NUMERICAL SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES

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ABSTRACT

NUMERICAL SOLUTIONS OF DYNAMIC EQUATIONS ON TIME SCALES

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The aim of this thesis is to discuss some numerical methods for solving dynamic equations on time scales. For this purpose, the Euler's method and Taylor series method of order 2 are analyzed and described for an arbitrary time scale. The error and convergence analysis for the two methods are also given.

The numerical method known as Trapezoidal Rule is deduced from the Taylor Series method of order2. Both methods are applied to several examples of initial value problems associated with first and second order dynamic equations. The examples contain many parameters which makes it possible to investigate one initial value problem on different time scales and impose different initial conditions.

The numerical results are computed with Matlab and the related approximate and exact solutions are computed both by tabulating their values and by sketching their graphs.

Finally, the conclusion and some additional remarks are given.

Keywords: Time scale, Dynamic equation, Euler method, Trapezoidal rule

ÖZ

ZAMAN SKALASINDA DİNAMİK DENKLEMLERİN SAYISAL ÇÖZÜMLERİ

Sulaiman, Sarah

Yüksek Lisans, Matematik

Tez Yöneticisi : Prof. Dr. İnci ERHAN

Eylül 2021, 94 sayfa

Bu tezin amacı, zaman skalasında dinamik denklemler için bazı sayısal yöntemleri incelemektir. Bu nedenle, keyfi zaman skalası için Euler yöntemi ve ikinci mertebeden Taylor serisi yöntemi analiz edilmiş ve açıklanmıştır. Her iki yöntem için hata ve yakınsaklık analizleri de verilmiştir.

Trapezoid (Yamuk) kuralı olarak bilinen sayısal yöntem, ikinci mertebeden Taylor serisi yönteminden elde edilmiştir. Her iki yöntem, birinci ve ikinci mertebeden dinamik denklemler için başlangıç değer problemlerine uygulanmıştır. Örneklerin parametreler içermesi sayesinde, başlangıç değer problemlerinin çeşitli zaman skalalarında ve farklı başlangıç değerleri verilerek incelenebilmesi olanağı vardır.

Sayısal sonuçlar Matlab kullanılarak hesaplanmıştır ve ilgili yaklaşık ve tam çözümler, hem değerleri tablolanarak, hem de grafikleri çizilerek karşılaştırılmıştır.

Son olarak, incelenen yöntemlerle ilgili sonuçlar ve bazı ek yorumlar verilmiştir.

Anahtar Kelimeler: Zaman skalası, Dinamik denklem, Euler yöntemi, Trapezoid kuralı

To my father who gave me the greatest gift anyone could give another person 'He believed in me'.

To my mother, the woman I aspire to be.

Finally, to my sisters and soul-mates Marwah and Yara.

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$\mathbb{N}$	:	the set of natural numbers
$\mathbb{N}_0$	:	the set of non-negative integers
$\mathbb{Z}$	:	the set of integers
$\mathbb{R}$	:	the set of real numbers
$\mathbb{C}$	:	the set of complex numbers
$\mathbb{T}$	:	a time scale
σ	:	forward jump operator
ρ	:	backward jump operator
Δ	:	delta derivative

CHAPTER 1

INTRODUCTION

Time scale theory is a consolidation of the theory of difference equations with that of differential equations, unifying integral and differential calculus with finite difference calculus, and providing a formalism for the study of hybrid discrete-continuous dynamical systems. It has applications in any field that requires simultaneous modeling of discrete and continuous data.

Time scales have been originally introduced by the German mathematician Stefan Hilger in his PhD thesis in 1988 [1] which was published later [2, 3] and attracted the attention of many scientists. After this pioneering work, many studies on calculus on time scales appeared in the literature [4, 5, 6, 7, 8]. The two main properties of the time scale calculus are **Unification** and **Extension**.

The time scales has a huge potential for real life applications. In particular, the life span of many insect species can be represented by continuous and discrete time intervals which mathematically can be best modeled by a time scale. For example, one can model insect populations that evolve continuously throughout the season, die out in winter while their eggs are incubating or resting, and then hatch in a new season, creating a non-overlapping population.

The continuous processes are described by differential equations (ordinary or partial) while the discrete processes are modeled by difference equations. On time scales the dynamic equations unify these two types of equations, and in this sense, avoids proving the theoretical results twice - once for differential equations and another time for difference equations. Many results involving differential equations are quite easily transferred to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. This makes the studies on time scales and especially the studies on dynamic equations very valuable. This fact can be understood from the enormous amount of publications on the subject [6, 9, 10, 11, 12].

The general idea is to prove a result for a dynamic equation where the domain of the unknown function is a so-called time scale (also called a time set), which can be any closed subset of the real. In this way, the results hold not only for the set of real numbers or the set of integers, but also for more general time scales.

We first give the definitions of basic notions on time scales.

Definition 1.0.1 *[13] A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, and usually a time scale is denoted by the symbol $\mathbb{T}$.*

The sets $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{N}_0$ and the closed intervals of real numbers are time scales.
The sets $\mathbb{Q}$, $\mathbb{R}/\mathbb{Q}$, $\mathbb{C}$, and the open or half open intervals such as (2,5), (3,5] and [3,5) are not time scales.

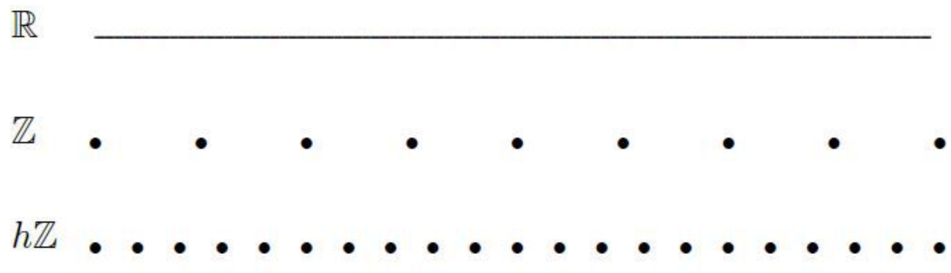

Figure 1.1: Some examples of time scales

Definition 1.0.2 *[13] Let $\mathbb{T}$ be a time scale.*

1. *For $t \in \mathbb{T}$ we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ as follows*

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$

and we note that $\sigma(t) \geq t$ for any $t \in \mathbb{T}$.

2. *For $t \in \mathbb{T}$ we define the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ as follows*

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

and we note that $\rho(t) \leq t$ for any $t \in \mathbb{T}$.

3. *The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by*

$$\mu(t) = \sigma(t) - t.$$

Remark 1.0.3 *We set*

$$\inf \emptyset = \sup \mathbb{T},$$

$$\sup \emptyset = \inf \mathbb{T},$$

where $\emptyset$ denotes the empty set.

Definition 1.0.4 *[13] For any $t \in \mathbb{T}$ we have the following cases.*

1. *If $\sigma(t) > t$, then t is called right-scattered.*
2. *If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense.*
3. *If $\rho(t) < t$, then t is called left-scattered.*
4. *If $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense.*
5. *If t is left-scattered and right-scattered at the same time, then t is called isolated.*
6. *If t is left-dense and right-dense at the same time, the t is called dense.*

Example 1.0.5 *Let $\mathbb{T} = \{0\} \cup \{1, 1/2, 1/3, 1/4, ...\}$.*

At the point $t = 0$,

$$\sigma(0) = \inf\{s \in \mathbb{T} : s > 0\} = \inf\{1, 1/2, 1/3, 1/4, ...\} = 0,$$

and

$$\rho(0) = \sup\{s \in \mathbb{T} : s < 0\} = \sup \emptyset = \inf \mathbb{T} = 0.$$

Also, $\sigma(1) = \inf\{s \in \mathbb{T} : s > 1\} = \inf\ \emptyset = \ \sup \mathbb{T} = \ 1.$

For $t = \dfrac{1}{n}, \quad n = 2, 3, \cdots,$ *we have*

$$\sigma(t) = \inf\left\{\frac{1}{2}, ..., \frac{1}{(n-1)}\right\} = \frac{1}{(n-1)} = \frac{t}{(1-t)}.$$

Also, for $t = \dfrac{1}{n}, \quad n = 1, 2, 3, \cdots,$

$$\rho(t) = \sup\left\{\frac{1}{(n+1)}, \frac{1}{(n+2)}, \frac{1}{(n+3)}, ...\right\} = \frac{1}{(n+1)} = \frac{t}{(1+t)}.$$

It becomes clear that

$$\sigma(t) = \begin{cases} 0 & , t = 0 \\ \dfrac{t}{1-t} & , t \in \left\{\frac{1}{n}\right\}_{n=2}^{\infty} \\ 1 & , t = 1 \end{cases}$$

and similarly,

$$\rho(t) = \begin{cases} 0 & , t = 0 \\ \dfrac{t}{1+t} & , t \in \{1, 1/2, 1/3, ...\}. \end{cases}$$

It suffices to realize that for dense points, the forward jump operator and backward jump operator returns the same time scale element that is drawn from the domain. For the isolated points the forward and backward jump operators return the next and the previous time scale element, respectively.

Example 1.0.6 *Let* $\mathbb{T} = \left\{\sqrt{2n+1} : n \in \mathbb{N}\right\} = \left\{\sqrt{3},\ \sqrt{5},\ \sqrt{7}, ...\right\}.$

Then we have $t = \ \sqrt{2n+1}$ *so that* $n = \dfrac{t^2 - 1}{2}.$
We compute

$$\begin{aligned} \sigma(t) &= \inf\left\{\sqrt{2l+1}\ \middle|\ \sqrt{2l+1} > \ \sqrt{2n+1}\right\} \\ &= \ \sqrt{2(n+1)+1} \\ &= \ \sqrt{2n+3} \end{aligned}$$

$$= \sqrt{t^2+2}.$$

For $t \neq \sqrt{3}$,

$$\rho(t) = \sup\left\{ \sqrt{2l+1} \;\middle|\; \sqrt{2l+1} < \sqrt{2n+1} \right\}$$
$$= \sqrt{2(n-1)+1}$$
$$= \sqrt{2n-1}$$
$$= \sqrt{t^2-2}.$$

For $t = \sqrt{3}$,

$$\rho(\sqrt{3}) = \sup\left\{ \sqrt{2l+1} \;\middle|\; \sqrt{2l+1} < \sqrt{3} \right\}$$
$$= \sup \emptyset$$
$$= \inf \mathbb{T}$$
$$= \sqrt{3}.$$

Then we obtain:

$$\sigma(t) = \sqrt{t^2+2}.$$

and

$$\rho(t) = \begin{cases} \sqrt{t^2+2} & if\ t \neq \sqrt{3} \\ \sqrt{3} & if\ t = \sqrt{3} \end{cases}.$$

The rest of the thesis is organized as follows. In Chapter 2, we present the necessary preliminaries on time scale calculus, such as differentiation, integration and define some elementary functions. In Chapter 3, we introduce the Euler method on arbitrary time scale, derive the local and global truncation errors of the method and apply it to numerical examples. In Chapter 4, we discuss the Taylor series method of order 2, perform its convergence analysis and derive the Trapezoidal rule as a particular case. We also give numerical examples of Taylor series method and Trapezoidal rule. Chapter 5 contains conclusion, remarks and further comments.

CHAPTER 2

BASIC CALCULUS ON TIME SCALES

In this chapter, the basic notions of calculus such as differentiation and integration on time scales are reviewed and discussed. Also, some elementary functions and time scales monomials on time scales are presented together with the Taylor Series theorem. Finally, the first order dynamic equation and the conditions for existence and uniqunesses of their solutions are given.

2.1 Delta derivative

We define first the delta derivative on time scales.

Definition 2.1.1 *[13] Let $\mathbb{T}$ be a time scale. We define the set $\mathbb{T}^k$ as follows*

$$\mathbb{T}^k = \begin{cases} \mathbb{T}/(\rho(\sup\mathbb{T}), \sup\mathbb{T}) & if \sup\mathbb{T} < \infty \\ \mathbb{T} & if \sup\mathbb{T} = \infty \end{cases}.$$

Definition 2.1.2 *[13] Let $t \in \mathbb{T}$ and let $y : \mathbb{T} \to \mathbb{R}$. Assign $y^{\Delta}(t)$ to be the function (if it exists) such that for any $\epsilon > 0$ there exists $\delta > 0$ for which*

$$\left|[y(\sigma(t)) - y(s)] - y^{\Delta}(t)[\sigma(t) - s]\right| \leq \epsilon|\sigma(t) - s|, \quad \textit{for all } s \in U_s = (t-\delta, t+\delta) \subset \mathbb{T}.$$

If $y^{\Delta}(t)$ exists, then it is called the delta derivative of $y(t)$, and y is said to be delta differentiable at t.

Some basic examples of delta derivative are the following.

- If $\mathbb{T} = \mathbb{R}$, then $y^{\Delta}(t) = y'(t)$.

- If $\mathbb{T} = \mathbb{Z}$, then $y^{\Delta}(t) = y(t+1) - y(t)$.

Next we give some properties of delta derivative .

Definition 2.1.3 *[14] Let $y : \mathbb{T} \to \mathbb{R}$, and $t \in \mathbb{T}^k$. Then, the following hold.*

1. *If y is delta differentiable at t, then y is continuous at t.*

2. *If y is continuous at t and t is right-scattered ($\sigma(t) > t$), then y is delta differentiable at t with*
$$y^{\Delta}(t) = \frac{f(\sigma(t)) - y(t)}{\sigma(t) - t} = \frac{y(\sigma(t)) - y(t)}{\mu(t)}.$$

3. *If t is right-dense ($\sigma(t) = t$), then y is delta differentiable at t with*
$$y^{\Delta}(t) = y'(t) = \lim_{s \to t} \frac{y(t) - y(s)}{t - s}$$
if this limit exists.

4. *If y is delta differentiable at t, then*
$$y(\sigma(t)) = y(t) + \mu(t) y^{\Delta}(t).$$

Example 2.1.4

1. *If $\mathbb{T} = \mathbb{R}$, then $y : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if y is differentiable in the ordinary sense at t. Then $y^{\Delta}(t) = y'(t)$.*

2. *If $\mathbb{T} = h\mathbb{Z}$, and $h > 0$, then any continuous function $y : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable at $t \in \mathbb{Z}$, with $y^{\Delta}(t) = \dfrac{y(t+h) - yf(t)}{h}$.*

3. *If $\mathbb{T} = q^{\mathbb{N}_0}$, and $q > 1$, then $y^{\Delta}(t) = \dfrac{y(qt) - y(t)}{(q-1)t}$, and this is known as the q-derivative of quantum calculus.*

.

Example 2.1.5 *Let $y : \mathbb{T} \to \mathbb{R}$, be given as $y(t) = \sigma(t)$ for $t \in \mathbb{T}$, where* $\mathbb{T} = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$. *We compute $y^{\Delta}(t)$. Since $y(t) = \sigma(t)$, then*

$$y^{\Delta}(t) = \frac{\sigma(\sigma(t)) - \sigma(t)}{\sigma(t) - t}$$

For the given time scale $\mathbb{T} = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$, put $n = \frac{1}{t}$. Then $t = \frac{1}{n}$ and $\sigma(t) = \dfrac{1}{n-1} = \dfrac{1}{\frac{1}{t} - 1} = \dfrac{t}{1-t}$. Then $y^{\Delta}(t)$ will be obtained as

$$\begin{aligned} y^{\Delta}(t) &= \frac{\sigma(\frac{t}{1-t}) - t}{\frac{t}{1-t} - t} \\ &= \frac{\left[\frac{\frac{t}{1-t}}{1-(\frac{t}{1-t})} - t\right]}{\frac{t-t(1-t)}{1-t}} \\ &= 1 + \frac{1}{1-2t}. \end{aligned}$$

Example 2.1.6 *Let $y(t) = t^2$, for $t \in \mathbb{T}$, where $\mathbb{T}$ will be taken as*

- *Case 1: $\mathbb{T} = \mathbb{N}_0^{\frac{1}{2}} = \{\sqrt{n} : n \in \mathbb{N}_0\}$. and*
- *Case 2: $\mathbb{T} = \{\frac{n}{2} : n \in \mathbb{N}_0\}$*

We compute the delta derivative as follows .

- *Case 1: On the given $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$, put $\sqrt{n} = t$, so $n = t^2$ and $\sigma(t) = \sqrt{t^2 + 1}$, since the elements of $\mathbb{T}$ form an increasing sequence. We have*
$$y^{\Delta}(t) = \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} = \sqrt{t^2 + 1} + t.$$
- *Case 2: On the given $\mathbb{T} = \left\{\frac{n}{2} : n \in \mathbb{N}\right\}$ put $\frac{n}{2} = t$ so $n = 2t$ and $\sigma(t) = \frac{2t+1}{2}$, since the elements of $\mathbb{T}$ form an increasing sequence. Then*
$$y^{\Delta}(t) = \frac{(\sigma(t))^2 - t^2}{\sigma(t) - t} = \sigma(t) + t = 2t + \frac{1}{2}.$$

In the following theorem is given the calculation of the delta derivative for algebraic operations of functions.

Theorem 2.1.7 *[14] Let $f, g : \mathbb{T} \to \mathbb{R}$, be delta differentiable at $t \in \mathbb{T}^k$. Then*

1. *Addition Rule:* $f + g : \mathbb{T} \to \mathbb{R}$ *is differentiable at* $t \in \mathbb{T}^k$ *and*

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

2. *Subtraction Rule:* $f - g : \mathbb{T} \to \mathbb{R}$ *is differentiable at* $t \in \mathbb{T}^k$ *and*

$$(f-g)^{\Delta}(t) = f^{\Delta}(t) - g^{\Delta}(t).$$

3. *Constant Multiple Rule : if* α *is any constant,* $\alpha f : \mathbb{T} \to \mathbb{R}$ *is differentiable at* $t \in \mathbb{T}^k$ *and*

$$(\alpha f)^{\Delta}(t) = \alpha f^{\Delta}(t).$$

4. *Product Rule:* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable at* $t \in \mathbb{T}^k$ *with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t)$$

$$or \quad (fg)^{(}t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

5. *Reciprocal Rule : if* $f(t)f(\sigma(t)) \neq 0$*, then* $\frac{1}{f}$ *is differentiable at* $t \in \mathbb{T}^k$

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}.$$

6. *Quotient Rule : if* $g(t)g(\sigma(t)) \neq 0$ *then* $\frac{f}{g}$ *is differentiable at* $t \in \mathbb{T}^k$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

2.2 Integration on time scale

In this section we discuss the integration on general time scales.

Definition 2.2.1 *[14] A function* $f : \mathbb{T} \to \mathbb{R}$ *whose right limits exists at all right-dense point in* $\mathbb{T}$*, and left limits exists at all left-dense point in* $\mathbb{T}$ *is called regulated.*

Definition 2.2.2 *[14] A function* $f : \mathbb{T} \to \mathbb{R}$ *which is continuous at right-dense points in* $\mathbb{T}$*, and whose left limits exist at left-dense points of* $\mathbb{T}$ *is called rd-continuous. The set of rd-continuous function is denoted by*

$$C_{rd} \text{ or } C_{rd}(\mathbb{T}) \text{ or } C_{rd}(\mathbb{T}, \mathbb{R}).$$

In the theorem below, the relations between rd-continuous and regulated functions are given.

Theorem 2.2.3 *[14] Let $f : \mathbb{T} \to \mathbb{R}$ be a function.*

1. *If f is continuous, then f is rd-continuous.*
2. *If f is rd-continuous, then f is regulated.*
3. *The forward jump operator σ is rd-continuous.*
4. *If f is regulated or rd-continuous, then so is $f^{\sigma} = f \circ \sigma$.*
5. *If f is continuous and $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, then $f \circ g$ is continuous.*

Definition 2.2.4 *A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called pre-differentiable with region of differentiation $\boldsymbol{D}$ if*

1. *$\boldsymbol{D} \subset \mathbb{T}^k$.*
2. *$\mathbb{T}^K/\boldsymbol{D}$ is countable and contains no right-scattered elements of $\mathbb{T}$.*
3. *f is differentiable at each $t \in \boldsymbol{D}$ and*

$$F^{\Delta}(t) = f(t), \; for \; all \; t \in \boldsymbol{D}.$$

   Such a function F is called pre-antiderivative of f.

4. *The indefinite integral of a regulated function f is defined as*

$$\int f(t)\Delta t = F(t) + \mathrm{C},$$

   where C is an arbitrary constant.

5. *The Cauchy integral of the function f is defined as*

$$\int_r^s f(t)\Delta t = F(s) - F(r), \quad r, s \in \mathbb{T},$$

where

$$F^{\Delta}t = f(t) \ \ for\ all\ t \in \mathbb{T}^k.$$

The function F is called antiderivative of f.

Example 2.2.5 *Let* $\mathbb{T} = \mathbb{Z}$*, so that* $\sigma(t) = t+1$*, and let* $f(t) = 3t^2+5t+2,\ g(t) = t^3+t^2.$ *For* $t \in \mathbb{T},$ *we have*

$$\begin{aligned} g^{\Delta}(t) &= (t^2)^{\Delta} + (t^2)^{\Delta} \\ &= \sigma^2(t) + t\sigma(t) + t^2 + \sigma(t) + t \\ &= (t+1)^2 + t(t+1) + t^2 + t + 1 + t \\ &= t^2 + 2t + 1 + t^2 + t + t^2 + 2t + 1 \\ &= 3t^2 + 5t + 2. \end{aligned}$$

Hence,

$$\int (3t^2 + 5t + 2)\Delta t = \int g^{\Delta}(t)\Delta t = g(t) + C = t^3 + t^2 + C.$$

Example 2.2.6 *Let* $\mathbb{T} = 2^{\mathbb{N}}$*, so that* $\sigma(t) = 2t$*, and let* $f(t) = \frac{2}{t}\sin(\frac{t}{2})\cos(\frac{3t}{2}),\ g(t) = \sin(t),$ *for* $t \in \mathbb{T}$*. Then we have*

$$\begin{aligned} g^{\Delta}(t) &= \frac{\sin(\sigma(t)) - \sin(t)}{\sigma(t) - t} \\ &= \frac{\sin(2t) - \sin(t)}{2t - t} \\ &= \frac{2\sin\frac{t}{2}\cos\frac{3t}{2}}{t} \\ &= f(t). \end{aligned}$$

Hence,

$$\int \frac{2}{t}\sin(\frac{t}{2})\cos(\frac{3t}{2})\Delta t = \int g^{\Delta}(t)\Delta t = \sin(t) + C.$$

Example 2.2.7 *Let $\mathbb{T} = \mathbb{N}_0^2$, so that $\sigma(t) = (\sqrt{t}+1)^2$, and let $f(t) = \frac{1}{1+2\sqrt{t}} \ln\left(\frac{(\sqrt{t}+1)^2}{t}\right)$, $g(t) = \ln(t)$, for $t \in \mathbb{T}$. Then we have*

$$g^{\Delta}(t) = \frac{\ln(\sigma(t)) - \ln(t)}{\sigma(t) - t}$$

$$= \frac{\ln(\sqrt{t}+1)^2 - \ln(t)}{(\sqrt{t}+1)^2 - t}$$

$$= \frac{1}{t + 2\sqrt{t} + 1 - t} \ln\left(\frac{(\sqrt{t}+1)^2}{t}\right)$$

$$= \frac{1}{1 + 2\sqrt{t}} \ln\left(\frac{(\sqrt{t}+1)^2}{t}\right).$$

Hence,

$$\int \frac{1}{1+2\sqrt{t}} \ln\left(\frac{(\sqrt{t+1})^2}{t}\right)\Delta t = \int g^{\Delta}(t)\Delta t = \ln(t) + C.$$

Theorem 2.2.8 *[13] Every rd-continuous function $f : \mathbb{T} \to \mathbb{R}$, has an antiderivative, In particular, if $t_0 \in \mathbb{T}$, then F defined by:*

$$F(t) = \int_{t_0}^{t} f(\tau)\Delta\tau, \ for\ t \in \mathbb{T}$$

is an antiderivative of f.

