has the form (7) can be obtained. Such an approach has been proposed by Kus et al. (2022) to study the lifetime properties of the classical $\delta$-shock model.■

Following the method proposed by Kus et al. (2022), in the following, we obtain an alternative LST representation for (6) in the form of (7).

**Corollary 6.** Let $X_0, X_1, ...$ have the exponential distribution with the density function $f(x) = \lambda e^{-\lambda x}, x > 0$. Then, the LST given by (6) can be approximated by

$$\tilde{L}(t) = \frac{b_{q+1}t^q + b_q t^{q-1} + ... + b_2 t + b_1}{t^{q+1} + a_{q+1}t^q ... + a_2 t + a_1}, \tag{8}$$

where $a_1 = \lambda^2 \frac{(q-1)!}{\delta_2^{q-1}} \left[1 - (1-p)e^{-\lambda\delta_1} - pe^{-\lambda\delta_2}\right]$,

$$a_2 = \frac{(q-1)!}{\delta_2^{q-1}} \left[\lambda^2\delta_2 + 2\lambda - p\lambda e^{-\lambda\delta_2} - (1-p)\lambda^2(\delta_2-\delta_1)e^{-\lambda\delta_1} - (1-p)\lambda e^{-\lambda\delta_1}\right],$$

$$\begin{aligned} a_3 &= \frac{(q-1)!}{\delta_2^{q-1}} \left[1 + \lambda^2\frac{\delta_2^2}{2} + 2\lambda\delta_2 - (1-p)\lambda^2\frac{(\delta_2-\delta_1)^2}{2}e^{-\lambda\delta_1}\right. \\ &\quad \left. -(1-p)\lambda(\delta_2-\delta_1)e^{-\lambda\delta_1}\right], \end{aligned}$$

and for $i = 4, 5, ..., q$,

$$\begin{aligned} a_i &= \frac{(q-1)!}{\delta_2^{q-1}} \left[\lambda^2\frac{\delta_2^{i-1}}{(i-1)!} + 2\lambda\frac{\delta_2^{i-2}}{(i-2)!} + \frac{\delta_2^{i-3}}{(i-3)!}\right. \\ &\quad \left. -(1-p)\lambda^2\frac{(\delta_2-\delta_1)^{i-1}}{(i-1)!}e^{-\lambda\delta_1} - (1-p)\lambda\frac{(\delta_2-\delta_1)^{i-2}}{(i-2)!}e^{-\lambda\delta_1}\right], \end{aligned}$$

$$a_{q+1} = \frac{(q-1)!}{\delta_2^{q-1}} \left[2\lambda\frac{\delta_2^{q-1}}{(q-1)!} + \frac{\delta_2^{q-2}}{(q-2)!} - (1-p)\lambda\frac{(\delta_2-\delta_1)^{q-1}}{(q-1)!}e^{-\lambda\delta_1}\right],$$

and $b_1 = \lambda^2 \frac{(q-1)!}{\delta_2^{q-1}} \left[1 - (1-p)e^{-\lambda\delta_1} - pe^{-\lambda\delta_2}\right]$,

$$b_i = \lambda^2\frac{(q-1)!}{\delta_2^{q-1}} \left[\frac{\delta_2^{i-1}}{(i-1)!} - (1-p)\frac{(\delta_2-\delta_1)^{i-1}}{(i-1)!}e^{-\lambda\delta_1}\right],$$

for $i = 2, 3, ..., q, b_{q+1} = 0$.■

By using the coefficients given by Corollary 6, the survival function of the system can be approximated by

$$\tilde{P}\{T > t\} = -\boldsymbol{\alpha} \exp(\boldsymbol{S}t)\boldsymbol{S}^{-1}\boldsymbol{s}, \tag{9}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{S}$ depend on the coefficients in (8). In particular, $\boldsymbol{\alpha} = (b_1, b_2, ..., b_{q+1})$

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & -a_3 & -a_4 & \cdots & -a_{q+1} \end{bmatrix}, \quad \boldsymbol{s} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The equation (9) is the survival function corresponding to the matrix-exponential distribution (see, e.g. Asmussen and Bladt (1997)).

The MTTF of the system can be immediately obtained from Corollary 5.

**Corollary 7.** Let $X_0, X_1, ...$ have the exponential distribution with the density function $f(x) = \lambda e^{-\lambda x}, x > 0$. Then,

$$E(T) = \frac{2 - 3(1-p)e^{-\lambda\delta_1} - 3pe^{-\lambda\delta_2} + (1-p)^2 e^{-2\lambda\delta_1} + p^2 e^{-2\lambda\delta_2} + 2p(1-p)e^{-\lambda(\delta_1+\delta_2)}}{\lambda\left(1 - e^{-\lambda\delta_1} + pe^{-\lambda\delta_1} - pe^{-\lambda\delta_2}\right)^2}.■$$

Theorem 2 can also be used to study the system's lifetime when the shock arrival process is different from Poisson. Let $X_0, X_1, ...$ have the common distribution with the density function $f(x) = \lambda^2 x e^{-\lambda x}, x > 0$. That is, the shock arrival process is a renewal process with Erlang distributed interarrival times. Then, $E(e^{-tX_0}) = \left(\frac{\lambda}{\lambda+t}\right)^2$, and

$$\bar{L}_i(t) = E(e^{-tX} | X > \delta_i) = \frac{\lambda^2 e^{-\delta_i t}(1 + \delta_i(t + \lambda))}{(\lambda + t)^2(1 + \delta_i \lambda)},$$

and

$$L_i(t)P\{X < \delta_i\} + \bar{L}_i(t)P\{X > \delta_i\} = \left(\frac{\lambda}{\lambda + t}\right)^2,$$

for $i = 1, 2$.