Theorem 2.2.9 *[13] Let $f \in C_{rd}(\mathbb{T})$. Then there exists an antiderivative F of f, and*

$$F(t) = \int_{t}^{\sigma(t)} f(\tau)\Delta\tau = f(t)\mu(t).$$

Proof: *we have*

$$\int_{t}^{\sigma(t)} f(\tau)\Delta\tau = F(\sigma(t)) - F(t)$$

$$= \frac{F(\sigma(t)) - F}{\sigma(t) - t}(\sigma(t) - t)$$

$$= F^{\Delta}(t)\mu(t)$$

$$= f(t)\mu(t).$$

2.3 Elementary functions on time scales

Some elementary functions such as exponential function, hyperbolic functions, trigonometric functions and monomials on time scales will be presented in this section. together with some examples.

2.3.1 Exponential function on time scale

First we will recall some notions of complex numbers and functions on complex numbers.

Definition 2.3.1 *[15] Let $h > 0$.*

1. *The Hilger complex numbers are defined and denoted by*
$$\mathbb{C}_h = \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}.$$
2. *The set $\mathbb{Z}_h$ is defined as the strip*
$$\mathbb{Z}_h = \left\{ z \in \mathbb{C} | -\frac{\pi}{h} < Im(z) \leq \frac{\pi}{h} \right\}.$$
For $h = 0$, we set $\mathbb{C}_0 = \mathbb{Z}_0 = \mathbb{C}$.

Definition 2.3.2 *[15] For $h > 0$, we define the cylindrical transformation $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by*
$$\xi_h(z) := \frac{1}{h} Log(1 + zh),$$

where Log is the principal branch oh the logarithm function on $\mathbb{C}$. Moreover, we define $\xi_0(z) = z$, for all $z \in \mathbb{C}$.
We note that
$$\xi_h^{-1}(z) = \frac{e^{zh} - 1}{h},$$
for $z \in \mathbb{Z}_h$.

Definition 2.3.3 *[15] The circle plus addition $\oplus$ on $\mathbb{C}_h$ is defined by*

$$z \oplus w = z + w + zwh.$$

Theorem 2.3.4 *[14]* $(\mathbb{C}_h, \oplus)$ *is an Abelian group.*

Proof

Let $z, w \in \mathbb{C}_h$, *that is* $z, w \neq -\frac{1}{h}$.

1. *Closure : we will show that* $\mathbb{C}_h$ *is closed with respect to* $\oplus$. *Therefore* $z \oplus w \in \mathbb{C}_h$.
   We have $z \oplus w = z + w + hwz$ *and*

$$h(z \oplus w) + 1 = h(z + w + zwh) + 1$$

$$= 1 + hz + hw + zwh^2$$

$$= 1 + hz + hw(1 + hz)$$

$$= (1 + hw)(1 + hz)$$

$$\neq 0,$$

   we conclude that $z \oplus w \in \mathbb{C}_h$.

2. *Associativity : we will show that* $(\mathbb{C}_h, \oplus)$ *is associative, So* $(z \oplus w) \oplus v = z \oplus (w \oplus v)$

$$(z \oplus w) \oplus v = (z + w + zwh) \oplus v$$

$$= z + w + zwh + v + (z + w + zwh)vh$$

$$= z + w + zwh + v + zvh + wvh + zwvh^2.$$

   and

$$z \oplus (w \oplus v) = z + (w \oplus v) + z(w \oplus v)h$$

$$= z + w + v + wvh + z(w + v + wvh)h$$

$$= z + w + v + wvh + zwh + zvh + zwvh^2,$$

   Consequently,

$$z \oplus (w \oplus v) = (z \oplus w) \oplus v.$$

   so $(\mathbb{C}_h, \oplus)$ *is associative.*

3. *Identity element : we will show the existence of an identity element*
   Note that $0 \in \mathbb{C}_h$ *and for any* $z \in \mathbb{C}$

$$z \oplus 0 = z + 0 + z.0.h = z$$

   i.e., 0 is the additive identity for $\oplus$.

4. *Inverse element : we will show the existence of an inverse element*
   For every $z \in \mathbb{C}_h$ *there exists* $z^{-1} \in \mathbb{C}_h$. *We have* $z^{-1} = -\dfrac{z}{1+zh}$, *is the invers of* z. *Clearly,*

$$\begin{aligned} z \oplus z^{-1} &= z + \left(-\frac{z}{1+zh}\right) + z\left(-\frac{z}{1+zh}\right).h \\ &= \frac{z(1+zh) - z - z^2h}{1+zh} \\ &= 0. \end{aligned}$$

   Note that

$$1 + hz^{-1} = 1 + h.(-\frac{z}{1+zh}) = \frac{1+hz-hz}{1+hz} = \frac{1}{1+hz} \neq 0.$$

   Therefore, $z^{-1} \in \mathbb{C}_h$.

5. *Commutativity : we will show that* $(\mathbb{C}_h, \oplus)$, *is commutative*

$$z \oplus w = z + w + zwh = w + z + wzh = w \oplus z.$$

   So, $(\mathbb{C}_h, \oplus)$ *is an Abelian group.*

Definition 2.3.5 *[15] Let* $z \in \mathbb{C}_h$, *The circle minus* $\ominus$ *of* z *is defined as*

$$\ominus z = \frac{-z}{1+zh}$$

.

Theorem 2.3.6 *[15] Let* $z \in \mathbb{C}_h$. *Then* $\ominus(\ominus z) = z$.

Proof: *We have*

$$\begin{aligned} \ominus(\ominus z) &= -\frac{\ominus z}{1 + (\ominus z)h} \\ &= -\frac{\frac{-z}{1+zh}}{1 + \frac{-z}{1+zh}h} \\ &= \frac{\frac{z}{1+zh}}{\frac{1+zh-zh}{1+zh}} \\ &= z. \end{aligned}$$

Remark 2.3.7 *[15] Let $z, w \in \mathbb{C}_h$. Then the circle minus $\ominus$ subtraction is written as*

$$z \ominus w = z \oplus (\ominus w).$$

Then we note that

$$\begin{aligned} z \ominus w &= z \oplus (\ominus w) \\ &= z + (\ominus w) + z(\ominus w)h \\ &= z - \frac{w}{1+wh} - \frac{zwh}{1+wh} \\ &= \frac{z + zwh - w - zwh}{1+wh} \\ &= \frac{z-w}{1+wh}. \end{aligned}$$

Then

$$z \ominus w = \frac{z-w}{1+hw}.$$

Definition 2.3.8 *[15] A function $f : \mathbb{T} \to \mathbb{R}$ is called regressive if*

$$1 + \mu(t) f(t) \neq, 0 \quad for\ all\ t \in \mathbb{T}^k$$

holds. The set of all regressive and rd-continuous function $f : \mathbb{T} \to \mathbb{R}$, is denoted by $\mathcal{R}$ or $\mathcal{R}(\mathbb{T})$ or $\mathcal{R}(\mathbb{T}, \mathbb{R})$.

Remark 2.3.9 *In $\mathcal{R}$, the circle plus addition is defined as*

$$f \oplus g = f + g + \mu f g, \quad f, g \in \mathcal{R},$$

the circle minus is defined as

$$\ominus f = -\frac{f}{1+\mu f},$$

and the circle minus subtraction $\ominus$ is defined as

$$f \ominus g = f \oplus (\ominus g), \quad f, g \in \mathcal{R}.$$

Theorem 2.3.10 *[15] Let $f, g \in \mathcal{R}$. Then*

1. $f \ominus f = 0$,
2. $\ominus(\ominus f) = f$,

3. $f \ominus g \in \mathcal{R}$,

4. $\ominus(f \ominus g) = g \ominus f$,

5. $\ominus(f \oplus g) = (\ominus f) \oplus (\ominus g)$,

6. $f \oplus \frac{g}{1+\mu f} = f + g$.

Definition 2.3.11 *[15] The generalized exponential function for $f \in \mathcal{R}$ is defined as*

$$e_f(t,s) = e^{\int_s^t \xi_{\mu(\tau)}(f(\tau))\Delta\tau} = e^{\int_s^t \frac{1}{\mu(\tau)} Log(1+\mu(\tau)f(\tau))\Delta\tau}, \ for\ s,t \in \mathbb{T}.$$

Some properties of the exponential function $e_f(t,s)$ **:**

Let $f \in \mathcal{R}$. Then we have:

- $e_f(t,r)e_f(r,s) = e_f(t,s)$ for all $t,r,s \in \mathbb{T}$.
- $e_0(t,s) = 1$ and $e_f(t,t) = 1$.
- $e_f^{\Delta}(t,t_0) = f(t).e_f(t,t_0)$.
- $e_f(t,t_0)$, is the solution of the Cauchy problem

$$y^{\Delta}(t) = f(t)y(t), \quad y(t_0) = 1.$$

- $e_f(\sigma(t),s) = (1+\mu(t)f(t)).e_f(t,s)$.
- $e_f(t,s) = \frac{1}{e_f(s,t)} = e_{\ominus f}(s,t)$.
- $e_f(t,s).e_g(t,s) = e_{f\oplus g}(t,s)$.
- $\dfrac{e_f(t,s)}{e_g(t,s)} = e_{f\ominus g}(t,s)$.
- $e_f(t,\sigma(s)).e_f(s,r) = \dfrac{1}{1+\mu(s)f(s)}.e_f(t,r)$.
- $e_{f\ominus g}^{\Delta}(t,t_0) = \dfrac{(f(t)-g(t)).e_f(t,t_0)}{e_g(\sigma(t),t_0)}$.
- $(e_f(c,t))^{\Delta} = -f(t)e_f(c,\sigma(t))$.
- $\int_a^b f(t)e_f(c,\sigma(t))\Delta t = e_f(c,b) - e_f(c,a)$.

Below we give some examples on exponential functions.

Example 2.3.12 *Let $\alpha : \mathbb{T} \to \mathbb{R}$ be regulated, that is , $1 + \alpha(t)\mu(t) \neq 0$, for all $t \in \mathbb{T}$, and let $t_0, t \in \mathbb{T}$, with $t_0 < t$.*

1. *If $\mathbb{T} = h\mathbb{Z}$ with $h > 0$, then every point in $\mathbb{T}$ is isolated and $\mu(t) = h$, for every $t \in \mathbb{T}$.*

   Then

$$
\begin{aligned}
e_\alpha(t, t_0) &= e^{\int_{t_0}^{t} \frac{1}{\mu(\tau)} Log\,[1 + \alpha(\tau)\mu(\tau)]\,\Delta\tau} \\
&= e^{\sum_{s\in[t_0,t)} \frac{1}{\mu(s)} Log\,[(1 + \alpha(s)\mu(s)]\,\mu(s)} \\
&= e^{\sum_{s\in[t_0,t)} Log\,[(1 + h\alpha(s)]} \\
&= \prod_{s\in[t_0,t)} (1 + h\alpha(s)).
\end{aligned}
$$

   If, in particular α is a constant, then

$$
\begin{aligned}
e_\alpha(t, t_0) &= \prod_{s\in[t_0,t)} (1 + h\alpha) \\
&= (1 + h\alpha)^{t-t_0}.
\end{aligned}
$$

2. *If $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$, then every point in $\mathbb{T}$ is isolated and $\mu(t) = (q-1)t$, for every $t \in \mathbb{T}$.*

   Then

$$
\begin{aligned}
e_\alpha(t, t_0) &= e^{\int_{t_0}^{t} \frac{1}{\mu(\tau)} Log\,(1 + \alpha(\tau)\mu(\tau))\,\Delta\tau} \\
&= e^{\sum_{s\in[t_0,t)} \frac{1}{\mu(s)} Log\,[1 + \alpha(s)\mu(s)]\,\mu(s)} \\
&= e^{\sum_{s\in[t_0,t)} Log\,[1 + \alpha(s)\mu(s)]} \\
&= e^{\sum_{s\in[t_0,t)} Log\,[1 + (q-1)s\alpha(s)]} \\
&= \prod_{s\in[t_0,t)} (1 + (q-1)s\alpha(s)).
\end{aligned}
$$

3. *If* $\mathbb{T} = \mathbb{N}_0^k$ *with* $k \in \mathbb{N}$, *then every point in* $\mathbb{T}$ *is isolated and* $\mu(t) = (\sqrt[k]{t} + 1)^k$, *for every* $t \in \mathbb{T}$.

   Then

$$
\begin{aligned}
e_\alpha(t, t_0) &= e^{\int_{t_0}^{t} \frac{1}{\mu(\tau)} Log\,[1 + \alpha(\tau)\mu(\tau)]\,\Delta\tau} \\
&= e^{\sum_{s\in[t_0,t)} \frac{1}{\mu(s)} Log\,[1 + \alpha(s)\mu(s)]\,\mu(s)} \\
&= e^{\sum_{s\in[t_0,t)} Log\,[1 + \alpha(s)\mu(s)]} \\
&= \prod_{s\in[t_0,t)} [1 + \alpha(s)\mu(s)] \\
&= \prod_{s\in[t_0,t)} \Big(1 + \big((\sqrt[k]{s} + 1)^k - s\big)\alpha(s)\Big).
\end{aligned}
$$

2.3.2 Hyperbolic and Trigonometric Functions

Definition 2.3.13 *[15] (Hyperbolic Functions) If* $f \in \mathcal{R}$ *and* $-\mu f^2 \in \mathcal{R}$, *then we define the hyperbolic functions* $\cosh_f$ *and* $\sinh_f$ *as:*

$$\cosh_f(t, s) = \frac{e_f(t, s) + e_{-f}(t, s)}{2} \quad \textit{and} \quad \sinh_f(t, s) = \frac{e_f(t, s) - e_{-f}(t, s)}{2}.$$

Theorem 2.3.14 *[15] If* $f \in \mathcal{R}$ *and* $-\mu f^2 \in \mathcal{R}$, *then :*

1. $\cosh_f^\Delta(t, s) = f \sinh_f(t, s)$,

2. $\sinh_f^\Delta(t, s) = f \cosh_f(t, s)$,

3. $\cosh_f^2(t, s) - \sinh_f^2(t, s) = e_{-\mu f^2}(t, s)$.

Proof

1.

$$
\begin{aligned}
\cosh_f^\Delta(t, s) &= \left(\frac{e_f(t, s) + e_{-f}(t, s)}{2}\right)^\Delta \\
&= \frac{f e_f(t, s) - f e_{-f}(t, s)}{2} \\
&= f \sinh_f(t, s).
\end{aligned}
$$

2.

$$
\begin{aligned}
\sinh_f^{\Delta}(t,s) &= \left(\frac{e_f(t,s) - e_{-f}(t,s)}{2}\right)^{\Delta} \\
&= \frac{f e_f(t,s) + f e_{-f}(t,s)}{2} \\
&= f \cosh_f(t,s).
\end{aligned}
$$

3.

$$
\begin{aligned}
\cosh_f^2(t,s) - \sinh_f^2(t,s) &= \left(\frac{e_f(t,s) + e_{-f}(t,s)}{2}\right)^2 - \left(\frac{e_f(t,s) - e_{-f}(t,s)}{2}\right)^2 \\
= \frac{e_f^2(t,s) + 2e_f(t,s)e_{-f}(t,s) + e_{-f}^2(t,s)}{4} &- \frac{e_f^2(t,s) - 2e_f(t,s)e_{-f}(t,s) + e_{-f}^2(t,s)}{4} \\
&= e_f(t,s)e_{-f}(t,s) \\
&= e_{f\oplus(-f)}(t,s) \\
&= e_{-\mu f^2}(t,s).
\end{aligned}
$$

Definition 2.3.15 *[15] (Trigonometric Functions) If $f \in \mathcal{R}$ and $-\mu f^2 \in \mathcal{R}$, then we define the trigonometric functions $\cos_f$ and $\sin_f$ by :*

$$
\cos_f(t,s) = \frac{e_{if}(t,s) + e_{-if}(t,s)}{2} \quad \textit{and} \quad \sin_f(t,s) = \frac{e_{if}(t,s) - e_{-if}(t,s)}{2i}.
$$

Theorem 2.3.16 *[15] If $f \in \mathcal{R}$ and $-\mu f^2 \in \mathcal{R}$, then*

1. $\cos_f^{\Delta}(t,s) = -f \sin_f(t,s)$,

2. $\sin_f^{\Delta}(t,s) = f \cos_f(t,s)$,

3. $\cos_f^2(t,s) + \sin_f^2(t,s) = e_{\mu f^2}(t,s)$.

Proof

1.

$$
\begin{aligned}
\cos_f^{\Delta}(t,s) &= \left(\frac{e_{if}(t,s) + e_{i-f}(t,s)}{2}\right)^{\Delta} \\
&= \frac{if e_{if}(t,s) - if e_{-if}(t,s)}{2} \\
&= -f \frac{e_{if}(t,s) - e_{-if}(t,s)}{2i} \\
&= -f \sin_f(t,s).
\end{aligned}
$$

2.

$$\sin_f^\Delta(t,s) = \left(\frac{e_{if}(t,s) - e_{-if}(t,s)}{2i}\right)^\Delta$$
$$= \frac{ife_{if}(t,s) + ife_{-if}(t,s)}{2i}$$
$$= f\cos_f(t,s).$$

3.

$$\cos_f^2(t,s) + \sin_f^2(t,s) = \left(\frac{e_{if}(t,s) + e_{-if}(t,s)}{2}\right)^2 + \left(\frac{e_{if}(t,s) - e_{-if}(t,s)}{2i}\right)^2$$
$$= \frac{e_{if}^2(t,s) + 2e_{if}(t,s)e_{-if}(t,s) + e_{-if}^2(t,s) - e_{if}^2(t,s) - 2e_{if}(t,s)e_{-if}(t,s) + e_{-if}^2(t,s)}{4}$$
$$= e_{if}(t,s)e_{-if}(t,s)$$
$$= e_{if\oplus(-if)}(t,s)$$
$$= e_{-\mu f^2}(t,s).$$

2.3.3 Monomials on time scales

Definition 2.3.17 *[15] Let $\mathbb{T}$ be a time scale and $s,t \in \mathbb{T}$. Define the polynomials $p_n(t,s)$ and $h_n(t,s)$ as:*

$$p_0(t,s) = h_0(t,s) = 1,$$

$$p_{k+1}(t,s) = \int_s^t p_k(\sigma(\tau),s)\Delta\tau, \quad h_{k+1}(\tau,s) = \int_s^t h_k(\tau,s)\Delta\tau, \textit{for } k = 0,1,2,....$$

Now, compute $p_1(t,s), h_1(t,s), p_2(t,s)$ and $h_2(t,s)$ as:

$$p_1(t,s) = \int_s^t p_0(\sigma(\tau),s)\Delta\tau = \int_s^t \Delta\tau = t - s,$$

$$p_2(t,s) = \int_s^t p_1(\sigma(\tau),s)\Delta\tau = \int_s^t (\sigma(\tau) - s)\Delta\tau,$$

$$h_1(t,s) = \int_s^t h_0(\tau,s)\Delta\tau = \int_s^t \Delta\tau = t - s,$$

$$h_2(t,s) = \int_s^t h_1(\tau,s)\Delta\tau = \int_s^t (\tau - s)\Delta\tau.$$

Note that:

$$p_k^\Delta(t,s) = p_{k-1}(\sigma(t),s), \quad h_k^\Delta(t,s) = h_{k-1}(t,s), \quad k \in \mathbb{N}.$$

Example 2.3.18 *Let* $\mathbb{T} = 2^{\mathbb{N}_0}$. *Then* $\sigma(t) = 2t$, $\mu(t) = \sigma(t) - t = t$.

We have

$$h_0(t, s) = 1, \quad p_0(t, s) = 1,$$

$$h_1(t, s) = t - s, \quad p_1(t, s) = t - s, \quad h_2(t, s) = \int_s^t (\tau - s)\Delta\tau.$$

Note that $\left(\frac{x^2}{3}\right)^{\Delta} = \frac{1}{3}(\sigma(x) + x) = \frac{3x}{3} = x.$

$$\begin{aligned} h_2(t, s) &= \int_s^t \left\{\left(\frac{\tau^2}{3}\right)^{\Delta} - (s\tau)^{\Delta}\right\}\Delta\tau = \left(\frac{\tau^2}{3} - s\tau\right)\Big|_s^t \\ &= \left(\frac{t^2}{3} - st\right) - \left(\frac{s^2}{3} - s^2\right) = \frac{t^2 - s^2}{3} - s(t - s) \\ &= (t - s)\left(\frac{t + s}{3} - s\right) \\ &= \frac{(t - s)(t - 2s)}{3}, \end{aligned}$$

and

$$p_2(t, s) = \int_s^t (\sigma(\tau) - s)\Delta\tau = \int_s^t (2\tau - s)\Delta\tau$$

Note that $\left(\frac{2x^2}{3}\right)^{\Delta} = \frac{2}{3}(\sigma(x) + x) = 2x$, *So*

$$\begin{aligned} p_2(t, s) &= \int_s^t \left\{\left(\frac{2\tau^2}{3}\right)^{\Delta} - (s\tau)^{\Delta}\right\}\Delta\tau = \left(\frac{2\tau^2}{3} - s\tau\right)\Big|_s^t \\ &= \left(\frac{2t^2}{3} - st\right) - \left(\frac{2s^2}{3} - s^2\right) \\ &\quad \frac{(2t - s)(t - s)}{3}. \end{aligned}$$

Definition 2.3.19 *[15] (Taylor series on* $\mathbb{T}$*)*

Let f *be infinitely many times delta differentiable at some* $\alpha \in \mathbb{T}$. *Then the Taylor series of* f *about* α *is defined as*

$$\sum_{n=0}^{\infty} f^{\Delta^n}(\alpha) h_n(t, \alpha)$$

and converges to f *on some interval containing* α.

Theorem 2.3.20 *[15] (Taylor's Formula)*

Let f *is n-times differentiable on* $\mathbb{T}^{k^n}$, $n \in \mathbb{N}$ *and* $\alpha \in \mathbb{T}^{k^{n-1}}$, $t \in \mathbb{T}$. *Then*

$$f(t) = \sum_{k=0}^{n-1} h_k(t, \alpha) f^{\Delta^k}(\alpha) + \int_{\alpha}^{\rho^{n-1}(t)} h_{n-1}(t, \sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau,$$

where $\int_{\alpha}^{\rho^{n-1}(t)} h_{n-1}(t,\sigma(\tau)) f^{\Delta^n}(\tau)\Delta\tau$, *is the remainder term.*

Next, we give some examples for computing Taylor series on a general time scale $\mathbb{T}$.

Example 2.3.21 *Let* $f(t) = e_c(t,\alpha)$, *where c is a constant, Now we will find the Taylor series of* $f(t)$. *Since we have* $f(t) = e_c(t,\alpha)$, *then:*

$$f^{\Delta}(t) = (e_c(t,\alpha))^{\Delta} = ce_c(t,\alpha), \quad f^{\Delta}(\alpha) = c$$

$$f^{\Delta^2}(t) = c^2 e_c(t,\alpha), \quad f^{\Delta^2}(\alpha) = c^2$$

$$f^{\Delta^3}(t) = c^3 e_c(t,\alpha), \quad f^{\Delta^3}(\alpha) = c^3$$

$$f^{\Delta^n}(t) = c^n e_c(t,\alpha), \quad f^{\Delta^n}(\alpha) = c^n.$$

So we come to the conclusion that Taylor series of $e_c(t,\alpha)$ *about* α *is*

$$\sum_{n=0}^{\infty} c^n h_n(t,\alpha) = h_0(t,\alpha) + ch_1(t,\alpha) + ... + c^n h_n(t,\alpha) + ...$$

Example 2.3.22 *Let* $f(t) = \cos_c(t,\alpha)$ *or* $f(t) = \sin_c(t,\alpha)$ *where c is a constant and* $\alpha \in \mathbb{T}$, *Now we will find the Taylor series of both* $\sin_c(t,\alpha)$ *and* $\cos_c(t,\alpha)$.

(1) *Take* $f(t) = \cos_c(t,\alpha)$

$$f(t) = \cos_c(t,\alpha), \quad f(\alpha) = \cos_c(\alpha,\alpha) = 1 \quad \textit{where c is a constant .}$$

$$f^{\Delta}(t) = -c\sin_c(t,\alpha), \quad f^{\Delta}(\alpha) = 0.$$

$$f^{\Delta^2}(t) = -c^2\cos_c(t,\alpha), \quad f^{\Delta^2}(\alpha) = -c^2.$$

$$f^{\Delta^3}(t) = c^3\sin_c(t,\alpha), \quad f^{\Delta^3}(\alpha) = 0.$$

$$f^{\Delta^4}(t) = c^4\cos_c(t,\alpha), \quad f^{\Delta^4}(\alpha) = c^4.$$

$$\cdots$$

$$f^{\Delta^n}(\alpha) = \begin{cases} 0, & if \quad n = 2k+1 \\ (-1)^k c^{2k}, & if \quad n = 2k \end{cases}.$$

So, the Taylor series of $\cos_c(t,\alpha)$ *is*

$$\sum_{k=0}^{\infty} f^{\Delta^{2k}}(\alpha) h_{2k}(t,\alpha) = \sum_{k=0}^{\infty} (-1)^k.c^{2k}.h_{2k}(t,\alpha)$$

$$= h_0(t,\alpha) - c^2 h_2(t,\alpha) + c^4 h_4(t,\alpha) + \ldots.$$

(2) *Take* $f(t) = \sin_c(t, \alpha)$

$$f(t) = \sin_c(t, \alpha), \quad f(\alpha) = \sin_c(\alpha, \alpha) = 0 \quad \textit{where c is a constant}$$

$$f^{\Delta}(t) = c\cos_c(t, \alpha), \quad f^{\Delta}(\alpha) = c.$$

$$f^{\Delta^2}(t) = -c^2 \sin_c(t, \alpha), \quad f^{\Delta^2}(\alpha) = 0.$$

$$f^{\Delta^3}(t) = -c^3 \cos_c(t, \alpha), \quad f^{\Delta^3}(\alpha) = -c^3.$$

$$f^{\Delta^4}(t) = c^4 \sin_c(t, \alpha), \quad f^{\Delta^4}(\alpha) = 0.$$

$$\ldots$$

$$f^{\Delta^n}(\alpha) = \begin{cases} 0, & if \quad n = 2k \\ (-1)^k c^{2k}, & if \quad n = 2k+1 \end{cases}.$$

So, the Taylor series of $\sin_c(t, \alpha)$ *is*

$$\sum_{k=0}^{\infty} f^{\Delta^{2k+1}}(\alpha) h_{2k+1}(t, \alpha) = \sum_{k=0}^{\infty} (-1)^k . c^{2k+1} . h_{2k+1}(t, \alpha)$$

$$= ch_1(t, \alpha) - c^3 h_3(t, \alpha) - c^5 h_5(t, \alpha) - \ldots.$$

2.4 Dynamic Equations On Time Scales

First we discuss the linear dynamic equations of first order and the existence and uniqueness of their solutions.

Definition 2.4.1 *[14] A first order linear dynamic equation is an equation of the form*

$$y^{\Delta} = p(t)y + f(t), \tag{2.1}$$

where $p(t)$ *and* $f(t)$ *are given functions.*

The adjoint equation is an equation of the form.

$$x^{\Delta} = -p(t)x^{\sigma} + f(t). \tag{2.2}$$

The next theorem gives the form of the solutions of the equations of equations (2.1) and (2.2).

Theorem 2.4.2 *[14] Let $p : \mathbb{T} \to \mathbb{R}$ be rd-continuous and $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$, and also let, $t_0 \in \mathbb{T}$ and $y_0 = y(t_0)$. Then the initial value problem IVP*

$$y^{\Delta} = p(t)y, \quad y(t_0) = y_0. \tag{2.3}$$

has a unique solution given by

$$y(t) = y_0 e_p(t, t_0).$$

Proof

Let $y(t) = y_0 e_p(t, t_0)$. Then we figure out $y^{\Delta}(t)$ as

$$y^{\Delta}(t) = y_0(e_p(t, t_0))^{\Delta} = y_0 p(t) e_p(t, t_0) = p(t)y(t),$$

This means that y satisfies the dynamic equation in (2.3). Also,

$$y(t_0) = y_0 e_p(t_0, t_0) = y_0.$$

This means that y also satisfies the initial condition. Then $y(t) = y_0 e_p(t, t_0)$ is a solution of the initial value problem (2.3).

For uniqueness, we assume that $y_1(t), \quad y_2(t)$ are two solutions of (2.3). Then

$$y_1^{\Delta} = p(t)y_1, \quad y_2^{\Delta} = p(t)y_2,$$

and hence

$$(y_1 - y_2)^{\Delta} = p(t)(y_1 - y_2).$$

On the other hand,

$$(y_1 - y_2)(t_0) = y_1(t_0) - y_2(t_0) = y_0 - y_0 = 0.$$

Then the function:

$$\psi(t) = y_1(t) - y_2(t)$$

satisfies

$$\psi^{\Delta}(t) = p(t)\psi, \quad \psi(t_0) = 0.$$

In the end, we will have $\psi(t) = 0e_p(t, t_0) \equiv 0$, for all $t \in \mathbb{R}$, that means $y_1(t) - y_2(t) \equiv 0$, which completes the proof of the uniqueness.

Theorem 2.4.3 *[14] Consider the IVP*

$$y^{\Delta} = -p(t)y(\sigma(t)) + f(t), \quad y(t_0) = y_0. \tag{2.4}$$

where f, p are rd-continuous function such that $f, p : \mathbb{T} \to \mathbb{R}$, and $1 + \mu(t)p(t) \neq 0$, for all $t \in \mathbb{T}$, $\quad y_0 \in \mathbb{R}$. Then the solution of the IVP (2.4.3) *is unique and is given as*

$$y(t) = e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau.$$

Proof

Note that

$$\begin{aligned} y^{\Delta}(t) &= \left(e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau\right)^{\Delta} \\ &= y_0 \ominus p(t)e_p(t, t_0) + \left(\int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau\right)^{\Delta} \end{aligned}$$

Differentiation under integral rule is given as

$$\begin{aligned} &\left(\int_{t_0}^{t} f(t, u)\Delta u\right)^{\Delta} \\ &= f(\sigma(t), t) + \int_{t_0}^{t} f^{\Delta}(t, u)\Delta u. \end{aligned}$$

Applying this rule we get

$$\begin{aligned} y^{\Delta}(t) &= y_0(\ominus p(t))e_p(t, t_0) + e_{\ominus p}(\sigma(t), t)f(t) + \int_{t_0}^{t} e^{\Delta}_{\ominus p}(t, \tau)f(\sigma)\Delta\tau \\ &= y_0(\ominus p(t))e_p(t, t_0) + e_{\ominus p}(\sigma(t), t)f(t) + \int_{t_0}^{t} (\ominus p(t))e_{\ominus p}(t, \sigma)f(\sigma)\Delta\sigma. \end{aligned}$$

Since

$$\begin{aligned} e_{\ominus p}(\sigma(t), t)f(t) &= 1 + \mu(t)(\ominus p(t))e_{\ominus p}(t, t)f(t) \\ &= \frac{1}{1 + \mu(t)p(t)}f(t), \end{aligned}$$

we obtain

$$y^{\Delta}(t) = \frac{-p(t)}{1 + \mu(t)p(t)}y_0e_{\ominus p}(t, t_0) + \frac{1}{1 + \mu(t)p(t)}f(t) - \frac{p(t)}{1 + \mu(t)p(t)}\int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau.$$

Multiplying by $1 + \mu(t)p(t)$ we have

$$\begin{aligned} (1 + \mu(t)p(t))y^{\Delta}(t) &= -p(t)e_{\ominus p}(t, t_0)y_0 + f(t) - p(t)\int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau \\ &= -p(t)\left(e_{\ominus p}(t, t_0)y_0 + \int_{t_0}^{t} e_{\ominus p}(t, \tau)f(\tau)\Delta\tau\right) + f(t) \end{aligned}$$

$$= -p(t)y(t) + f(t).$$

Thus

$$y^{\Delta}(t) = -\mu(t)p(t)y^{\Delta}(t) - p(t)y(t) + f(t)$$

$$= -p(t)(\mu(t)y^{\Delta}(t) + y(t)) + f(t).$$

Due to the fact that

$$y^{\Delta}(t) = \frac{y(\sigma(t)) - y(t)}{\mu(t)},$$

we get

$$y^{\Delta}(t) = -p(t)y(\sigma(t)) + f(t).$$

Then, $y(t)$ is a solution of a dynamic equation. Also,

$$y(t_0) = e_{\ominus p}(t_0, t_0)y_0 + \int_{t_0}^{t_0} e_{\ominus p}(t_0, \tau)f(\tau)\Delta\tau = y_0.$$

Now, to show the uniqueness of the solution, let y_1, y_2 be two solution of the IVP (2.4.3),
Then we have

$$y_1^{\Delta} = -p(t)y_1(\sigma(t)) + f(t),$$

$$y_2^{\Delta} = -p(t)y_2(\sigma(t)) + f(t).$$

Therefore,

$$(y_1 - y_2)^{\Delta} = -p(t)(y_1(\sigma(t)) - y_2(\sigma(t))).$$

Now define $\phi(t) = y_1(t) - y_2(t)$. Then $\phi(t)$ satisfies

$$\phi^{\Delta} = -p(t)\phi(t),$$

and

$$\phi(t_0) = y_1(t_0) - y_2(t_0) = y_0 - y_0 = 0.$$

Hence ϕ is the unique solution of

$$\phi^{\Delta} = -p(t)\phi(t) \quad \phi(t_0) = 0.$$

As a result, we conclude that

$$\phi(t) = 0.e_{-p(t)}(t, t_0) \equiv 0,$$

so that,

$$\phi(t) \equiv y_1(t) - y_2(t) \equiv 0,$$

which implies

$$y_1(t) = y_2(t).$$

The existence and uniqueness of solution of IVP for the first order non homogeneous dynamic equations is given in the next theorem.

Theorem 2.4.4 *[14] Consider the IVP*

$$y^{\Delta}(t) = p(t)y(t) + f(t), \quad y(t_0) = y_0,$$

where $p, f : \mathbb{T} \to \mathbb{R}$ *are rd-continuous function and* $1 + \mu(t)p(t) \neq 0, t_0 \in \mathbb{T}, y_0 \in \mathbb{R}$. *Then the IVP has a unique solution in the form*

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^{t} e_p(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Definition 2.4.5 *[14] On a time scale* $\mathbb{T}$, *a function* $f : \mathbb{T}\times\mathbb{R} \to \mathbb{R}$ *is called Lipschitz continuous if there exsits a constant* $L > 0$ *such that*

$$|f(t, x_1) - f(t, x_2)| \leq L|x_1 - x_2| \text{ for all } (t, x_1), (t, x_2) \in \mathbb{T} \times \mathbb{R}.$$

The following theorem gives condiions for the local existence and uniqueness of non-linear IVP for first order dynamic equations. The proof of the theorem can be found in [13].

Theorem 2.4.6 *[13] (Local Existence and Uniqueness) Let* $\mathbb{T}$ *be a time scale,* $t_0 \in \mathbb{T}$, $y_0 \in \mathbb{R}$, $a > 0$ *with* $\inf \mathbb{T} \leq t_0 - a$ *and* $\sup \mathbb{T} \geq t_0 + a$, *and let :*

$$I_a = (t_0 - a, t_0 + a) \text{ and } U_b = \{x \in \mathbb{R} : |x - x_0| < b\}.$$

Suppose that $f : I_a \times U_b \to \mathbb{R}$ *is rd-continuous, bounded (with bound* $M > 0$*), and Lipschitz continuous (with constant* $L > 0$*). Then the IVP*

$$x^{\Delta} = f(t, x),\ x(t_0) = x_0.$$

has exactly one solution on $[t_0 - \alpha, t_0 + \alpha]$, *where*

$$\alpha = min\left\{a, \frac{b}{M}, \frac{1-\epsilon}{L}\right\} \text{ for some } \epsilon > 0.$$

If t_0 *is right-scattered and* $\alpha < \mu(t_0)$, *then the unique solution exists on the interval* $[t_0 - \alpha, \sigma(t_0)]$.

Some example of IVP for first order dynamic equation are given below.

Example 2.4.7 *Consider the IVP*

$$y^{\Delta} = 2y + 3^{t}, \quad y(0) = 0,$$

where $\mathbb{T} = \mathbb{Z}$.

$$p(t) = 2, \quad f(t) = 3^{t}, \quad y_0 = 0,$$

and

$$\sigma(t) = t + 1, \quad \mu(t) = 1, \quad t \in \mathbb{T}$$

The unique solution is given as

$$y(t) = \int_0^t e_2(t, \sigma(\tau))3^{\tau}\Delta\tau = \int_0^t e_2(t, \tau + 1)3^{\tau}\Delta\tau.$$

By using the definition of cylinder transformation and generalized exponential function we have

$$e_2(t, \tau + 1) = e^{\int t^{\tau+1} log3\Delta\tau} = e^{(log3)(t-\tau-1)} = 3^{t-\tau-1}$$

Then

$$y(t) = \int_0^t 3^{t-\tau-1}3^{\tau}\Delta\tau = \int_0^t 3^{t-1}\Delta\tau$$
$$= 3^{t-1}\int_0^t \Delta\tau = t3^{t-1}.$$

is the unique solution of the IVP.

Example 2.4.8 *Consider the IVP*

$$y^{\Delta} = 4y + t, \quad y(0) = 1,$$

on the time scale $\mathbb{T} = 2\mathbb{Z}$. *Here*

$$p(t) = 4, \quad f(t) = t, \quad y_0 = 1.$$

By the existence-uniquess theorem, we have

$$y(t) = e_4(t, 0) + \int_0^t e_4(t, \sigma(\tau))\tau\Delta\tau = e_4(t, 0) + \int_0^4 e_4(t, \tau + 2)\tau\Delta\tau.$$

By using the definition of cylinder transformation and generalized exponential function we have

$$e_4(t, 0) = 9^{\frac{t}{2}},$$

and

$$e_4(t, \tau + 2) = 9^{\frac{t-\tau-2}{2}}.$$

Then we get the unique solution of the IVP as

$$\begin{aligned} y(t) &= 3^t + \int_0^t 9^{\frac{t-\tau-2}{2}} \tau \Delta\tau \\ &= 3^t + 9^{\frac{t}{2}-1} \int_0^t 9^{-\frac{\tau}{2}} \tau \Delta\tau. \end{aligned}$$

CHAPTER 3

THE EULER METHOD FOR DYNAMIC EQUATIONS ON TIME SCALES

The most basic method for finding approximate solution of initial value problems for first order differential equation is the so-called Euler method introduced by Leonhard Euler in his famous 3 volumes monograph ”Institutiones calculi integralis” published between 1786 and 1770. This method is based on the simple idea of constructing the tangent line at a point t to find the value of $x(t+h)$ for some step size $h > 0$, [16, 17, 18]. The Euler method on time scales has been recently introduced in [19].

This chapter analyzes the Euler method for dynamic equations on time scales, including the local truncation error and global truncation error with some numerical examples. Throughout this chapter we assume that $\mathbb{T}$ is a time scale with the forward jump operator σ, the backward jump operator ρ and delta differentiation operator Δ.

3.1 Derivation of the Euler method

Consider the initial value problem

$$\begin{cases} x^{\Delta}(t) = f(t, x(t)) \\ \quad x(t_0) = x_0, \end{cases} \tag{3.1}$$

where $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is a given function, $x_0 \in \mathbb{R}$, $\quad t_0 \in \mathbb{T}$ are given numbers and $x : \mathbb{T} \to \mathbb{R}$ is unknown function. Suppose that $t \in [t_0, t_N]_{\mathbb{T}}$, and take the points

$$t_0 < t_1 < \ldots < t_N, \quad t_i \in \mathbb{T}, \quad i = 1, \ldots, N,$$

where

$$t_i = t_{i-1} + r_i = \begin{cases} \sigma^{l_{i-1}}(t_{i-1}), & l_{i-1} \in \mathbb{N} \quad if\ t_{i-1}\ is\ right-scattered \\ t_{i-1} + q_{i-1}, & q_{i-1} \in \mathbb{R} \quad if\ t_{i-1}\ is\ right-dense \end{cases}. \tag{3.2}$$

The Taylor's formula of the first order gives

$$x(t_{i+1}) = x(t_i + r_i) = x(t_i) + h_1(t_i + r_i, t_i)x^{\Delta}(t_i)$$
$$+ \int_{t_i}^{\rho(t_i+r_i)} h_1(t_i + r_i, \sigma(\tau))x^{\Delta^2}(\tau)\Delta\tau.$$

Denote the integral remainder by

$$R_1(r_i) = \int_{t_i}^{\rho(t_i+r_i)} h_1(t_i + r_i, \sigma(\tau))x^{\Delta^2}(\tau)\Delta\tau.$$

Then the previous formula can be written by

$$x(t_{i+1}) = x(t_i) + r_i x^{\Delta}(t_i) + R_1(r_i).$$

Using the dynamic equation in (3.1) and neglecting the remainder $R_1(r_i)$ we get the approximate solution as

$$x_{i+1} = x_i + r_i f(t_i, x_i) = \begin{cases} x_i + (\sigma^{l_i}(t_i) - t_i)f(t_i, x_i) & if\ t_i\ is\ right-scattered \\ x_i + q_i f(t_i, x_i) & if\ t_i\ is\ right-dense, \end{cases} \tag{3.3}$$

where t_0, x_0 are given in the initial conditions and t_i, $i = 1, \ldots, N$ are defined in (3.2).

Definition 3.1.1 *The formula* (3.3) *is the Euler method for solving numerically the initial value problem* (3.1) *on arbitrary time scale.*

3.2 Local truncation error

In this section we give the derivation of the local truncation error. We study the cases when t_0 is right-dense and right-scattered separately.