## 3 Numerical results

In this section, we give some numerics on the survival probabilities and MTTF of the system for both discrete (geometric) and continuous (exponential) cases.

**Example 1.** Consider a system that is subject to external shocks and the interarrival times between shocks have the geometric distribution with mean $\frac{1}{p_1}$. Table 1 shows the mean time to failure (MTTF) of the system. Table 2 contains the pmf of the system's lifetime for $\delta_1 = 4$, $\delta_2 = 7$ and $p = 0.2$ and different values of the parameter $p_1 = 0.4, 0.6$. The MTTF of the system decreases with $p$. As expected, the smaller $p_1$ the larger MTTF.

| $\delta_1$ | $\delta_2$ | $p_1 = 0.4, p = 0.2$ | $p_1 = 0.4, p = 0.6$ | $p_1 = 0.6, p = 0.2$ | $p_1 = 0.6, p = 0.6$ |
|---|---|---|---|---|---|
| 4 | 10 | 5.529626 | 5.254659 | 3.423369 | 3.377397 |
|  | 12 | 5.524897 | 5.242963 | 3.423287 | 3.377165 |
| 7 | 10 | 5.102378 | 5.063338 | 3.338901 | 3.336332 |
|  | 12 | 5.098889 | 5.053207 | 3.338827 | 3.336111 |

Table 1. The MTTF of the system for geometrically distributed interarrival times

| $p_1$ | $t$ | $P\{T = t\}$ | $p_1$ | $t$ | $P\{T = t\}$ |
|---|---|---|---|---|---|
| 0.4 | 3 | 0.192 | 0.6 | 3 | 0.288 |
|  | 4 | 0.1728 |  | 4 | 0.1728 |
|  | 5 | 0.110592 |  | 5 | 0.073728 |
|  | 10 | 0.02957319 |  | 10 | 0.004968677 |
|  | 15 | 0.0080232521 |  | 15 | 0.0003170169 |
|  | 20 | 0.0021668474 |  | 20 | 0.00001998044 |
|  | 25 | 0.0005850001 |  | 25 | 0.000001257033 |

Table 2. The pmf of the system's lifetime when $\delta_1 = 4$, $\delta_2 = 7$ and $p = 0.2$.

**Example 2.** Consider a system that is subject to external shocks and the interarrival times between shocks have the exponential distribution with mean $\frac{1}{\lambda}$. In Table 3, we compute the MTTF of the system for selected values of the parameters.

| $\delta_1$ | $\delta_2$ | $\lambda = 0.1, p = 0.2$ | $\lambda = 0.1, p = 0.6$ | $\lambda = 0.2, p = 0.2$ | $\lambda = 0.2, p = 0.6$ |
|---|---|---|---|---|---|
| 4 | 10 | 35.629980 | 29.563946 | 13.150361 | 11.765284 |
|  | 12 | 34.782833 | 28.143701 | 13.033506 | 11.528799 |
| 7 | 10 | 28.898023 | 27.222428 | 11.446162 | 11.096371 |
|  | 12 | 28.433420 | 26.112157 | 11.372846 | 10.903673 |

Table 3. The MTTF of the system for exponentially distributed interarrival times

Table 4 contains simulated (S) and approximate (A) values of the survival function of the system for selected values of $t$ when the interarrival times between shocks have the exponential distribution with mean $\frac{1}{\lambda} = 10$, and $p = 0.2, \delta_1 = 4, \delta_2 = 10$. The approximation is based on the LST in Corollary 6 when $q = 4$. According to the results in Table 4, the approximation performs quite well.

|  | $P\{T > t\}$ |  |
|---|---|---|
| $t$ | S | A |
| 3.00 | 0.9617 | 0.9680 |
| 5.00 | 0.9123 | 0.9231 |
| 10.00 | 0.7861 | 0.7870 |
| 11.00 | 0.7676 | 0.7603 |
| 12.00 | 0.7445 | 0.7346 |
| 15.00 | 0.6797 | 0.6645 |
| 18.00 | 0.6212 | 0.6049 |

Table 4. Approximated and simulated values of $P\{T > t\}$

## 4 Concluding remarks

In this paper, a new extension of the well-known $\delta$-shock model has been proposed and studied. This new extension is more flexible to model real life situations since the recovery time of the system may depend on the size of the damage produced by the shock. On the other hand, as described in the paper, such a model is also useful to study a certain cold standby repairable system that consists of two components.

As a future work, this new extension can be studied under different shock arrival processes. The mixed version of the model that also involves a failure mechanism depending on the shock magnitude may be considered as well.

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