Case 1: Let t_0 be right-scattered.
After a single step of the Euler's method, the computed result is

$$x_1 = x_0 + (\sigma^{l_0}(t_0) - t_0)f(t_0, y(t_0)),$$

and it differs from the exact solution $x(t_1) = x(\sigma^{l_0}(t_0))$ by

$$x(t_1) - x_1 = x(\sigma^{l_0}(t_0)) - x(t_0) - (\sigma^{l_0}(t_0 - t_0))f(t_0, x(t_0))$$

$$= x(\sigma^{l_0}(t_0)) - x(t_0) - (\sigma^{l_0}(t_0 - t_0))x^{\Delta}(t_0).$$

If $l_0 = 1$,then $x(t_1) = x_1$. Assuming that x has continuous first-order and second-order derivatives, this can be written, using Taylor's formula, in the form

$$\int_{t_0}^{\rho(\sigma^{l_0}(t_0))} h_1(\sigma^{l_0}(t_0), \sigma(\tau))x^{\Delta^2}(\tau)\Delta\tau.$$

Another way of writing the error, assuming that the third derivative x^{Δ^3} also exists and is bounded, is

$$h_2(\sigma^{l_0}(t_0), t_0)x^{\Delta^2}(t_0) + O(h_3(\sigma^{l_0}(t_0), t_0)).$$

Case 2: Let t_0 be right-dense.
After a single step of the Euler's method, the computed result is

$$x_1 = x_0 + q_0 f(t_0, x(t_0))$$

and differs from the exact solution $x(t_1)$ by

$$x(t_1) - x_1 = x(t_0 + q_0) - x(t_0) - q_0 f(t_0, x(t_0))$$

$$= x(t_0 + q_0) - x(t_0) - q_0 x^{\Delta}(t_0).$$

Assuming that x has continuous first and second derivatives, this can be written in the form

$$\int_{t_0}^{\rho(t_0+q_0)} h_1((t_0 + q_0), \sigma(\tau))x^{\Delta^2}(\tau)\Delta\tau.$$

Another way of writing the error, assuming that the third derivative exists and is bounded, is

$$h_2(t_0 + q_0, t_0)x^{\Delta^2}(t_0) + O(h_3(t_0 + q_0, t_0)).$$

Example 3.2.1 *Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and consider the initial value problem*

$$\begin{cases} x^{\Delta}(t) &= -\dfrac{1}{1+2t}x(t), \quad t > 1, \\ x(1) &= \dfrac{1}{2}. \end{cases}$$

Take

$$t_0 = 1, \quad t_1 = 8, \quad t_2 = 16, \quad t_3 = 32.$$

We will compute the local truncation error. We have $t_1 = \sigma^3(t_0)$, i.e., $l_0 = 3$. Let

$$g_1(t) = \frac{t^2}{3} - t,$$

$$g_2(t) = \frac{t^3}{21} - \frac{t^2}{3} + \frac{2}{3}t,$$

$$x(t) = \frac{1}{1+t}, \quad t \in \mathbb{T}.$$

Then

$$\begin{aligned}
g_1^{\Delta}(t) &= \frac{\sigma(t)+t}{3} - 1 \\
&= \frac{2t+t}{3} - 1 \\
&= t - 1, \\
g_2^{\Delta}(t) &= \frac{(\sigma(t))^2 + t\sigma(t) + t^2}{21} - \frac{\sigma(t)+t}{3} + \frac{2}{3} \\
&= \frac{4t^2 + 2t^2 + t^2}{21} - \frac{2t+t}{3} + \frac{2}{3}, \\
&= \frac{t^2}{3} - t + \frac{2}{3}, \\
x^{\Delta}(t) &= -\frac{1}{(1+t)(1+\sigma(t))} \\
&= -\frac{1}{(1+t)(1+2t)} \\
&= -\frac{1}{2t^2+3t+1}, \\
x^{\Delta^2}(t) &= \frac{2(\sigma(t)+t)+3}{(2t^2+3t+1)(2(\sigma(t))^2+3\sigma(t)+1)} \\
&= \frac{6t+3}{(2t^2+3t+1)(8t^2+6t+1)}, \quad t \in \mathbb{T}, \\
x^{\Delta^2}(1) &= \frac{9}{6 \cdot 15} \\
&= \frac{1}{10}.
\end{aligned}$$

Hence,

$$\begin{aligned}
x^{\Delta}(t) &= -\frac{1}{(1+t)(1+2t)} \\
&= -\frac{1}{1+2t}x(t), \quad t > 1, \\
x(1) &= \frac{1}{2},
\end{aligned}$$

i.e., x is a solution of the considered IVP. Next,

$$
\begin{aligned}
h_2(t,t_0) &= \int_1^t (\tau-1)\Delta\tau \\
&= \int_1^t g_1^{\Delta}(\tau)\Delta\tau \\
&= g_1(t) - g_1(1) \\
&= \left(\frac{t^2}{3} - t\right) - \left(\frac{1}{3} - 1\right) \\
&= \frac{t^2}{3} - t + \frac{2}{3},
\end{aligned}
$$

$$
\begin{aligned}
h_2(\sigma^{l_0}(t_0),t_0) &= h_2(8,1) \\
&= \frac{64}{3} - 8 + \frac{2}{3} \\
&= 22 - 8 \\
&= 14, \\
h_3(t,t_0) &= \int_1^t = h_2(\tau,t_0)\Delta\tau \\
&= \int_1^t \left(\frac{\tau^2}{3} - \tau + \frac{2}{3}\right)\Delta\tau \\
&= \int_1^t g_2^{\Delta}(\tau)\Delta\tau \\
&= g_2(t) - g_1(1) \\
&= \left(\frac{t^3}{21} - \frac{t^2}{3} + \frac{2}{3}t\right) - \left(\frac{1}{21} - \frac{1}{3} + \frac{2}{3}\right) \\
&= \frac{t^3}{21} - \frac{t^2}{3} + \frac{2}{3}t - \frac{8}{21}, \quad x \in \mathbb{T}. \\
h_3(\sigma^{l_0}(t_0),t_0) &= h_3(8,1) \\
&= \frac{512}{21} - \frac{64}{3} + \frac{16}{3} - \frac{8}{21} \\
&= \frac{504}{21} - \frac{48}{3} \\
&= 24 - 16 \\
&= 8.
\end{aligned}
$$

Therefore, the local truncation error is

$$
\begin{aligned}
h_2\Big(\sigma^{l_0}(t_0),t_0\Big)x^{\Delta^2}(t_0) + O\Big(h_3(\sigma^{l_0}(t_0),t_0)\Big) &= \frac{14}{10} + O(8) \\
&= \frac{7}{5} + O(8).
\end{aligned}
$$

3.3 Global Truncation Error

Let $\bar{x}(t)$ denote the computed solution on the interval $[t_0, t_N]_{\mathbb{T}}$. At the points $t_0, t_1, \ldots, t_N$. $\bar{x}$ is computed by using (3.2). At the offstep points, $\bar{x}(t)$ is defined by linear interpolation, or other equivalent method. So, $\bar{x}(t)$ is evaluated using a partial step from the most recently computed step values. That is, if $t \in [t_{k-1}, t_k], \quad k = 1, \ldots, N$, then

$$\bar{x}(t) = x_{k-1} + h_1(t, t_{k-1}) f(t_{k-1}, x_{k-1}), \tag{3.4}$$

Define the maximum step size as

$$m := \underbrace{max}_{1 \le i \le N} \{t_i - t_{i-1}\}.$$

Let

$$\alpha(t) = x(t) - \bar{x}(t), \quad \beta(t) = f(t, x(t)) - f(t, \bar{x}(t)). \tag{3.5}$$

Suppose that
$|f(t, x) - f(t, z)| \le L\,|x - z\,|$ for all $t \in \mathbb{T}$ and $x, z \in \mathbb{R}$,
where $L > 0$. From the definition of $\alpha(t)$ and $\beta(t)$, we have

$$|\beta(t)| \le L\,|\alpha(t)|.$$

Define $E(t)$ so that the exact solution satisfies

$$x(t) = x(t_{k-1}) + h_1(t, t_{k-1}) f(t_{k-1}, x(t_{k-1})) + h_2(t, t_{k-1}) E(t), \quad t \in [t_{k-1}, t_k], \tag{3.6}$$

and assume that $|E(t)| \le p$. Subtracting (3.4) from (3.6) we get

$$x(t) - \bar{x}(t) = x(t_{k-1}) - x_{k-1} + h_1(t, t_{k-1})\Big(f(t_{k-1}, x(t_{k-1})) - f(t_{k-1}, x_{k-1})\Big) + h_2(t, t_{k-1}) E(t).$$

Hence, using (3.5), we get

$$\alpha(t) = \alpha(t_{k-1}) + h_1(t, t_{k-1}) \beta(t_{k-1}) + h_2(t, t_{k-1}) E(t).$$

We make the following estimation.

$$h_2(t, t_{k-1}) = \int_{t_{k-1}}^{t} h_1(\tau, t_{k-1}) \Delta\tau \le h_1(t, t_{k-1})(t - t_{k-1}) \le m h_1(t, t_{k-1}),$$

since $h_1(\tau, t_{k-1}) = \tau - t_{k-1}$ takes its maximum value at $\tau = t_k \in [t_{k-1}, t_k]$. Then

$$|\alpha(t)| \le |\alpha(t_{k-1})| + |h_1(t, t_{k-1})|\, L\,|\alpha(t_{k-1})| + p\,|h_2(t, t_{k-1})|$$

$$= (1 + L\,|h_1(t, t_{k-1})|)\,|\alpha(t_{k-1})| + |h_1(t, t_{k-1})|\,p$$

$$\le (1 + L\,|h_1(t, t_{k-1})|)\,|\alpha(t_{k-1})| + mp\,|h_1(t, t_{k-1})|\,.$$

If $L = 0$, then the previous inequality simplifies as

$$|\alpha(t)| \le |\alpha(t_{k-1})| + mph_1(t, t_{k-1}).$$

In particular,

$$|\alpha(t_k)| \le |\alpha(t_{k-1})| + mph_1(t_k, t_{k-1}), \quad for\ all\ k = 1, \ldots, N.$$

Therefore, we deduce

$$|\alpha(t)| \le |\alpha(t_{k-2})| + mp(h_1(t_{k-1}, t_{k-2}) + h_1(t, t_{k-1}))$$

$$= |\alpha(t_{k-2})| + mph_1(t, t_{k-2})$$

$$\le \ldots$$

$$\le |\alpha(t_0)| + mph_1(t, t_0).$$

If $L > 0$, then we have

$$|\alpha(t)| \le (1 + Lh_1(t, t_{k-1}))\,|\alpha(t_{k-1})| + mph_1(t, t_{k-1})$$

$$= (1 + Lh_1(t, t_{k-1}))\,|\alpha(t_{k-1})| + \frac{mp}{L}Lh_1(t, t_{k-1}) + \frac{mp}{L} - \frac{mp}{L},$$

that is,

$$|\alpha(t)| + \frac{mp}{L} \le (1 + Lh_1(t, t_{k-1}))\,|\alpha(t_{k-1})| + \frac{mp}{L}(1 + Lh_1(t, t_{k-1}))$$

$$= (1 + Lh_1(t, t_{k-1}))\Big(\frac{mp}{L} + |\alpha(t_{k-1})|\Big)$$

$$\le e_L(t, t_{k-1})\Big(\frac{mp}{L} + |\alpha(t_{k-1})|\Big).$$

In particular

$$|\alpha(t_k)| + \frac{mp}{L} \le e_L(t_k, t_{k-1})\Big(\frac{mp}{L} + |\alpha(t_{k-1})|\Big).$$

Hence, if $t \in [t_{k-1}, t_k]$, then we get

$$|\alpha(t)| + \frac{mp}{L} \le e_L(t, t_{k-1})\Big(\frac{mp}{L} + |\alpha(t_{k-1})|\Big)$$

$$\le e_L(t, t_{k-1})e_L(t_{k-1}, t_{k-2})\Big(\frac{mp}{L} + |\alpha(t_{k-2})|\Big)$$

$$\le \ldots$$

$$\leq e_L(t,t_0)\Big(\frac{mp}{L} + |\alpha(t_0)|\Big).$$

Combining the estimates found in the two cases and stating them formally, we have the following result.

Theorem 3.3.1 *Let $f : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ satisfy a Lipschitz condition with a constant L. The global truncation error is satisfies the inequality*

$$|x(t) - \bar{x}(t)| \leq \begin{cases} |x(t_0) - \bar{x}(t_0)| + mph_1(t,t_0) & if \;\; L = 0 \\ e_L(t,t_0)\left(\frac{mp}{L} + |\alpha(t_0)| - \frac{mp}{L}\right) & if \;\; L > 0. \end{cases}$$

Now we consider a sequence of approximations to $x(\bar{t})$. In each of these approximations, a computation using Euler method is performed, starting from an approximation to $x(t_0)$, and taking a sequence of positive steps. Denote approximation number r by $\bar{x}_r$. The only assumption we will make about $\bar{x}_r$, for each specific value of r, is that the initial error $x(t_0) - \bar{x}_r(t_0)$ is bounded by K_r and that the greatest step size is bounded by m_r. It is assumed that $K_r \to 0$ as $r \to \infty$. If $m_r \to 0$, then we get

$$|x(\bar{t}) - \bar{x}_r(\bar{t})| \to 0 \quad as \quad r \to \infty.$$

There are cases when m_r does not tend to zero as $r \to \infty$, for instance, when $\mathbb{T} = 2^{\mathbb{N}_0}$. When $\mathbb{T} = \mathbb{R}$, we have $m_r \to 0$ as $r \to \infty$.

3.4 Numerical Examples

Example 3.4.1 *Consider the first order dynamic equation*

$$x^{\Delta}(t) = \frac{1 + x(t)}{1 + x(t) + (x(t))^2}, \quad x(1) = a,$$

on the time scale

$$\mathbb{T} = \{1, 2, 3, 4\} \cup [5, 10].$$

First we define $t_0 = 1$, $t_1 = 2$, $t_2 = 3$, $t_3 = 4$, $t_4 = 5$ on the discrete part of $\mathbb{T}$. On the continuous part we take

$$t_i = 5 + q(i - 4), \quad i = 5, ..., N,$$

where $N = 5 + \frac{5}{q}$, *and* q *is the step size. The initial condition is*

$$x_0 = x(t_0) = a.$$

For the discrete part the Euler method defines the sequence

$$x_i = x_{i-1} + \frac{1 + x_{i-1}}{1 + x_{i-1} + x_{i-1}^2}, \quad i = 1, 2, 3, 4$$

and for the continuous part,

$$x_i = x_{i-1} + q\frac{1 + x_{i-1}}{1 + x_{i-1} + x_{i-1}^2}, \quad i = 5, ..., N.$$

Note that on $\{1, 2, 3, 4\}$ *the exact solution* $x_e(t)$ *coincides with the solution obtained with the Euler method. On* $[5, 10]$, *we have the differential equation*

$$x' = \frac{1 + x}{1 + x + x^2}$$

which is a separable first order equation and its exact solution is obtained implicitly as

$$\frac{x^2}{2} + \ln(1 + x) - t = c.$$

We compute the values of x *using Matlab for* $q = 0.5, 0.25$ *and* $x(1) = 1$, *and* $x(1) = 3$. *The exact and approximate solution are compared graphically in the Figures 3.1-3.4.*

It is clear that the exact solution and the approximate solution obtained by using the Euler method coincide on the discrete part of $\mathbb{T}$. *On the continuous part of* $\mathbb{T}$, *the error seem to be very small for the chosen small values of* q, *that is,* $q = 0.5$ *and* $q = 0.25$.

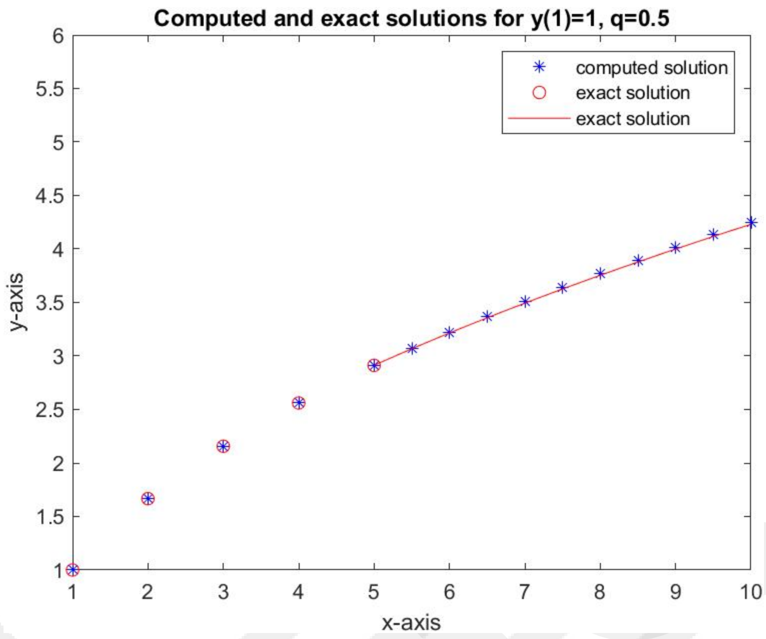

Figure 3.1: Computed and exact values of the solution with $x(1) = 1$ and $q = 0.5$.

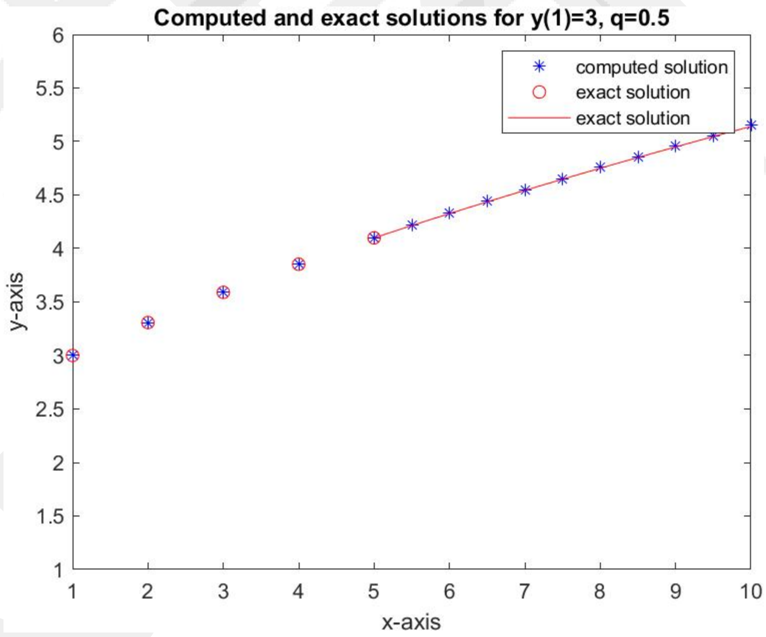

Figure 3.2: Computed and exact values of the solution with $x(1) = 3$ and $q = 0.5$.

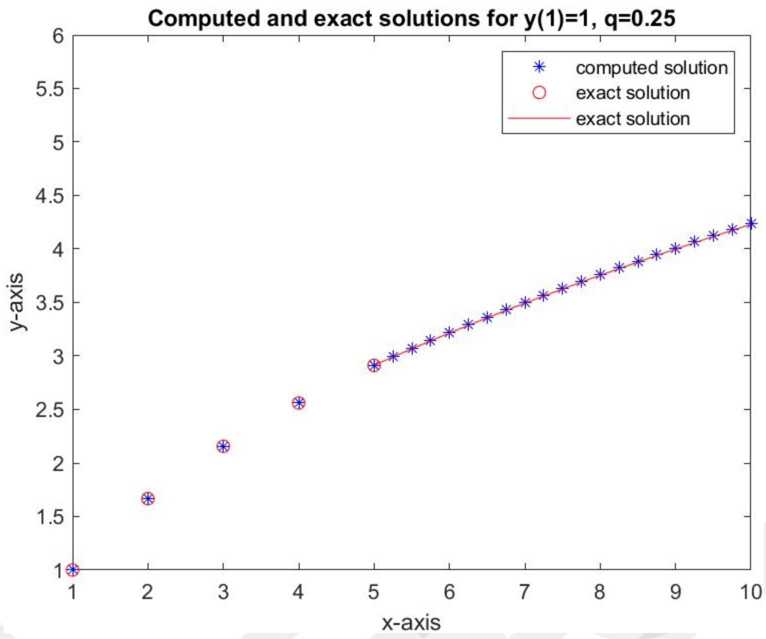

Figure 3.3: Computed and exact values of the solution with $x(1) = 1$ and $q = 0.25$.

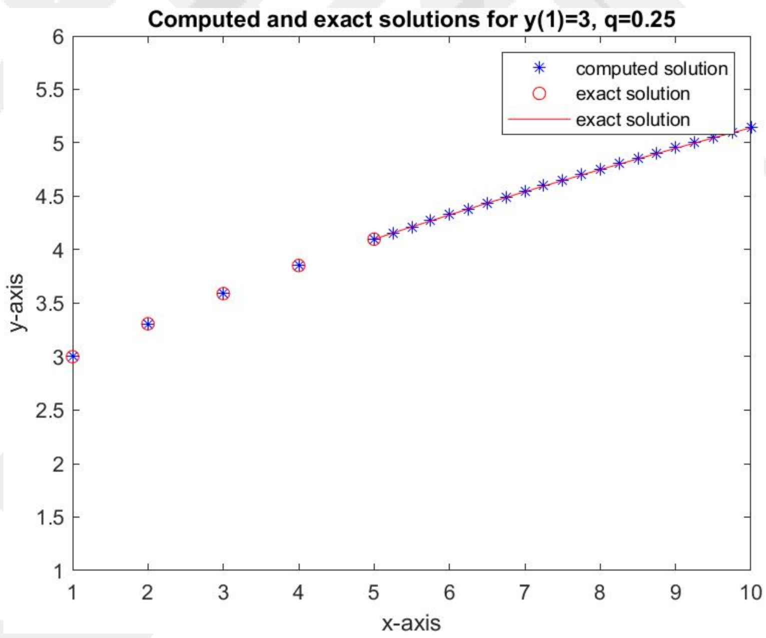

Figure 3.4: Computed and exact values of the solution with $x(1) = 3$ and $q = 0.25$.

Example 3.4.2 *Consider the IVP for the second order dynamic equation*

$$\begin{cases} x^{\Delta^2}(t) + (t+2)x^{\Delta}(t) + 2tx(t) = 0 \\ x(0) = a, \quad x^{\Delta}(0) = b, \end{cases}.$$

on the time scale $\mathbb{T} = \alpha\mathbb{N}_0$*, where* $t \in [0,5]$.

The exact solution can be computed in the following way. Rewrite the dynamic equation as

$$(x^{\Delta} + 2x)^{\Delta} + t(x^{\Delta} + 2x) = 0.$$

Let $u(t) = x^{\Delta}(t) + 2x(t)$*. Then we have*

$$u^{\Delta}(t) + tu(t) = 0$$

or

$$u^{\Delta}(t) = -tu(t).$$

On $\mathbb{T} = \alpha\mathbb{N}_0$,

$$u^{\Delta}(t) = \frac{u(t+\alpha) - u(t)}{\alpha},$$

so that we get

$$u(t+\alpha) = u(t) - \alpha tu(t)$$

or

$$u(t+\alpha) = (1-\alpha t)u(t), \quad t \in [0,5]$$

with

$$u(0) = b + 2a.$$

Then

$$x^{\Delta}(t) + 2x(t) = u(t)$$

or

$$x(t+\alpha) = x(t) + \alpha\,(u(t) - 2x(t))$$

which implies

$$x(t+\alpha) = \alpha u(t) + (1-2\alpha)x(t), \quad t \in [0,5]$$

with $x(0) = a$.

To apply the Euler Method, first, we transform the second order dynamic equation into a system of two first order dynamic equations.

Let $x_1(t) = x(t)$ *and* $x_2(t) = x^{\Delta}(t)$.

Then the dynamic equation

$$x^{\Delta^2}(t) + (t+2)x^{\Delta}(t) + 2tx(t) = 0$$

can be written as a system of first order dynamic equations as

$$\begin{cases} x_1^{\Delta}(t) = x_2(t) \\ x_2^{\Delta}(t) = -(t+2)x_2(t) - 2tx_1(t) \end{cases}.$$

with the initial conditions $x_1(0) = a, \quad x_2(0) = b$.

Let $r_n = q =$ *constant so that* $q \geq \alpha$. *The Euler method yields*

$$\begin{cases} x_{1,i+1} = x_{1,i} + qx_{2,i} \\ x_{2,i+1} = x_{2,i} + q\left(-(t_i+2)x_{2,i} - 2t_i x_{1,i}\right) \end{cases}.$$

for $i = 0, \ldots, N-1$ *where* $x_{1,0} = a$ *and* $x_{2,0} = b$.

We compute the approximate solution for $q = 0.5$ *and* $q = 0.2$, *and different values of* α, a *and* b *using Matlab.*

The graphs of computed and exact solutions are presented in Figures 3.5-3.8. The values of the computed solution are denoted by x_c *and those of the exact solution by* x_e *and are shown in Tables 3.1-3.4.*

Table 3.1: The values of x_c, and the exact solution x_e at points of the interval $[0, 5]$ for $q = 0.5$, $\alpha = 0.25$, $a = 1$ and $b = 1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	1.00000000	1.00000000
0.50	1.37500000	1.50000000
1.00	1.31054688	1.50000000
1.50	0.95248413	1.12500000
2.00	0.52808928	0.56250000
2.50	0.22263740	0.14062500
3.00	0.07300364	0.00000000
3.50	0.01999308	0.00000000
4.00	0.00506632	0.00000000
4.50	0.00126701	0.00000000

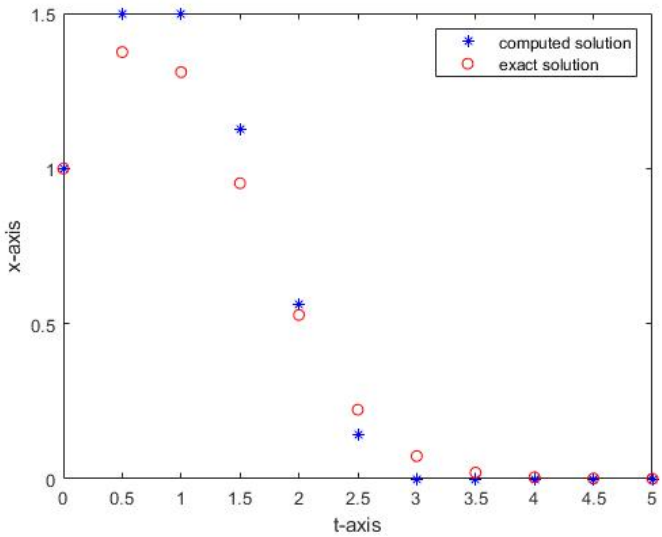

Figure 3.5: Computed and exact solutions with $q = 0.5$, $\alpha = 0.25$, $a = 1$ and $b = 1$.

Table 3.2: The values of x_c, and the exact solution x_e at points of the interval $[0, 5]$ for $q = 0.2$, $\alpha = 0.1$, $a = 1$ and $b = 1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	1.00000000	1.00000000
0.20	1.18000000	1.20000000
0.40	1.28386000	1.32000000
0.60	1.31856803	1.36800000
0.80	1.29190451	1.35072000
1.00	1.21397699	1.27676160
1.20	1.09730509	1.15777382
1.40	0.95586455	1.00803779
1.60	0.80358999	0.84298652
1.80	0.65284066	0.67726989
2.00	0.51323229	0.52296696
2.20	0.39106043	0.38840739
2.40	0.28934219	0.27782076
2.60	0.20834357	0.19176720
2.80	0.14637441	0.12809919
3.00	0.10062943	0.08311817
3.20	0.06790883	0.05262471
3.40	0.04513247	0.03267635
3.60	0.02963571	0.02000236
3.80	0.01928460	0.01212831
4.00	0.01246847	0.00731252
4.20	0.00802693	0.00439604
4.40	0.00515367	0.00263933
4.60	0.00330371	0.00158387
4.80	0.00211600	0.00095035

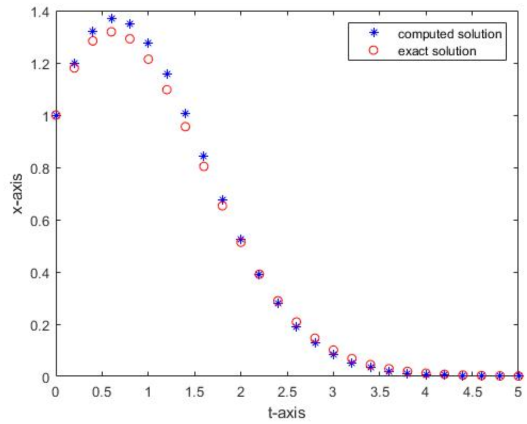

Figure 3.6: Computed and exact solution with $q = 0.2$, $\alpha = 0.1$, $a = 1$ and $b = 1$.

Table 3.3: The values of x_c, and the exact solution x_e at points of the interval $[0, 5]$ for $q = 0.5$, $\alpha = 0.25$, $a = 0.5$ and $b = -1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	0.50000000	0.50000000
0.50	0.12500000	0.00000000
1.00	0.03125000	0.00000000
1.50	0.00781250	0.00000000
2.00	0.00195312	0.00000000
2.50	0.00048828	0.00000000
3.00	0.00012207	0.00000000
3.50	0.00003052	0.00000000
4.00	0.00000763	0.00000000
4.50	0.00000191	0.00000000

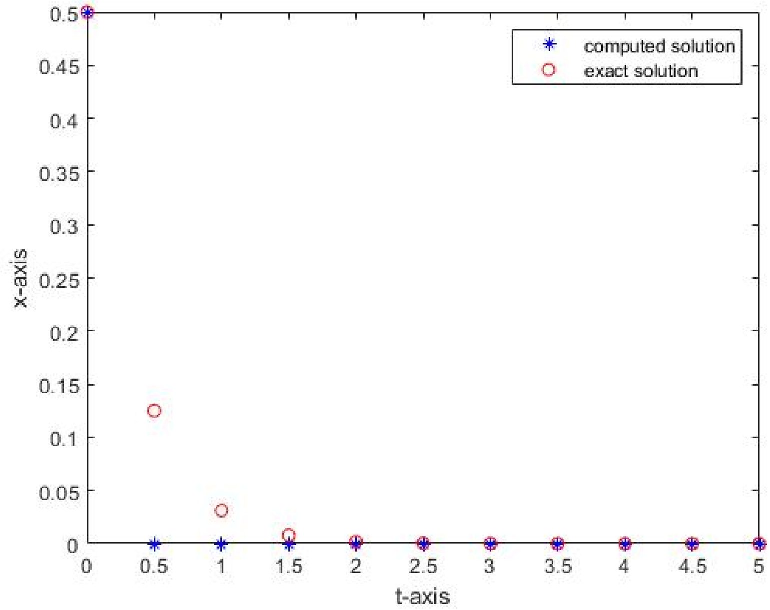

Figure 3.7: Computed and exact solution with $q = 0.5$, $\alpha = 0.25$, $a = 0.5$ and $b = -1$.

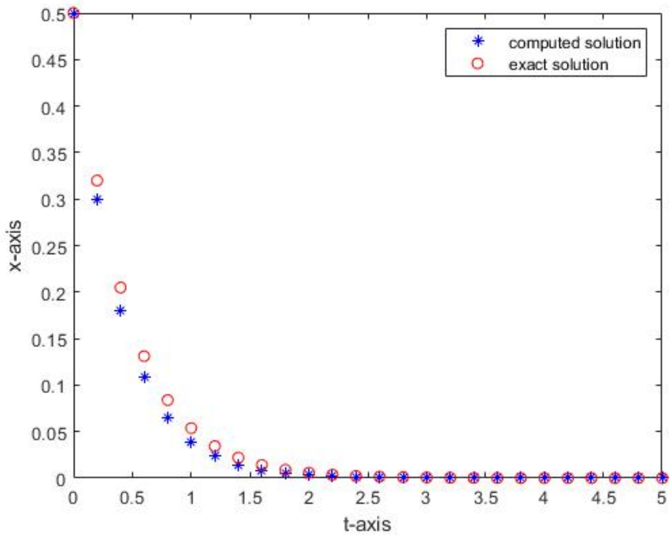

Figure 3.8: Computed and exact solution with $q = 0.2$, $\alpha = 0.1$, $a = 0.5$ and $b = -1$.

Table 3.4: The values of x_c, and the exact solution x_e at points of the interval $[0,5]$ for $q = 0.2$, $\alpha = 0.1$, $a = 0.5$ and $b = -1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	0.50000000	0.50000000
0.20	0.32000000	0.30000000
0.40	0.20480000	0.18000000
0.60	0.13107200	0.10800000
0.80	0.08388608	0.06480000
1.00	0.05368709	0.03888000
1.20	0.03435974	0.02332800
1.40	0.02199023	0.01399680
1.60	0.01407375	0.00839808
1.80	0.00900720	0.00503885
2.00	0.00576461	0.00302331
2.20	0.00368935	0.00181399
2.40	0.00236118	0.00108839
2.60	0.00151116	0.00065303
2.80	0.00096714	0.00039182
3.00	0.00061897	0.00023509
3.20	0.00039614	0.00014106
3.40	0.00025353	0.00008463
3.60	0.00016226	0.00005078
3.80	0.00010385	0.00003047
4.00	0.00006646	0.00001828
4.20	0.00004254	0.00001097
4.40	0.00002722	0.00000658
4.60	0.00001742	0.00000395
4.80	0.00001115	0.00000237

Example 3.4.3 *Consider the IVP for the first order dynamic equation*

$$x^{\Delta}(t) = \frac{t}{t^2+1} + \frac{1}{x^2+1}, \quad x(t_0) = x_0,$$

on the time scales $\mathbb{T} = \alpha\mathbb{N}_0$*. Let* $t \in [t_0, 10]_{\mathbb{T}}$*, and let* q *be a fixed step size. Then the Euler method for this problem can be written as*

$$t_{i+1} = t_i + q,$$
$$x_{i+1} = x_i + q\left[\frac{t_i}{t_i^2+1} + \frac{1}{1+x_i^2}\right].$$

for $i = 0, 1, \ldots, N-1$ *where* t_0 *and* x_0 *are given in the initial conditions,* $N = \dfrac{10}{q}$.

The exact solution of this IVP will be

$$x_e(t+\alpha) = x_e(t) + \alpha\left[\frac{t}{t^2+1} + \frac{1}{1+x_e(t)^2}\right], \quad t \in [t_0, 10]_{\mathbb{T}}.$$

We compute the approximate solution for $q = 1$ *and* $q = 0.5$, *and different values of* α, t_0 *and* x_0 *using Matlab.*

The graphs of computed and exact solutions are presented in Figures 3.9-3.12. The values of the computed solution are denoted by x_c *and those of the exact solution by* x_e *and are shown in Tables 3.5-3.8.*

Table 3.5: The values of x_c, and the exact solution x_e at points of the interval $[0, 10]$ with $q = 1$, $\alpha = 1$, $t_0 = 0$ and $x_0 = 1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	1.00000000	1.00000000
1.00	1.50000000	1.50000000
2.00	2.30769231	2.30769231
3.00	2.86578398	2.86578398
4.00	3.27432958	3.27432958
5.00	3.59493895	3.59493895
6.00	3.85906722	3.85906722
7.00	4.08415254	4.08415254
8.00	4.28071266	4.28071266
9.00	4.45553736	4.45553736

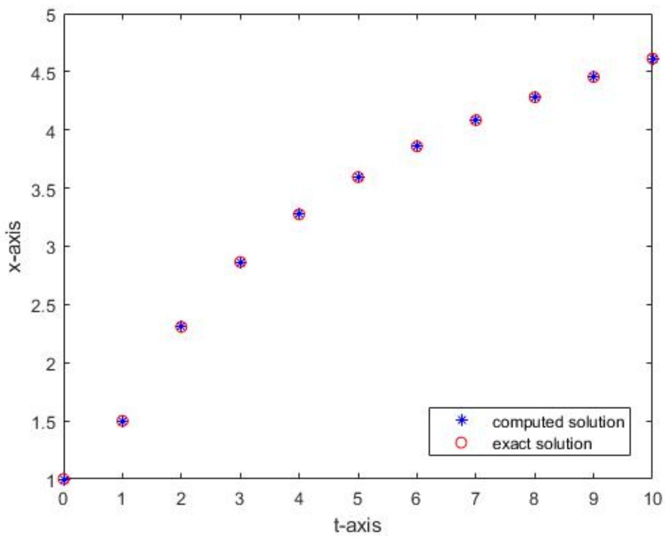

Figure 3.9: Computed and exact solution with $q = 1$, $\alpha = 1$, $t_0 = 0$ and $x_0 = 1$.

Table 3.6: The values of x_c, and the exact solution x_e at points of the interval $[0, 10]$ for $q = 1$, $\alpha = 0.5$, $t_0 = 0$ and $x_0 = 1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	1.00000000	1.00000000
1.00	1.64512195	1.50000000
2.00	2.35842934	2.30769231
3.00	2.86999968	2.86578398
4.00	3.25405149	3.27432958
5.00	3.56021668	3.59493895
6.00	3.81509146	3.85906722
7.00	4.03389026	4.08415254
8.00	4.22599490	4.28071266
9.00	4.39756224	4.45553736

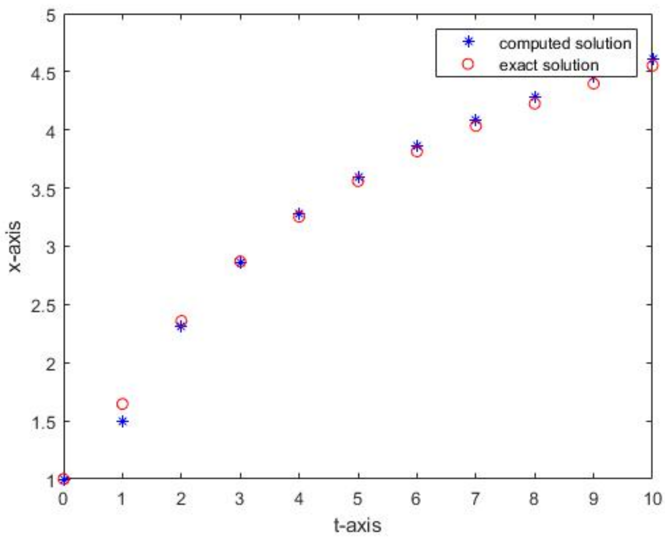

Figure 3.10: Computed and exact solutions with $q = 1$, $\alpha = 0.5$, $t_0 = 0$ and $x_0 = 1$.

Table 3.7: The values of x_c, and the exact solution x_e at points of the interval $[0, 10]$ for $q = 0.5$, $\alpha = 0.25$, $t_0 = 0$ and $x_0 = 1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	1.00000000	1.00000000
0.50	1.29416836	1.25000000
1.00	1.68543859	1.64512195
1.50	2.05282097	2.03002279
2.00	2.36613670	2.35842934
2.50	2.63119427	2.63462343
3.00	2.85846282	2.86999968
3.50	3.05659282	3.07413040
4.00	3.23197353	3.25405149
4.50	3.38925642	3.41484347
5.00	3.53186629	3.56021668
5.50	3.66237141	3.69293322
6.00	3.78273556	3.81509146
6.50	3.89448833	3.92831671
7.00	3.99884148	4.03389026
7.50	4.09676978	4.13283839
8.00	4.18906812	4.22599490
8.50	4.27639280	4.31404586
9.00	4.35929160	4.39756224
9.50	4.43822636	4.47702414

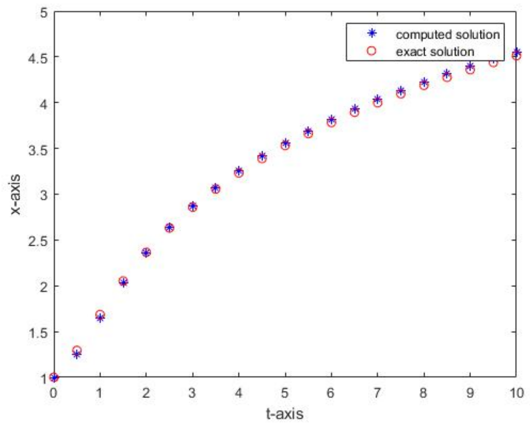

Figure 3.11: Computed and exact solutions with $q = 0.5$, $\alpha = 0.25$, $t_0 = 0$ and $x_0 = 1$.

Table 3.8: The values of x_c, and the exact solution x_e at points of the interval $[0, 10]$ for $q = 0.5$, $\alpha = 0.25$, $t_0 = 0$ and $x_0 = -2$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	-2.00000000	-2.00000000
0.50	-1.83912025	-1.90000000
1.00	-1.49678853	-1.59154013
1.50	-1.07926571	-1.20001734
2.00	-0.59533792	-0.76433358
2.50	0.01000896	-0.24871802
3.00	0.64972735	0.39456735
3.50	1.10883874	0.97721179
4.00	1.44336143	1.36504921
4.50	1.71029621	1.65731634
5.00	1.93496229	1.89664955
5.50	2.13028655	2.10156337
6.00	2.30381801	2.28187274
6.50	2.46040095	2.44350877
7.00	2.60335875	2.59038172
7.50	2.73508614	2.72523191
8.00	2.85737441	2.85006797
8.50	2.97160346	2.96641365
9.00	3.07886170	3.07545656
9.50	3.18002456	3.17814292

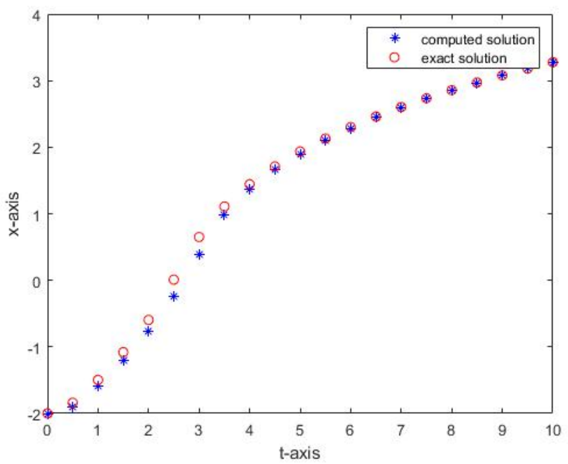

Figure 3.12: Computed and exact solutions with $q = 0.5$, $\alpha = 0.25$, $t_0 = 0$ and $x_0 = -2$.

CHAPTER 4

THE TAYLOR SERIES METHOD OF ORDER 2 FOR DYNAMIC EQUATIONS ON TIME SCALES

The Euler's method, being the most basic and simple methods for solving differential equations is not very efficient as its order of convergence and its accuracy are very small. More accurate methods can be obtained by using the Taylor polynomial approximations of orders higher than one. In this chapter we propose the Taylor series method of order 2 for computation of the approximate solution of IVPs for dynamic equations. The derivation of the method uses the so-called Pötzsche's Chain Rule which is given below.

4.1 Pötzsche's Chain Rule

Theorem 4.1.1 *[20] (Pötzsche's Chain Rule). For some fixed $t_0 \in \mathbb{T}^k$, let $g : \mathbb{T} \to X$, $f : \mathbb{T} \times X \to Y$ be functions such that $g, f(\cdot, g(t_0))$ are delta differentiable at t_0, and let $U \subseteq \mathbb{T}$ be a neighborhood of t_0 such that $f(t, \cdot)$ is delta differentiable for $t \in U \bigcup \{\sigma(t_0)\}$, $\frac{\partial}{\partial x} f(\sigma(t_0), \cdot)$ is continuous on the line segment*

$$\left\{ g(t_0) + h\mu(t_0) g^{\Delta}(t_0) \in X : h \in [0, 1] \right\}$$

and $\frac{\partial f}{\partial x}$ is continuous at $(t_0, g(t_0))$. Then the composition function $F : \mathbb{T} \to Y$, defined as $F(t) = f(t, g(t))$ is differentiable at t_0 with derivative

$$F^{\Delta}(t_0) = \Delta_1 f(t_0, g(t_0)) + \left(\int_0^1 \frac{\partial}{\partial x} f(\sigma(t_0), g(t_0) + h\mu(t_0) d^{\Delta}(t_0)) dh \right) g^{\Delta}(t_0).$$

Here $\Delta_1 f(\cdot, g(t_0))$ denotes the delta derivative of f with respect to its first variable , and $\frac{\partial}{\partial x} f(t, \cdot)$, the partial derivative of f with respect to its second variable.

Proof. We define the function $v(t,s) = t - s$. Then the graininess function $\mu(t) = \sigma(t) - t$ can be written as $v(\sigma(t), t)$. Let $U_0 \subseteq U$ be a neighborhood of t_0 such that

$$\mu(t_0) \leq |v(t, \sigma(t_0))| \quad for \quad t \in U_0.$$

Let

$$\varphi(t,h) = \frac{\partial}{\partial x} f(t, g(t_0) + h(g(t) - g(t_0))), \quad t \in U_0 \quad h \in [0,1].$$

Note that there exists a constant $C > 0$ such that

$$|\varphi(\sigma(t_0), h) - \varphi(t_0, h)| \leq C|v(t, \sigma(t_0))| \quad for \quad t \in U_0, \quad h \in [0,1].$$

Let $\epsilon > 0$ be arbitrarily chosen. We choose $\epsilon_1 > 0, \epsilon_2 > 0$ small enough such that

$$\epsilon_1\Big(1 + C + |\int_0^1 \varphi(\sigma(t_0), h)dh|\Big) + \epsilon_2\Big(\epsilon_1 + 2|g^\Delta(t_0)|\Big) \leq \epsilon.$$

Since g and $f(\cdot, g(t_0))$ are delta differentiable at t_0, there exists a neighborhood $U_1 \subseteq U_0$ of t_0 such that

$$|g(t) - g(t_0)| \leq \epsilon_1,$$

$$|g(t) - g(\sigma(t_0)) - v(t, \sigma(t_0))g^\Delta(t_0)| \leq \epsilon_1 |v(t, \sigma(t_0))|,$$

$$|f(t, g(t_0)) - f(\sigma(t_0), g(t_0)) - v(t, \sigma(t_0))\Delta_1 f(t_0, g(t_0))| \leq \epsilon_1 |v(t, \sigma(t_0))|$$

for $t \in U_1$. Hence,

$$\begin{aligned}
|g(t) - g(t_0)| &= |g(t) - g(\sigma(t_0)) - v(t, \sigma(t_0))g^\Delta(t_0) + g^\Delta(t_0)v(t, \sigma(t_0)) + g(\sigma(t_0)) - g(t_0)| \\
&\leq |g(t) - g(\sigma(t_0)) - v(t, \sigma(t_0))g^\Delta(t_0)| \\
&\quad + |g^\Delta(t_0)|\,|v(t, \sigma(t_0))| + |g(\sigma(t_0)) - g(t_0)| \\
&\leq \epsilon_1 |v(t, \sigma(t_0))| + |g^\Delta(t_0)|\,|v(t, \sigma(t_0))| \\
&\quad + |g\Delta(t_0)|\mu(t_0) \\
&= \Big(\epsilon_1 + |g^\Delta(t_0)|\Big)|v(t, \sigma(t_0))| + |g^\Delta(t_0)|\mu(t_0) \\
&\leq \Big(\epsilon_1 + 2|g^\Delta(t_0)|\Big)|v(t, \sigma(t_0))|, \quad t \in U_1.
\end{aligned}$$

Since g is continuous at t_0 and $\frac{\partial}{\partial x} f$ is continuous at $(t_0, g(t_0))$, there exists a neighborhood $U_2 \subseteq U$ of t_0 so that

$$|\varphi(t,h) - \varphi(t_0, h)| \leq \epsilon_2 \quad for \quad t \in U_2, \quad h \in [0,1].$$

Hence,

$$|F(t) - F(\sigma(t_0)) - v(t,\sigma(t_0))\Big(\Delta_1 f(t_0, g(t_0)) + \int_0^1 \varphi(\sigma(t_0), h)dhg^{\Delta}(t_0)\Big)|$$

$$= |f(t, g(t)) - f(\sigma(t_0), g(\sigma(t_0))) - f(\sigma(t_0), g(t_0)) + f(\sigma(t_0), g(t_0))$$

$$-f(t, g(t_0)) + f(t, g(t_0)) - v(t,\sigma(t_0))\Delta_1 f(t_0, g(t_0))$$

$$-v(t,\sigma(t_0)) \int_0^1 \varphi(\sigma(t_0), h)dhg^{\Delta}(t_0)$$

$$-\int_0^1 \varphi(\sigma(t_0), h)dh(g(t) - g(t_0)) + \int_0^1 \varphi(\sigma(t_0), h)dh(g(t) - g(t_0))|$$

$$\leq |f(t, g(t_0)) - f(\sigma(t_0), g(t_0)) - v(t,\sigma(t_0))\Delta_1 f(t_0, g(t_0))|$$

$$+|\int_0^1 \varphi(\sigma(t_0), h)dh\Big(g(t) - g(t_0) - v(t,\sigma(t_0))g^{\Delta}(t_0)\Big)|$$

$$+|f(t, g(t)) - f(t, g(t_0)) - (f(\sigma(t_0), g(\sigma(t_0))) - f(\sigma(t_0), g(t_0))$$

$$-\int_0^1 \varphi(\sigma(t_0), h)dh(g(t) - g(t_0))|$$

$$\leq |f(t, g(t_0)) - f(\sigma(t_0), g(t_0)) - v(t,\sigma(t_0))\Delta_1 f(t_0, g(t_0))|$$

$$+|\int_0^1 \varphi(\sigma(t_0), h)dh|\ |g(t) - g(t_0) - v(t,\sigma(t_0))g^{\Delta}(t_0)|$$

$$+|\int_0^1 (\varphi(t, h) - \varphi(\sigma(t_0), h))dh(g(t) - g(t_0))|$$

$$\leq |f(t, g(t_0)) - f(\sigma(t_0), g(t_0)) - v(t,\sigma(t_0))\Delta_1 f(t_0, g(t_0))|$$

$$+|\int_0^1 \varphi(\sigma(t_0), h)dh|\ |g(t) - g(t_0) - v(t,\sigma(t_0))g^{\Delta}(t_0)|$$

$$+|(\varphi(t, h) - \varphi(t_0, h))dh|\ |g(t) - g(t_0)|$$

$$+|\int_0^1 (\varphi(t_0, h) - \varphi(\sigma(t_0), h))dh|\ |g(t) - g(t_0)|$$

$$\leq \epsilon_1|v(t,\sigma(t_0))| + \epsilon_1|v(t,\sigma(t_0))|\ |\int_0^1 \varphi(\sigma(t_0), h)dh|$$

$$+\epsilon_2\Big(\epsilon_1 + 2|g^{\Delta}(t_0)|\Big)|v(t,\sigma(t_0))| + \epsilon_1 C|v(t,\sigma(t_0))|$$

$$= \left[\epsilon_2\Big(1 + C + |\int_0^1 \varphi(\sigma(t_0), h)dh|\Big) + \epsilon_2\Big(\epsilon_1 + 2|g^{\Delta}(t_0)|\Big)\right]|v(t,\sigma(t_0))|$$

$$\leq \epsilon|v(t,\sigma(t_0))|, \quad t \in U_1 \bigcap U_2.$$

This completes the proof.

4.2 Derivation of Taylor Series method of order 2

We start now with the necessary requirements for the Taylor series method. Suppose that $\mathbb{T}$ is a time scale and that $t_0, t_N \in \mathbb{T}$, $t_N < \infty$. Consider the initial value problem

$$\begin{cases} x^{\Delta}(t) = f(t, x(t)), \quad t \in [t_0, t_N] \\ \quad x(t_0) = x_0, \end{cases} \tag{4.1}$$

where $x_0 \in \mathbb{R}$. In the rest of the chapter we assume that the following conditions hold. For some given constant $K > 0$,

$$(C1)\begin{cases} |f(t,x)| \leq K, \quad t \in \mathbb{T}, x \in \mathbb{R}, \\ \text{there exist } \Delta_1 f(t,x) \text{ and } \dfrac{\partial}{\partial x} f(t,x) \text{ such that} \\ |\Delta_1 f(t,x)| \leq K, \quad \left|\frac{\partial}{\partial x} f(t,x)\right| \leq K, \quad t \in \mathbb{T}, x \in \mathbb{R}. \end{cases}.$$

In addition

$$(C2)\begin{cases} If\ g(t,x) = \Delta_1 f(t,x) + \left(\int_0^1 \frac{\partial}{\partial x} f(\sigma(t), x + s\mu(t) f(t,x)) ds \right) f(t,x), \\ t \in \mathbb{T}, x \in \mathbb{R}, \text{ there exist } \Delta_1 g(t,x) \text{ and } \frac{\partial}{\partial x} g(t,x) \text{ such that} \\ |\Delta_1 g(t,x)| \leq K, \quad \left|\frac{\partial}{\partial x} g(t,x)\right| \leq K, \quad t \in \mathbb{T}, x \in \mathbb{R}. \end{cases}.$$

Suppose that $r > 0$, and $t, t+r \in [t_0, t_f]$, $\rho^2(t+r) \in [t, t_N]$. Then by the Taylor's formula of the second order we compute

$$x(t+r) = x(t) + h_1(t+r,t)x^{\Delta}(t) + h_2(t+r,t)x^{\Delta^2}(t) + \int_t^{\rho^2(t+r)} h_2(t+r, \sigma(\tau))x^{\Delta^3}(\tau)\Delta\tau.$$

Denote the integral remainder by

$$R_2(r) = \int_t^{\rho^2(t+r)} h_2(t+r, \sigma(\tau))x^{\Delta^3}(\tau)\Delta\tau.$$

Then the previous formula can be written by

$$x(t+r) = x(t) + h_1(t+r,t)x^{\Delta}(t) + h_2(t+r,t)x^{\Delta^2}(t) + R_2(r). \tag{4.2}$$

Assume that $\{t_0 < t_1 < \ldots < t_N\}$ is a partition of the interval $[t_0, t_N]$ such that $t_{n+1} = t_n + r_{n+1} \in \mathbb{T}$, $r_{n+1} > 0$, $n \in \{0, \ldots, N-1\}$. As in Euler method, we use a variable step size. The nature of an arbitaray time scale makes it very difficult to use a constant step size as for the ordinary differential equations. For example, when $\mathbb{T} = 2^{\mathbb{N}_0}$ and $[t_0, t_f] = [1, 16]$, then for $t_0 = 1, \quad t_1 = 2, \quad t_1 = 4, \quad t_3 = 8, \quad t_4 = 16$ we have $r_1 = 1, r_2 = 2, r_3 = 4, r_4 = 8$.

Then

$$x(t_{n+1}) = x(t_n) + h_1(t_{n+1}, t_n)x^{\Delta}(t_n) + h_2(t_{n+1}, t_n)x^{\Delta^2}(t_n) + R_2(r_{n+1})$$

$$= x(t_n) + r_{n+1}x^{\Delta}(t_n) + h_2(t_{n+1}, t_n)x^{\Delta^2}(t_n) + R_2(r_{n+1})$$

for $n \in \{0, \dots, N-1\}$. Neglecting the remainder term $R_2(r_{n+1})$ we obtain the formula

$$x(t_{n+1}) = x(t_n) + r_{n+1}x^{\Delta}(t_n) + h_2(t_{n+1}, t_n)x^{\Delta^2}(t_n).$$

Let $x_n = x(t_n)$, $x_n^{\Delta} = x^{\Delta}(t_n)$ and $x_n^{\Delta^2} = x^{\Delta^2}(t_n)$. Then the above equation can be written as

$$x_{n+1} = x_n + r_{n+1}x_n^{\Delta}(t_n) + h_2(t_{n+1}, t_n)x_n^{\Delta^2}(t_n). \tag{4.3}$$

We will refer to the relation (4.3) as the Taylor series method of order 2. The value x_n^{Δ} in this relation can be computed from the initial value problem as

$$x_n^{\Delta} = f(t_n, x_n).$$

The determine $x_n^{\Delta^2}$ we have to differentiate both sides of the equation in the IVP (4.1). By applying the Pötzsche's chain rule given in Theorem 4.1 , we get

$$(f(t, x(t)))^{\Delta} = \Delta_1 f(t, x(t)) + \Big(\int_0^1 \frac{\partial}{\partial x} f(\sigma(t), x(t) + s\mu(t)x^{\Delta}(t))ds\Big)x^{\Delta}(t).$$

for $t \in \mathbb{T}^k$, whereupon

$$x^{\Delta^2}(t) = \Delta_1 f(t, x(t)) + \Big(\int_0^1 \frac{\partial}{\partial x} f(\sigma(t), x(t) + s\mu(t)x^{\Delta}(t))ds\Big)x^{\Delta}(t), \quad t \in \mathbb{T}^k.$$

Hence,

$$x_n^{\Delta^2} = \Delta_1 f(t_n, x_n) + \Big(\int_0^1 \frac{\partial}{\partial x} f(\sigma(t_n), x_n + s\mu(t_n)x_n^{\Delta})ds\Big)f(t_n, x_n).$$

Example 4.2.1 *Consider the IVP*

$$\begin{cases} x^{\Delta}(t) = g(t) + \dfrac{1}{(x(t))^2 + 1}, \quad t \in [t_0, t_N] \\ x(t_0) = x_0, \end{cases}$$

where $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable, $|g(t)| \leq B$, $|g^{\Delta}(t)| \leq B$ for some positive constant B and $x_0 \in \mathbb{R}$. Here

$$f(t, x(t)) = g(t) + \frac{1}{(x(t))^2 + 1}. \tag{4.4}$$

Then, if $x(t)$ is a solution of the IVP, we get

$$\begin{aligned} x^{\Delta^2}(t) &= g^{\Delta}(t) + \left(\int_0^1 \frac{2(x(t) + s\mu(t)x^{\Delta}(t))}{(1 + (x(t) + s\mu(t)x^{\Delta}(t))^2)^2} ds \right) x^{\Delta}(t) \\ &= g^{\Delta}(t) \\ &- \left(\int_0^1 \frac{2\left(x(t) + s\mu(t)\left(g(t) + \frac{1}{(x(t))^2+1}\right)\right)}{\left(1 + \left(x(t) + s\mu(t)\left(g(t) + \frac{1}{(x(t))^2+1}\right)\right)^2\right)^2} ds \right) \\ &\times \left(g(t) + \frac{1}{(x(t))^2 + 1} \right). \end{aligned}$$

4.3 Convergence of the Order-Two Taylor Series Method of order 2

In this section we discuss the convergence of the Taylor Series method of order 2 defined in the previous section. First we need the following result about the time scale monomials which is proved in [20].

Theorem 4.3.1 *[20] For all $i \in \mathbb{N}$ we have the following estimate*

$$0 \leq h_i(t, s) \leq \frac{(t-s)^i}{i!}, \quad t \geq s.$$

By the dynamic equation in the IVP and the condition (*C1*), we have

$$|x^{\Delta}(t)| \leq K, \quad t \in [t_0, t_N].$$

Then, by the Pötzsche's chain rule and the condition (*C2*), we get

$$\begin{aligned} |(f(t, x(t)))^{\Delta}| &\leq |\Delta_1 f(t, x(t))| + \left(\int_0^1 \left| \frac{\partial}{\partial x} f(\sigma(t), x(t) + s\mu(t)x^{\Delta}(t)) \right| ds \right) |x^{\Delta}(t)| \\ &\leq K + K^2, \quad t \in [t_0, t_N]. \end{aligned}$$

Therefore,

$$|x^{\Delta^2}| \leq K + K^2, \quad t \in [t_0, t_N].$$

by the Theorem 4.1. On the other hand, by the Theorem 4.3.1 we have

$$h_2(t+r,\sigma(\tau)) \le \frac{(t+r-\sigma(\tau))^2}{2}$$
$$\le \frac{(t+r-t)^2}{2}$$
$$= \frac{r^2}{2},$$

$$h_1(t+r,\sigma(\tau)) \le t+r-\sigma(\tau)$$
$$\le (t+r-t) = r$$

where $\tau \in [t, \rho^2(t+r)]$, and $t, t+r \in [t_0, t_N]$, $\quad r > 0$, and applying again the Pötzsche's Chain rule we get

$$x^{\Delta^3}(t) = \Delta_1 g(t,x(t)) + \left(\int_0^1 \frac{\partial}{\partial x} g(\sigma(t), x(t) + s\mu(t)x^{\Delta}(t))ds\right)x^{\Delta}(t),$$

so that,

$$|x^{\Delta^3}(t)| \le |\Delta_1 g(t,x(t))| + \left(\int_0^1 \left|\frac{\partial}{\partial x} g(\sigma(t), x(t) + s\mu(t)x^{\Delta}(t))\right|ds\right)|x^{\Delta}(t)|,$$
$$\le K + K^2, \quad t \in [t_0, t_f].$$

Then by the previous steps we estimate

$$|R_2(r) = \left|\int_t^{\rho^2(t+r)} h_2(t+r,\sigma(\tau))x^{\Delta^3}(\tau)\Delta\tau\right|$$
$$\le \int_t^{\rho^2(t+r)} h_2(t+r,\sigma(\tau))\left|x^{\Delta^3}(\tau)\right|\Delta\tau$$
$$\le \frac{r^2}{2}(K+K^2)(\rho^2(t+r)-t)$$
$$\le \frac{r^2}{2}(K+K^2)(t+r-t)$$
$$\le \frac{r^3}{2}(K+K^2), \quad t, t+r \in [t_0, t_N], \quad r > 0.$$

On the other hand we have

$$|R_1(r)| = \left|\int_t^{\rho^2(t+r)} h_1(t+r,\sigma(\tau))x^{\Delta^2}(\tau)\Delta\tau\right|$$
$$\le \int_t^{\rho^2(t+r)} h_1(t+r,\sigma(\tau))\left|x^{\Delta^2}(\tau)\right|\Delta\tau$$

$$\le r(K+K^2)(\rho(t+r)-t)$$

$$\le r(K+K^2)(t+r-t)$$

$$\le r^2(K+K^2), \quad t, t+r \in [t_0, t_N], \quad r > 0.$$

i.e.

$$R_2(r) = O(r^3), \quad R_1(r) = O(r^2).$$

Now, denote

$$e_n = x(t_n) - x_n.$$

By the Taylor formula, we have

$$x(t_{n+1}) = x(t_n) + r_{n+1} f(t_n, x(t_n)) + h_2(t_{n+1}, t_n) g(t_n, x(t_n)) + R_2(r_{n+1}).$$

Recall the relation (4.3)

$$x_{n+1} = x_n + r_{n+1} f(t_n, x_n) + h_2(t_{n+1}, t_n) g(t_n, x_n).$$

We subtract the above equations side by side and apply the Mean Value Theorem in the classical case. This yields

$$x(t_{n+1}) - x_{n+1} = x(t_n) - x_n + r_{n+1} f(t_n, x(t_n)) - f(t_n, x(t_n))$$

$$h_2(t_{n+1}, t_n)(g(t_n.x(t_n)) - g(t_n, x_n)) + R_2(r_{n+1})$$

$$= x(t_n) - x_n + r_{n+1} \frac{\partial}{\partial x} f(t_n, \xi)(x(t_n) - x_n)$$

$$+ h_2(t_{n+1}, t_n) \frac{\partial}{\partial x} g(t_n, \eta)(x(t_n) - x_n) R_2(r_{n+1}).$$

$$= e_n + \left(r_{n+1} \frac{\partial}{\partial x} f(t_n, \xi) + h_2(t_{n+1}, t_n) \frac{\partial}{\partial x} g(t_n, \eta) \right) e_n + R_2(r_{n+1}),$$

where η and ξ are between $x(t_n)$ and x_n. Let

$$K_n = r_{n+1} \frac{\partial}{\partial x} f(t_n, \xi) + h_2(t_{n+1}, t_n) \frac{\partial}{\partial x} g(t_n, \eta).$$

We have

$$|K_n| \le r_{n+1} \left| \frac{\partial}{\partial x} f(t_n, \xi) \right| + h_2(t_{n+1}, t_n) \left| \frac{\partial}{\partial x} g(t_n, \eta) \right|$$

$$\le K r_{n+1} + K \frac{r_{n+1}^2}{2}$$

$$= r_{n+1} \left(1 + \frac{r_{n+1}^2}{2} \right) K.$$

Then

$$e_0 = 0,$$

$$e_1 = R_2(r_1),$$

$$e_{n+1} = (1 + K_n)e_n + R_2(r_{n+1}), \quad n \in \mathbb{N}.$$

In particular,

$$e_2 = (1 + K_1)e_1 + R_2(r_2),$$

$$e_3 = (1 + K_2)e_2 + R_2(r_3)$$

$$= (1 + K_2)((1 + K_1)e_1 + R_2(r_2)) + R_2(r_3)$$

$$= (1 + K_2)(1 + K_1)e_1 + (1 + K_2)R_2(r_2)) + R_2(r_3)$$

$$e_4 = (1 + K_3)e_3 + R_2(r_4),$$

$$= (1 + K_3)((1 + K_2)(1 + K_1)e_1 + (1 + K_2)R_2(r_2) + R_2(r_3)) + R_2(r_4)$$

$$= (1 + K_3)(1 + K_2)(1 + K_1)e_1 + (1 + K_3)(1 + K_2)R_2(r_2) + (1 + K_3)R_2(r_3) + R_2(r_4),$$

and so on.

Let $r_{max} = \max\{r_1, \ldots, r_{N+1}\}$. Then since we have $0 < r_j \leq t_N - t_0$, $j \in \{1, \ldots, N + 1\}$, and $0 < t_N - t_0 < \infty$, we get that $R_2(r_j) = O(r_{max}^3)$, $j \in \{1, \ldots, N + 1\}$. Since $0 < t_N - t_0 < \infty$ and $t_{j-1} + r_j \in [t_0, t_N]$, $j \in \{1, \ldots, N + 1\}$, there exists a constant $0 < M < \infty$ such that $Nr_{max} \leq M$. Therefore, we estimate

$$|e_2| \leq \left(1 + r_{max}\left(1 + \frac{r_{max}}{2}\right)k\right)R_2(r_{max}) + R_2(r_{max}),$$

$$|e_3| \leq \left(1 + r_{max}\left(1 + \frac{r_{max}}{2}\right)k\right)^2 R_2(r_{max})$$

$$+\left(1 + r_{max}\left(1 + \frac{r_{max}}{2}\right)K\right)R_2(r_{max}) + R_2(r_{max})$$

$$\ldots$$

$$|e_N| \leq \sum_{j=0}^{N-1} \left(1 + r_{max}\left(1 + \frac{r_{max}}{2}\right)K\right)^j R_2(r_{max})$$

$$\leq \sum_{j=0}^{N-1} e^{r_{max}(1+\frac{r_{max}}{2})Kj} R_2(r_{max})$$

$$\leq Ne^{Nr_{max}\left(1+\frac{r_{max}}{2}\right)K} R_2(r_{max})$$

$$\leq Ne^{M(1+\frac{r_{max}}{2})K} R_2(r_{max})$$

$$\leq Ne^{M(1+\frac{t_N-t_0}{2})K}R_2(r_{max})$$

$$\leq Nr_{max}r_{max}^2 e^{(1+\frac{t_N-t_0}{2})K}C$$

$$\leq r_{max}^2 e^{M(1+\frac{t_N-t_0}{2})K}MC,$$

where C is some positive constant. Since $t_N < \infty$, we conclude that

$$e_m = O(r_{max}^2).$$

That is, the order of convergence of the order two Taylor series method is two.

4.4 Trapezoidal Rule

In this section we will derive the Trapezoidal rule for IVPs associated with first order dynamic equations on time scales. We write the Taylor's formula for x^Δ which is

$$x^\Delta(t+r) = x^\Delta(t)+rx^{\Delta^2}(t)+\int_t^{p(t+r)} h_1(t+r,\sigma(\tau))x^{\Delta^3}(\tau)\Delta\tau = x^\Delta(t)+rx^{\Delta^2}(t)+R_1(r), \quad (4.5)$$

whereupon

$$rx^{\Delta^2}(t) = x^\Delta(t+r) - x^\Delta(t) - R_1(r), \quad t, t+r \in [t_0, t_N], \; r > 0.$$

Now, we substitute the last relation into the equation (4.2)

$$\begin{aligned}
x(t+r) &= x(t) + rx^\Delta(t) + h_2(t+r,t)x^{\Delta^2}(t) + R_2(r)\\
&= x(t) + rx^\Delta(t) + \frac{h_2(t+r,t)}{r}(rx^{\Delta^2}(t)) + R_2(r)\\
&= x(t) + rx^\Delta(t) + \frac{h_2(t+r,t)}{r}\Big(x^\Delta(t+r) - x^\Delta(t) - R_1(r)\Big) + R_2(r)\\
&= x(t) + \Big(r - \frac{h_2(t+r,t)}{r}\Big)x^\Delta(t) + \frac{h_2(t+r,t)}{r}x^\Delta(t+r)\\
&\quad -\frac{h_2(t+r,t)}{r}R_1(r) + R_2(r)\\
&= x(t) + \Big(r - \frac{h_2(t+r,t)}{r}\Big)f(t,x(t)) + \frac{h_2(t+r,t)}{r}f(t+r,x(t+r))\\
&\quad -\frac{h_2(t+r,t)}{r}R_1(r) + R_2(r), \quad ,t+r \in [t_0, t_f], \; r > 0.
\end{aligned}$$

Evaluating this at $t = t_n$ and neglecting the remainder terms leads to the following definition of the Trapezoidal rule.

Definition 4.4.1 *The formula*

$$x_{n+1} = x_n + \left(r_{n+1} - \frac{h_2(t_{n+1}, t_n)}{r_{n+1}}\right) f(t_n, x_n) + \frac{h_2(t_{n+1}, t_n)}{r_{n+1}} f(t_{n+1}, x_{n+1}) \tag{4.6}$$

is called Trapezoidal rule.

Example 4.4.2 *Let $\mathbb{T} = \mathbb{R}$ and $t_{n+1} - t_n = r$ be constant. Then*

$$h_2(t_{n+1}, t_n) = \frac{1}{2} r^2,$$

and the Trapezoidal rule takes the form

$$x_{n+1} = x_n + \frac{1}{2} r(f(t_n, x_n) + f(t_{n+1}, x_{n+1})),$$

which is the Trapezoidal rule in classical sense.

Example 4.4.3 *Let $\mathbb{T} = 2^{\mathbb{N}_0}$. Then*

$$h_2(t_{n+1}, t_n) = \frac{(t_{n+1} - t_n)(t_{n+2} - 2t_n)}{3}$$

$$\frac{r_{n+1}}{3}(t_{n+1} - 2t_n)$$

and the Trapezoidal rule takes the form

$$x_{n+1} = x_n + \left(r_{n+1} - \frac{1}{3}(t_{n+1} - 2t_n)\right) f(t_n, x_n) + \frac{1}{3}(t_{n+1} - 2t_n) f(t_{n+1}, x_{n+1}).$$

4.5 Numerical Examples

Example 4.5.1 *Consider the initial value problem*

$$\begin{cases} x^{\Delta}(t) = g(t) + \dfrac{1}{1 + x^2}, \\ x(t_0) = x_0 \end{cases},$$

on the time scale $\mathbb{T} = \alpha\mathbb{N}_0$. We will solve this problem with both Taylor series of order 2 formula and with the Trapezoidal rule. Recall that by the Example 4.2.1 we have

$$x^{\Delta^2} = g^{\Delta}(t) - \left\{ \int_0^1 \frac{2(x(t) + s\alpha(g(t) + \frac{1}{x^2(t)+1}))}{\left[1 + (x(t) + s\alpha(g(t) + \frac{1}{x^2(t)+1}))^2\right]^2} ds \right\}$$

$$\times\left(g(t) + \frac{1}{x^2(t)+1}\right).$$

Let

$$A = g(t) + \frac{1}{x^2(t)+1}.$$

Then the expression above becomes

$$x^{\Delta^2} = g^{\Delta}(t) - \left(\int_0^1 \frac{2(x(t)+s\alpha A)}{\left[1+(x(t)+s\alpha A)^2\right]^2} ds\right) \times A.$$

Let $x(t) + \alpha s A = u$, *so that* $\alpha A ds = du$ *and the integral above becomes*

$$\int_{x(t)}^{x(t)+\alpha A} \frac{2u}{(1+u^2)^2}\frac{du}{\alpha A} = \frac{1}{\alpha A}\left[-(1+u^2)^{-1}\Big|_{x(t)}^{x(t)+\alpha A}\right]$$
$$= -\frac{1}{\alpha A}\left(\frac{1}{1+(x(t)+\alpha A)^2} - \frac{1}{1+x(t)^2}\right).$$

This yields

$$x^{\Delta^2} = g^{\Delta}(t) + \frac{1}{\alpha}\left(\frac{1}{1+(x(t)+\alpha A)^2} - \frac{1}{1+x(t)^2}\right).$$

On the time scale $\mathbb{T} = \alpha\mathbb{N}_0$, *we have*

$$h_0(t,s) = 1, \quad h_1(t,s) = t - s$$

and from

$$\left(\frac{t^2}{2} - \frac{\alpha t}{2} - st\right)^{\Delta} = \frac{t+\alpha+t}{2} - \frac{\alpha}{2} - s = t - s$$

we compute

$$h_2(t,s) = \int_s^t (\tau - s)\Delta\tau = \left(\frac{\tau^2}{2} - \frac{\alpha\tau}{2} - s\tau\right)\Big|_s^t$$
$$= \left(\frac{t^2}{2} - \frac{\alpha t}{2} - st\right) - \left(\frac{s^2}{2} - \frac{\alpha s}{2} - s^2\right) = \frac{(t-s)(t-s-\alpha)}{2}.$$

Choosing constant step size $q \geq \alpha$ *the Taylor seies of order 2 formula gives*

$$x_{i+1} = x_i + q\left(g(t_i) + \frac{1}{x_i^2+1}\right) + \frac{q(q-\alpha)}{2}x_i^{\Delta^2},$$

where

$$x_i^{\Delta^2} = g^{\Delta}(t_i) + \frac{1}{\alpha}\left(\frac{1}{1+(x_i+\alpha A)^2} - \frac{1}{1+x_i^2}\right).$$

On the other hand, the Trapezoidal rule yields

$$x_{i+1} = x_i + \frac{q+\alpha}{2}\left(g(t_i) + \frac{1}{x_i^2+1}\right) + \frac{q-\alpha}{2}\left(g(t_{i+1}) + \frac{1}{x_{i+1}^2+1}\right).$$

The second formula is an implicit relation and to find x_{i+1} we rewrite it as

$$\frac{a_3 x_{i+1}^3 + a_2 x_{i+1}^2 + a_1 x_{i+1} + a_0}{2(1 + x_{i+1}^2)} = 0$$

where

$$\begin{aligned} a_3 &= 2, \\ a_2 &= -\left(2x_i + (q+\alpha)(g(t_i) + \frac{1}{x_i^2+1}) + (q-\alpha)g(t_{i+1})\right), \\ a_1 &= 2, \\ a_0 &= -\left(2x_i + (q+\alpha)(g(t_i) + \frac{1}{x_i^2+1}) + (q-\alpha)g(t_{i+1}) + q - \alpha\right). \end{aligned}$$

We apply the Newton's method to find x_{i+1} at each step $i = 0, \ldots, N-1$. In what follows, we take $g(t)$ in two different forms and compute the approximate solutions.

Case 1: *Take $g(t) = 2t$. Then we have $g^{\Delta}(t) = 2$. The computations are done with Matlab for different values of q, α, t_0 and x_0. In Tables 4.1 - 4.3, the values of computed and exact solutions obtained with Taylor series method are given. These solutions are shown graphically in Figures 4.1 - 4.6 .*

Case 2: *Take $g(t) = \frac{t}{t^2+1}$. Then*

$$g^{\Delta}(t) = \frac{\frac{t+\alpha}{(t+\alpha)^2+1} - \frac{t}{t^2+1}}{\alpha} = \frac{-t^2 - \alpha t + 1}{(t^2+1)\left[(t+\alpha)^2+1\right]}$$

The computations are done with Matlab for different values of q, α, t_0 and x_0. In Tables 4.4 - 4.6, the values of computed and exact solutions obtained with Taylor series method are given. These solutions are shown graphically in Figures 4.7 - 4.12 .

Table 4.1: The values of the exact solution x_e, solution obtained with Taylor series method x_{ts} and solution obtained with Trapezoidal rule x_{tr} for $g(t) = 2t$ with $q = 1$, $\alpha = 1$, $t_0 = 1$ and $x_0 = 1$.

t_i	$x_e(t_i)$	$x_{ts}(t_i)$	$x_{tr}(t_i)$
1.00	1.00000000	1.00000000	1.00000000
2.00	3.50000000	3.50000000	3.50000000
3.00	7.57547170	7.57547170	7.57547170
4.00	13.59259857	13.59259857	13.59259857
5.00	21.59798190	21.59798190	21.59798190
6.00	31.60012106	31.60012106	31.60012106
7.00	43.60112149	43.60112149	43.60112149
8.00	57.60164724	57.60164724	57.60164724
9.00	73.60194854	73.60194854	73.60194854
10.00	91.60213310	91.60213310	91.60213310

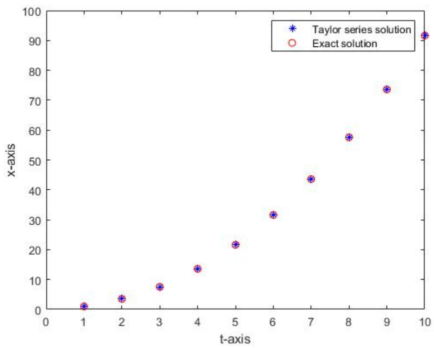

Figure 4.1: Computed and exact solution with Taylor series method for $g(t) = 2t$ with $q = 1$, $\alpha = 1$, $t_0 = 1$ and $x_0 = 1$.

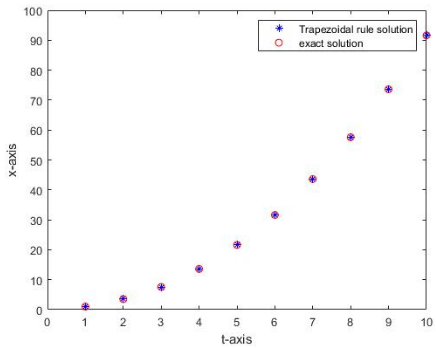

Figure 4.2: Computed and exact solution with the Trapezoidal rule for $g(t) = 2t$ with $q = 1$, $\alpha = 1$, $t_0 = 1$ and $x_0 = 1$.

Table 4.2: The values of the exact solution x_e, solution obtained with Taylor series method x_{ts} and solution obtained with Trapezoidal rule x_{tr} for $g(t) = 2t$ with $q = 1$, $\alpha = 0.5$, $t_0 = 1$ and $x_0 = 1$.

t_i	$x_e(t_i)$	$x_{ts}(t_i)$	$x_{tr}(t_i)$
1.00	1.00000000	1.00000000	1.00000000
2.00	3.83247423	3.83247423	3.89049337
3.00	8.37847407	8.37847407	8.44043411
4.00	14.88932428	14.88932428	14.95192932
5.00	23.39296660	23.39296660	23.45572276
6.00	33.89449804	33.89449804	33.95730012
7.00	46.39524684	46.39524684	46.45806575
8.00	60.89565433	60.89565433	60.95848034
9.00	77.39589444	77.39589444	77.45872378
10.00	95.89604487	95.89604487	95.95887591

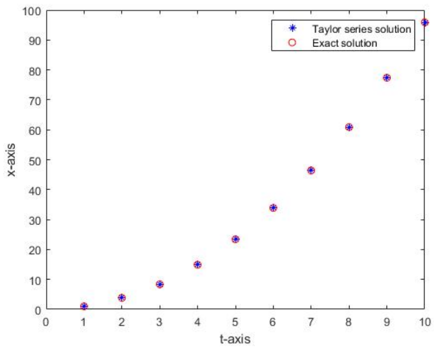

Figure 4.3: Computed and exact solution with Taylor series method for $g(t) = 2t$ with $q = 1$, $\alpha = 0.5$, $t_0 = 1$ and $x_0 = 1$.

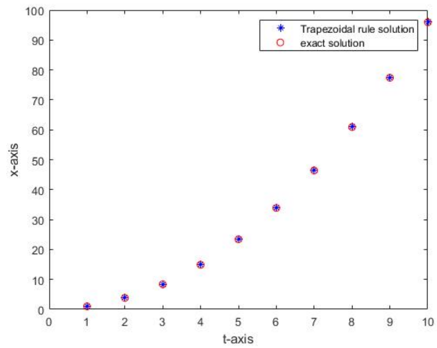

Figure 4.4: Computed and exact solution with Trapezoidal rule for $g(t) = 2t$ with $q = 1$, $\alpha = 0.5$, $t_0 = 1$ and $x_0 = 1$.

Table 4.3: The values of the exact solution x_e, solution obtained with Taylor series method x_{ts} and solution obtained with Trapezoidal rule x_{tr} for $g(t) = 2t$ with $q = 0.5$, $\alpha = 0.25$, $t_0 = 1$ and $x_0 = 1$.

t_i	$x_e(t_i)$	$x_{ts}(t_i)$	$x_{tr}(t_i)$
1.00	1.00000000	1.00000000	1.00000000
1.50	2.31866953	2.31866953	2.33191647
2.00	4.00633223	4.00633223	4.02244175
2.50	6.15553259	6.15553259	6.17246640
3.00	8.79143018	8.79143018	8.80864784
3.50	11.92196037	11.92196037	11.93929010
4.00	15.55003699	15.55003699	15.56741616
4.50	19.67687559	19.67687559	19.69427864
5.00	24.30303849	24.30303849	24.32045396
5.50	29.42880855	29.42880855	29.44623088
6.00	35.05433808	35.05433808	35.07176441
6.50	41.17971388	41.17971388	41.19714264
7.00	47.80498780	47.80498780	47.82241810
7.50	54.93019208	54.93019208	54.94762339
8.00	62.55534749	62.55534749	62.57277946
8.50	70.68046779	70.68046779	70.69790022
9.00	79.30556234	79.30556234	79.32299510
9.50	88.43063767	88.43063767	88.44807066
10.00	98.05569843	98.05569843	98.07313159
10.50	108.18074796	108.18074796	108.19818125

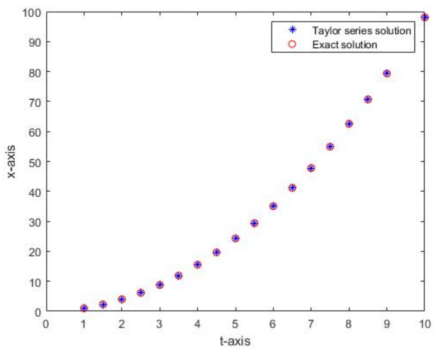

Figure 4.5: Computed and exact solution with Taylor series method for $g(t) = 2t$ with $q = 0.5$, $\alpha = 0.25$, $t_0 = 1$ and $x_0 = 1$.

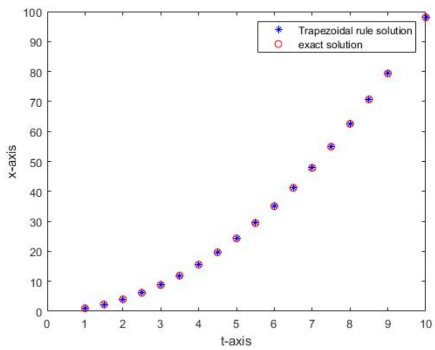

Figure 4.6: Computed and exact solution with Trapezoidal rule for $g(t) = 2t$ with $q = 0.5$, $\alpha = 0.25$, $t_0 = 1$ and $x_0 = 1$.

Table 4.4: The values of the exact solution x_e, solution obtained with Taylor series method x_{ts} and solution obtained with Trapezoidal rule x_{tr} for $g(t) = \frac{t}{t^2+1}$ with $q = 1$, $\alpha = 1$, $t_0 = 1$ and $x_0 = 1$.

t_i	$x_e(t_i)$	$x_{ts}(t_i)$	$x_{tr}(t_i)$
0.00	1.00000000	1.00000000	1.00000000
1.00	1.50000000	1.50000000	1.50000000
2.00	2.30769231	2.30769231	2.30769231
3.00	2.86578398	2.86578398	2.86578398
4.00	3.27432958	3.27432958	3.27432958
5.00	3.59493895	3.59493895	3.59493895
6.00	3.85906722	3.85906722	3.85906722
7.00	4.08415254	4.08415254	4.08415254
8.00	4.28071266	4.28071266	4.28071266
9.00	4.45553736	4.45553736	4.45553736

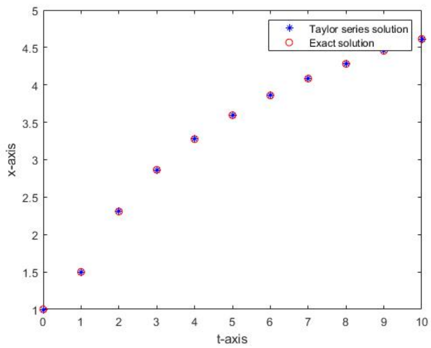

Figure 4.7: Computed and exact solution with Taylor series method for $g(t) = \frac{t}{t^2+1}$ with $q = 1$, $\alpha = 0.5$, $t_0 = 1$ and $x_0 = 1$.

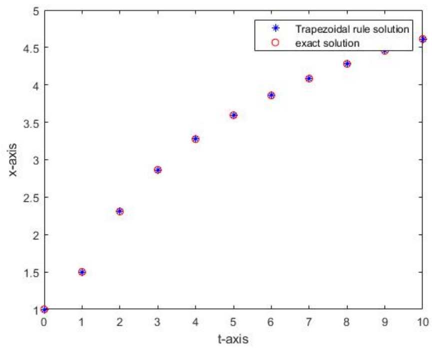

Figure 4.8: Computed and exact solution with Trapezoidal rule for $g(t) = \frac{t}{t^2+1}$ with $q = 1$, $\alpha = 0.5$, $t_0 = 1$ and $x_0 = 1$.

Table 4.5: The values of the exact solution x_e, solution obtained with Taylor series method x_{ts} and solution obtained with Trapezoidal rule x_{tr} for $g(t) = \frac{t}{t^2+1}$ with $q = 1$, $\alpha = 0.5$, $t_0 = 1$ and $x_0 = 1$.

t_i	$x_e(t_i)$	$x_{ts}(t_i)$	$x_{tr}(t_i)$
0.00	1.00000000	1.00000000	1.00000000
1.00	1.64512195	1.64512195	1.57202022
2.00	2.35842934	2.35842934	2.30274677
3.00	2.86999968	2.86999968	2.82458937
4.00	3.25405149	3.25405149	3.21401314
5.00	3.56021668	3.56021668	3.52339411
6.00	3.81509146	3.81509146	3.78042471
7.00	4.03389026	4.03389026	4.00079323
8.00	4.22599490	4.22599490	4.19411132
9.00	4.39756224	4.39756224	4.36665890

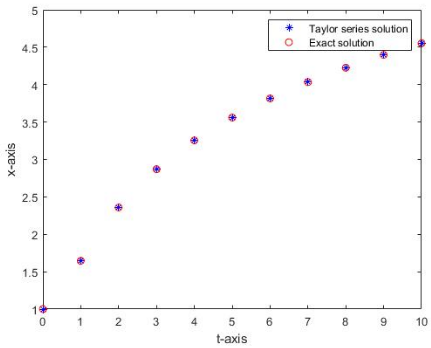

Figure 4.9: Computed and exact solution with Taylor series method for $g(t) = \frac{t}{t^2+1}$ with $q = 1$, $\alpha = 0.5$, $t_0 = 1$ and $x_0 = 1$.

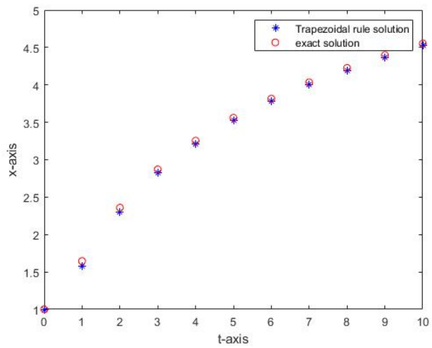

Figure 4.10: Computed and exact solution with Trapezoidal rule for $g(t) = \frac{t}{t^2+1}$ with $q = 1$, $\alpha = 0.5$, $t_0 = 1$ and $x_0 = 1$.

Table 4.6: The values of the exact solution x_e, solution obtained with Taylor series method x_{ts} and solution obtained with Trapezoidal rule x_{tr} for $g(t) = \frac{t}{t^2+1}$ with $q = 0.5$, $\alpha = 0.25$, $t_0 = 1$ and $x_0 = 1$.

t_i	$x_e(t_i)$	$x_{ts}(t_i)$	$x_{tr}(t_i)$
0.00	1.00000000	1.00000000	1.00000000
0.50	1.29416836	1.29416836	1.28466340
1.00	1.68543859	1.68543859	1.67159828
1.50	2.05282097	2.05282097	2.03984486
2.00	2.36613670	2.36613670	2.35468250
2.50	2.63119427	2.63119427	2.62096986
3.00	2.85846282	2.85846282	2.84914236
3.50	3.05659282	3.05659282	3.04793822
4.00	3.23197353	3.23197353	3.22382162
4.50	3.38925642	3.38925642	3.38149504
5.00	3.53186629	3.53186629	3.52441668
5.50	3.66237141	3.66237141	3.65517683
6.00	3.78273556	3.78273556	3.77575405
6.50	3.89448833	3.89448833	3.88768814
7.00	3.99884148	3.99884148	3.99219805
7.50	4.09676978	4.09676978	4.09026373
8.00	4.18906812	4.18906812	4.18268390
8.50	4.27639280	4.27639280	4.27011773
9.00	4.35929160	4.35929160	4.35311521
9.50	4.43822636	4.43822636	4.43213987

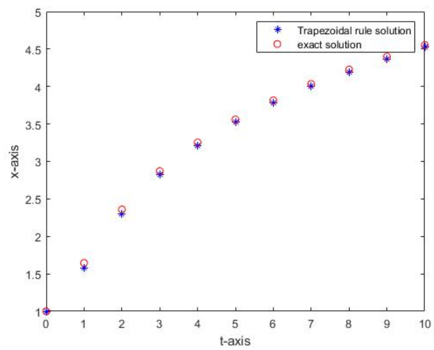

Figure 4.11: Computed and exact solution with Taylor series method for $g(t) = \frac{t}{t^2+1}$ with $q = 0.5$, $\alpha = 0.25$, $t_0 = 1$ and $x_0 = 1$.

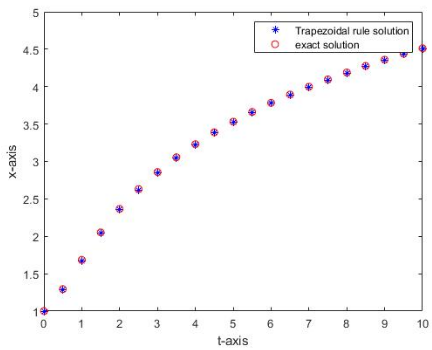

Figure 4.12: Computed and exact solution with Trapezoidal rule for $g(t) = \frac{t}{t^2+1}$ with $q = 0.5$, $\alpha = 0.25$, $t_0 = 1$ and $x_0 = 1$.

Example 4.5.2 *Consider the IVP given in Example (3.4.2), that is,*

$$\begin{cases} x^{\Delta^2}(t) + (t+2)x^{\Delta}(t) + 2tx(t) = 0 \\ x(0) = a, \ x^{\Delta}(0) = b \end{cases}.$$

The IVP can be written as

$$\begin{cases} x_1^{\Delta}(t) = x_2(t) \\ x_2^{\Delta}(t) = -(t+2)x_2(t) - 2tx_1(t) \\ x_1(0) = a, \ x_2(0) = b \end{cases}.$$

From the Example 4.5.1 we have $h_2(t_{i+1}, t_i) = \dfrac{q(q-\alpha)}{2}$ *for* $r_n = q = constant.$

Then, the Trapezoidal rule yields

$$\begin{cases} x_{1,i+1} = x_{1,i} + \left(q - \frac{q(q-\alpha)}{2q}\right) x_{2,i} + \frac{q(q-\alpha)}{2q} x_{2,i+1} \\ x_{2,i+1} = x_{2,i} + \left(q - \frac{q(q-\alpha)}{2q}\right)(-(t_i+2)x_{2,i} - 2t_i x_{1,i}) \\ +\frac{q(q-\alpha)}{2q}\left(-(t_{i+1}+2)x_{2,i+1} - 2t_{i+1}x_{1,i+1}\right) \end{cases}$$

which can be simplified as

$$\begin{cases} x_{1,i+1} = x_{1,i} + \frac{q+\alpha}{2} x_{2,i} + \frac{q-\alpha}{2} x_{2,i+1} \\ x_{2,i+1} = x_{2,i} + \frac{q+\alpha}{2}\left(-(t_i+2)x_{2,i} - 2t_i x_{1,i}\right) \\ +\frac{q-\alpha}{2}\left(-(t_{i+1}+2)x_{2,i+1} - 2t_{i+1}x_{1,i+1}\right) \end{cases}. \tag{4.7}$$

We solve the second equation for $x_{2,i+1}$ *as*

$$x_{2,i+1} = \frac{1}{1 + \frac{(q-\alpha)}{2}(2+t_{i+1})}\left[-t_i(q+\alpha)x_{1,i} + \left(1 - \frac{2+t_i}{2}(q+\alpha)\right)x_{2,i} - t_{i+1}(q-\alpha)x_{1,i+1}\right],$$

and we insert $x_{2,i+1}$ *from the above relation into the first equation of the system* (4.7).

Then we obtain

$$\begin{aligned} x_{1,i+1} &= x_{1,i} + \tfrac{q+\alpha}{2} x_{2,i} \\ &+ \tfrac{q-\alpha}{2\left(1+\frac{q-\alpha}{2}(2+t_{i+1})\right)}\left[-t_i(q+\alpha)x_{1,i} + \left(1 - \tfrac{2+t_i}{2}(q+\alpha)\right)x_{2,i} - t_{i+1}(q-\alpha)x_{1,i+1}\right]. \end{aligned}$$

Then we get

$$\left[1 + \frac{(q-\alpha)^2 t_{i+1}}{2\left[1 + \frac{(q-\alpha)}{2}(2+t_{i+1})\right]}\right] x_{1,i+1}$$

$$= x_{1,i} + \frac{q+\alpha}{2} x_{2,i} - \frac{t_i(q-\alpha)(q+\alpha)}{2\left(1 + \frac{(q-\alpha)}{2}(2+t_{i+1})\right)} x_{2,i}$$

$$+\frac{(q-\alpha)}{2\left(1+\frac{(q-\alpha)}{2}(2+t_{i+1})\right)}\left(1-\frac{2+t_i}{2}(q+\alpha)\right)x_{2,i}.$$

Let

$$s = 1+\frac{q-\alpha}{2}(2+t_{i+1})$$
$$p = 1-\frac{2+t_i}{2}(q+\alpha).$$

Finally we obtain

$$\begin{cases} x_{1,i+1} = \frac{1}{1+\frac{(q+\alpha)^2 t_{i+1}}{2s}}\left[x_{1,i}+\frac{q+\alpha}{2}x_{2,i}-\frac{t_i(q^2-\alpha^2)}{2s}x_{1,i}\right]+\frac{(q+\alpha)p}{2s}x_{2,i} \\ x_{2,i+1} = \frac{1}{s}\left[-t_i(q+\alpha)x_{1,i}+px_{2,i}-t_{i+1}(q-\alpha)x_{1,i+1}\right], \end{cases}$$

for $i = 0,\ldots,N-1$. *We compute the approximate solution for different values of* a, b, q *and* α *using Matlab. Note that this example has been solved in Chapter 3 as Example 3.4.2. In order to compare the results of Euler method and Trapezoidal rule we use the same values for the parameters as in the Example 3.4.2.*

The graphs of computed and exact solutions are presented in Figures 4.13-4.16. The values of the computed solution are denoted by x_c *and those of the exact solution by* x_e *and are shown in Tables 4.7-4.10.*

Table 4.7: The values of the solution obtaind with Trapizodal rule x_c, and the exact solution x_e at points of the interval $[0, 5]$ for $q = 0.5$, $\alpha = 0.25$, $a = 1$ and $b = 1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	1.00000000	1.00000000
0.50	1.37500000	1.38235294
1.00	1.31054688	1.32745098
1.50	0.95248413	0.98458204
2.00	0.52808928	0.55646233
2.50	0.22263740	0.23114111
3.00	0.07300364	0.06801888
3.50	0.01999308	0.01455120
4.00	0.00506632	0.00283129
4.50	0.00126701	0.00058205

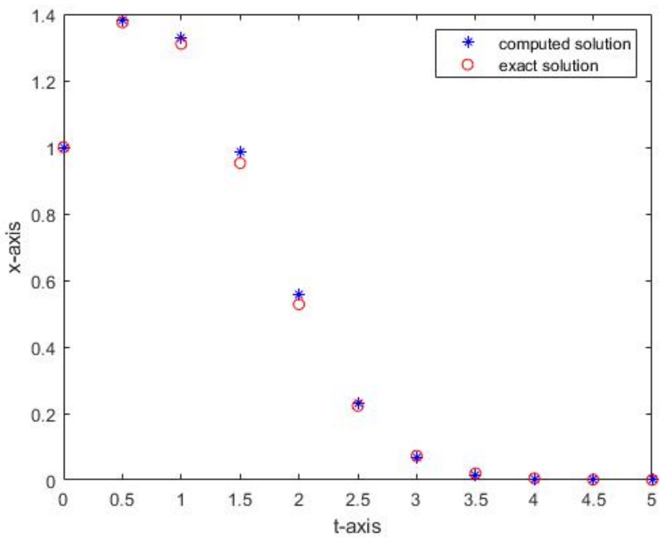

Figure 4.13: Computed and exact solution with $q = 0.5$, $\alpha = 0.25$, $t_0 = 1$ and $x_0 = 1$.

Table 4.8: The values of the solution obtaind with Trapizodal rule x_c, and the exact solution x_e at points of the interval $[0, 5]$ for $q = 0.2$, $\alpha = 0.1$, $a = 1$ and $b = 1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	1.00000000	1.00000000
0.20	1.18000000	1.18046805
0.40	1.28386000	1.28464264
0.60	1.31856803	1.31986163
0.80	1.29190451	1.29396970
1.00	1.21397699	1.21695199
1.20	1.09730509	1.10111712
1.40	0.95586455	0.96023347
1.60	0.80358999	0.80810258
1.80	0.65284066	0.65705947
2.00	0.51323229	0.51679912
2.20	0.39106043	0.39376349
2.40	0.28934219	0.29113384
2.60	0.20834357	0.20931411
2.80	0.14637441	0.14670002
3.00	0.10062943	0.10051550
3.20	0.06790883	0.06754488
3.40	0.04513247	0.04466623
3.60	0.02963571	0.02916568
3.80	0.01928460	0.01886539
4.00	0.01246847	0.01212233
4.20	0.00802693	0.00775581
4.40	0.00515367	0.00494916
4.60	0.00330371	0.00315357
4.80	0.00211600	0.00200794

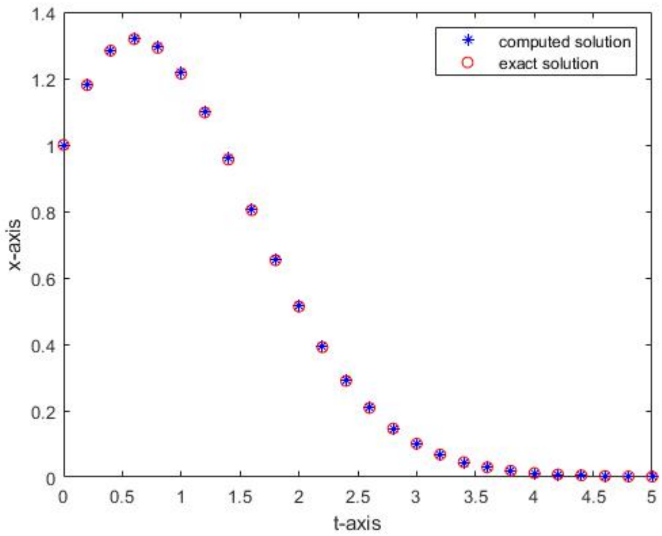

Figure 4.14: Computed and exact solution with $q = 0.2$, $\alpha = 0.1$, $t_0 = 1$ and $x_0 = 1$.

Table 4.9: The values of the solution obtaind with Trapizodal rule x_c, and the exact solution x_e at points of the interval $[0, 5]$ for $q = 0.5$, $\alpha = 0.25$, $a = 0.5$ and $b = -1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	0.50000000	0.50000000
0.50	0.12500000	0.10000000
1.00	0.03125000	0.02000000
1.50	0.00781250	0.00400000
2.00	0.00195312	0.00080000
2.50	0.00048828	0.00016000
3.00	0.00012207	0.00003200
3.50	0.00003052	0.00000640
4.00	0.00000763	0.00000128
4.50	0.00000191	0.00000026

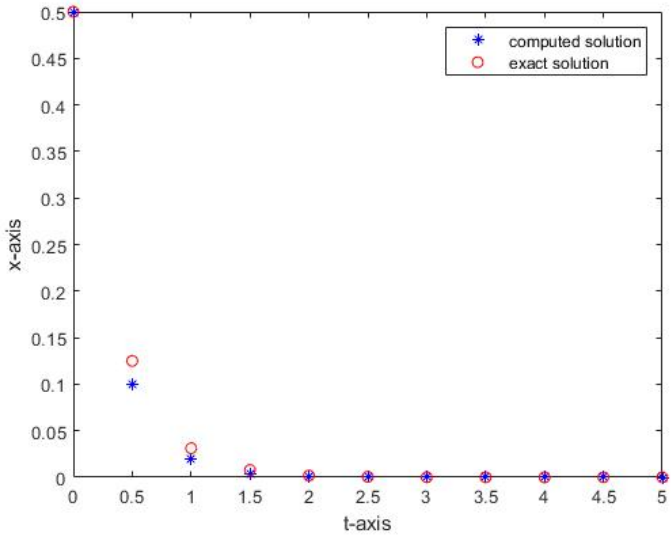

Figure 4.15: Computed and exact solution with $q = 0.5$, $\alpha = 0.25$, $t_0 = 0.5$ and $x_0 = -1$.

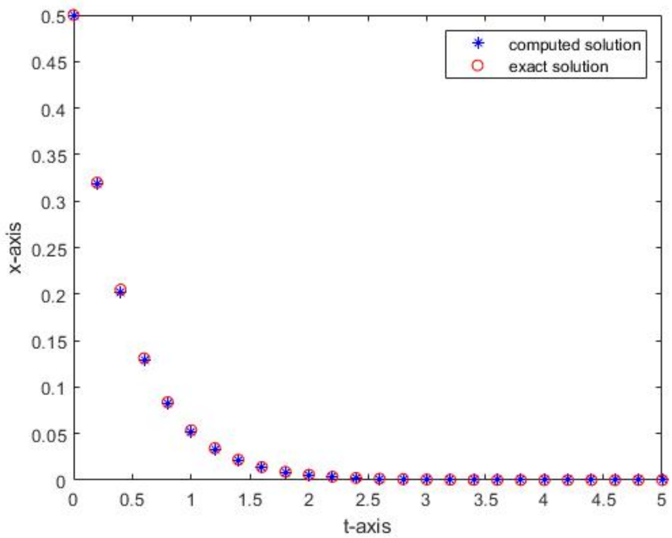

Figure 4.16: Computed and exact solution with $q = 0.2$, $\alpha = 0.1$, $t_0 = 0.5$ and $x_0 = -1$.

Table 4.10: The values of the solution obtaind with Trapizodal rule x_c, and the exact solution x_e at points of the interval $[0, 5]$ for $q = 0.2$, $\alpha = 0.2$, $a = 0.5$ and $b = -1$.

t_i	$x_e(t_i)$	$x_c(t_i)$
0.00	0.50000000	0.50000000
0.20	0.32000000	0.31818182
0.40	0.20480000	0.20247934
0.60	0.13107200	0.12885049
0.80	0.08388608	0.08199577
1.00	0.05368709	0.05217912
1.20	0.03435974	0.03320490
1.40	0.02199023	0.02113039
1.60	0.01407375	0.01344661
1.80	0.00900720	0.00855693
2.00	0.00576461	0.00544532
2.20	0.00368935	0.00346520
2.40	0.00236118	0.00220513
2.60	0.00151116	0.00140326
2.80	0.00096714	0.00089299
3.00	0.00061897	0.00056826
3.20	0.00039614	0.00036162
3.40	0.00025353	0.00023012
3.60	0.00016226	0.00014644
3.80	0.00010385	0.00009319
4.00	0.00006646	0.00005930
4.20	0.00004254	0.00003774
4.40	0.00002722	0.00002402
4.60	0.00001742	0.00001528
4.80	0.00001115	0.00000973

CHAPTER 5

CONCLUSION

The concept of time scale has been put forward for the first time about 35 years ago. Since then, the theory on time scales, including one and multivariable calculus, dynamic and partial dynamic equations and the relevant theoretical results have been almost completely developed [7, 2, 3, 13, 21].

On the other hand, very little has been done about the numerical analysis on time scales. On very recent studies the Euler method and Taylor Series method of order 2 have been considered on time scales [22, 23]. However, there is a little information on the numerical applications of these method to particular examples.

In this thesis we applied both the Euler and the Taylor series method of order 2 to various examples on different time scales. We computed the approximate solution and compared it with the exact solutions of the initial value problems under consideration. For the first time we applied both Taylor series of order 2 and the Trapezoidal rule to a given initial value problem. We also applied both Euler method and Trapezoidal rule to an initial value problem for a second order dynamic equation for the first time.

The numerical computations show that as in the case of differential equations, the Euler's method produces "poor" approximate solution in the sense that it results in a large error.

The Taylor series method of order 2 is much more efficient since it is convergent of order 2. The drawback of the method is the difficulty of the computation of x^{Δ^2} involved in the formula, which requires applying the Pötzsche Chain Rule.

The Trapezoidal rule, in which computation of x^{Δ^2} is not required is more appropriate.

However, it also has a drawback due to the fact that it is an implicit formula. Therefore, computation of the approximate solution requires use of root finding methods such as Newton's method.

In conclusion the numerical methods discussed in this thesis initiate the theory of numerical methods for dynamic equations on time scales and provide directions for further studies on this subject.

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APPENDIX A

ALGORITHMS

In the following we give the MATLAB codes for the programs used in the numerical examples.

A.1 MATLAB Code 1

The following is a MATLAB code for the Example 3.4.1.

```
clear all
q=input('enter the step size =');
y(1)=input('enter the initial value = ');
a=y(1)
ye(1)=a
x(1)=1;
xe(1)=1
for i=1:4
    x(i+1)=x(i)+1;
    y(i+1)=y(i)+(1+y(i))/(1+y(i)+y(i)^2);
    xe(i+1)=xe(i)+1
    ye(i+1)=y(i+1)
end
p=5/q;
b=ye(5)
for i=5:p+4
```

```
    x(i+1)=x(i)+q;
y(i+1)=y(i)+q*(1+y(i))/(1+y(i)+y(i)^2);
end
c=5-(b^2/2)-log(1+b)
f=@(s,t) (t^2/2)+log(1+t)-s+c
for i=6:p+4
    v=x(i);
    t=fzero(@(t) f(t,v),y(i))
    yee(i)=t
end
fileID = fopen('dataex1.txt','w')
for i=1:5
    fprintf(fileID,'%2d %4.2f %10.8f %10.8f \ n',i,x(i),ye(i),y(i))
end
for i=5:p+4
    fprintf(fileID,'%2d %4.2f %10.8f %10.8f \ n',i,x(i),yee(i),y(i))
end
fclose(fileID)
figure
fig1=plot(x,y,'b*')
xlim([1,10]);
ylim([1,6]);
xlabel('x-axis')
ylabel('y-axis')
hold on
plot(xe,ye,'ro')
hold on
fimplicit(f,[5 10 1 6],"Color",'r')
legend('computed solution','exact solution','exact solution')
```

A.2 MATLAB Code 2

The following is a MATLAB code for the Example 3.4.2.

```
clear all
q=input('enter the step size =');
alp=input('enter the value of alpha =');
a=input('enter value of x_0 = ');
b=input('enter value of x^D_0 = ');
x1(1)=a;
x2(1)=b;
xe(1)=a;
xe1(1)=a;
u(1)=b+2*a;
t(1)=0;
te(1)=0;
tlast=fix(5/q);
telast=fix(5/alp);
for i=1:tlast
    t(i+1)=t(i)+q;
end
for i=1:telast
    te(i+1)=te(i)+alp;
    u(i+1)=(1-alp*te(i))*u(i);
    xe(i+1)=alp*u(i)+(1-2*alp)*xe(i);
end
for i=1:tlast
    x1(i+1)=x1(i)+q*x2(i);
    x2(i+1)=x2(i)-q*((t(i)+2)*x2(i)+2*t(i)*x1(i));
end
m=q/alp;
for i=1:tlast
    p=i*m+1
```

```
    xe1(i+1)=xe(p);
end
fileID = fopen('dataeulerex2.txt','w')
for i=1:tlast
    fprintf(fileID,'%4.2f %10.8f %10.8f \ n',t(i),xe1(i),x1(i))
end
fclose(fileID)
figure
fig1=plot(t,x1,'b*')
xlim([0,5]);
xlabel('t-axis')
ylabel('x-axis')
hold on
plot(t,xe1,'ro')
legend('computed solution','exact solution')
```

A.3 MATLAB Code 3

The following is a MATLAB code for the Example 3.4.3.

```
clear all
q=input('enter the step size =');
alp=input('enter the value of alpha =')
a=input('enter value of t_0 = ');
b=input('enter value of x_0 = ');
t(1)=a
x(1)=b
tlast=fix(10/q)
for i=1:tlast
    t(i+1)=t(i)+q;
end
for i=1:tlast
    x(i+1)=x(i)+q*((t(i)/(t(i)^2+1))+1/(1+x(i)^2));
```

```
end
telast=fix(10/alp)
xe(1)= b;
te(1)=a
for i=1:telast
  te(i+1)=te(i)+alp;
  xe(i+1)=xe(i)+alp*((te(i)/(te(i)^2+1))+1/(1+xe(i)^2));
end
xe1(1)=b
m=q/alp
for i=1:tlast+1
    p=(i-1)*m+1
    xe1(i)=xe(p);
end
fileID = fopen('dataeuler3.txt','w')
for i=1:tlast
    fprintf(fileID,'%4.2f %10.8f %10.8f \ n',t(i),xe1(i),x(i))
end
fclose(fileID)
figure
fig1=plot(t,x,'b*')
xlim([0,10]);
xlabel('t-axis')
ylabel('x-axis')
hold on
plot(t,xe1,'ro')
legend('computed solution','exact solution')
```

A.4 MATLAB Code 5

The following is a MATLAB code for the Example 4.5.1, Case 1.

```
clear all
```

```
q=input('enter the step size =');
alp=input('enter the value of alpha =')
a=input('enter value of t_0 = ');
b=input('enter value of x_0 = ');
t(1)=a
x(1)=b
xt(1)=b
tlast=fix(10/q)
for i=1:tlast
    t(i+1)=t(i)+q;
end
for i=1:tlast
    bc(i)=2*t(i)+1/(1+x(i)^2);
    r(i)=1/(1+x(i)^2);
    xs(i)=2+(1/(alp*bc(i)))*(1/(1+(x(i)+alp*bc(i))^2)-r(i))*bc(i);
    x(i+1)=x(i)+q*bc(i)+(q*(q-alp)/2)*xs(i);
end
for i=1:tlast
    s(1)=xt(i);
    c=xt(i)+(q+alp)*t(i)+0.5*(q+alp)*(1/(1+xt(i)*xt(i)))+(q-alp)*t(i+1);
    for j=1:150
        d1=2*(s(j)^3)-2*c*(s(j)^2)+2*s(j)-2*c-q+alp;
        d2=6*(s(j)^2)-4*c*s(j)+2;
        s(j+1)=s(j)-(d1/d2);
    end
    xt(i+1)=s(151);
end
telast=fix(10/alp)
xe(1)= b;
te(1)=a
for i=1:telast
  te(i+1)=te(i)+alp;
  xe(i+1)=xe(i)+alp*(2*te(i)+1/(1+xe(i)^2));
```

```
end
xe1(1)=b
s=q/alp
for i=1:tlast+1
    l=(i-1)*s+1
    xe1(i)=xe(l);
end
fileID = fopen('datataylorex1.txt','w')
for i=1:tlast
    fprintf(fileID,'%4.2f & %10.8f & %10.8f & %10.8f \  n',

t(i),xe1(i),x(i),xt(i))
end
fclose(fileID)
figure
fig1=plot(t,x,'b*')
xlim([0,10]);
xlabel('t-axis')
ylabel('x-axis')
hold on
plot(t,xe1,'ro')
legend('Taylor series solution','Exact solution')
```

A.5 MATLAB Code 6

The following is a MATLAB code for the Example 4.5.1, Case 2.

```
clear all
q=input('enter the step size =');
alp=input('enter the value of alpha =')
a=input('enter value of t_0 = ');
b=input('enter value of x_0 = ');
t(1)=a
```

```
x(1)=b
xt(1)=b
tlast=fix(10/q)
for i=1:tlast
    t(i+1)=t(i)+q;
end
for i=1:tlast
    bc(i)=(t(i)/(t(i)^2+1))+1/(1+x(i)^2);
    r(i)=1/(1+x(i)^2);
    v=t(i)^2+t(i)*alp-1;
    w=(t(i)^2+1)*((t(i)+alp)^2+1);
    xs(i)=-(v/w)+(1/(alp*bc(i)))*(1/(1+(x(i)+alp*bc(i))^2)-r(i))*bc(i);
    x(i+1)=x(i)+q*bc(i)+(q*(q-alp)/2)*xs(i);
end
for i=1:tlast
    s(1)=xt(i);
    c=xt(i)+0.5*(q+alp)*(t(i)/(t(i)*t(i)+1))...

+0.5*(q+alp)*(1/(1+xt(i)*xt(i)));
    c1=0.5*(q-alp)*(t(i+1)/(t(i+1)*t(i+1)+1));
    for j=1:150
        d1=2*(s(j)^3)-2*(c+c1)*(s(j)^2)+2*s(j)-2*(c+c1)-q+alp;
        d2=6*(s(j)^2)-4*(c+c1)*s(j)+2;
        s(j+1)=s(j)-(d1/d2);
    end
    xt(i+1)=s(151);
end
telast=fix(10/alp)
xe(1)= b;
te(1)=a
for i=1:telast
  te(i+1)=te(i)+alp;
  xe(i+1)=xe(i)+alp*(te(i)/(te(i)*te(i)+1)+1/(1+xe(i)*xe(i)));
```

```
end
xe1(1)=b
s=q/alp
for i=1:tlast+1
    l=(i-1)*s+1
    xe1(i)=xe(l);
end
fileID = fopen('datataylorex4.txt','w')
for i=1:tlast
    fprintf(fileID,'%4.2f & %10.8f & %10.8f & %10.8f \ n',

t(i),xe1(i),x(i),xt(i))
end
fclose(fileID)
figure
fig1=plot(t,x,'b*')
xlim([0,10]);
xlabel('t-axis')
ylabel('x-axis')
hold on
plot(t,xe1,'ro')
legend('Taylor series solution','Exact solution')
```

A.6 MATLAB Code 7

The following is a MATLAB code for the Example 4.5.2.

```
clear all
q=input('enter the step size =');
alp=input('enter the value of alpha =');
a=input('enter value of x_0 = ');
b=input('enter value of x^D_0 = ');
x1(1)=a;
```

```
x2(1)=b;
xe(1)=a;
xe1(1)=a;
u(1)=b+2*a;
t(1)=0;
te(1)=0;
tlast=fix(5/q);
telast=fix(5/alp);
for i=1:tlast
    t(i+1)=t(i)+q;
end
for i=1:telast
    te(i+1)=te(i)+alp;
    u(i+1)=(1-alp*te(i))*u(i);
    xe(i+1)=alp*u(i)+(1-2*alp)*xe(i);
end
for i=1:tlast
    s=1+0.5*(q-alp)*(2+t(i+1));
    r=1-0.5*(2+t(i))*(q+alp);
    r1=1/(1+((q-alp)*(q-alp)*t(i+1))/(2*s))
    x1(i+1)=r1*(x1(i)+0.5*(q+alp)*x2(i)-(t(i)*x1(i)*(q*q-alp*alp))/(2*s)...

+0.5*(q-alp)*(r/s)*x2(i));
    x2(i+1)=(1/s)*(-t(i)*(q+alp)*x1(i)+r*x2(i)-t(i+1)*x1(i+1)*(q-alp));
end
m=q/alp;
for i=1:tlast
    p=i*m+1
    xe1(i+1)=xe(p);
end
fileID = fopen('datatrapnew1.txt','w')
for i=1:tlast
    fprintf(fileID,'%4.2f %10.8f %10.8f \ n',t(i),xe1(i),x1(i))
```

```
end
fclose(fileID)
figure
fig1=plot(t,x1,'b*')
xlim([0,5]);
xlabel('t-axis')
ylabel('x-axis')
hold on
plot(t,xe1,'ro')
legend('computed solution','exact solution')
```