ON PROPERTIES OF HERMITE AND $q-$HERMITE I POLYNOMIALS AND THEIR LIMIT RELATIONS

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ABSTRACT

On properties of Hermite and $q-$Hermite I polynomials and their limit relations

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In this thesis, some important properties of the Hermite polynomials and discrete q-Hermite I polynomials are presented. Their properties will be considered in the same manner. The discrete q-Hermite I polynomials are the q-analogues of the Hermite polynomials. Such polynomials are an important class of the classical orthogonal polynomials and their q-analogues. The central idea in this thesis is to study the differential and q-difference equation of hypergeometric type, three terms recurrence relations, Rodrigues formulas, orthogonalities and generating functions that the Hermite polynomials and its discrete version have. Hermite polynomials are obtained from the discrete q-Hermite I polynomials in the limiting case as $q \to 1$. Such limit relation between the Hermite polynomials and the discrete q-Hermite I polynomials on each properties that is introduced in the thesis are considered in detailed.

Keywords: Classical ortohogonal polynomials, Hermite polynomials, q-classical orthogonal polynomials, discrete q-Hermite I polynomials, differential equation of hypergeometric type, q-difference equation of hypergeometric type, Rodrigues formula, three terms recurrence relation, generating function

ÖZ

Hermite ve $q-$Hermite I polinomlarının özellikleri ve aralarındaki limit ilişkileri üzerine

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Bu tezde Hermite polinomları ve ayrık q-Hermite I polinomlarının bazı önemli özellikleri sunulmaktadır. Bu polinomların özellikleri aynı tarzda ele alınacaktır. Ayrık q-Hermite I polinomları, Hermite polinomlarının q-analoğudur. Bu tip polinomlar klasik ortogonal polinomlar ve q-analoğunun önemli bir sınıfıdır. Bu tezdeki temel düşünce, Hermite polinomları ve bunların ayrık versiyonlarının sahip oldukları hipergeometrik tipte diferansiyel ve q-fark denklemleri, üç terimli yineleme bağıntısı, Rodrigues formülü, ortogonal ilişkileri, üreteç fonksiyon özellikleri üzerine çalışmaktır. Hermite polinomları, $q \to 1$ limit durumunda ayrık q-Hermite I polinomlarından elde edilmektedir. Bu tezde sunulan her bir özellik için Hermite polinomları ve ayrık q-Hermite I polinomları arasındaki limit ilişkisi ayrıntılı olarak ele alınacaktır.

Anahtar Kelimeler: Klasik ortogonal polinomlar, Hermite polinomları, q-klasik ortogonal polinomlar, ayrık q-Hermite I polinomları, hipergeometrik tipte diferansiyel denklem, hipergeometrik tipte q-fark denklemi, Rodrigues formülü, üç terimli yineleme bağıntısı, üreteç fonksiyon

To my mother, Fatima,

and

to the memory of my late father, Abdusalam

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CHAPTER 1

INTRODUCTION and PRELIMINARIES

1.1 Introduction

Hermite polynomials were defined by Laplace in 1810 which are not in recognizable form, and in 1859, they have been studied in detail by Chebyshev. However, in 1864, Charles Hermite studied Chebyshev's work and described the polynomials as new family.

Hermite polynomials are one of the important orthogonal family of the classical orthogonal polynomials which have enormous applications in mathematics and physics. They satisfy the following differential equation of hypergeometric type

$$y'' - 2zy' + 2ny = 0, \tag{1.1}$$

where $n \in \mathbb{N}_0$.

Discrete version of the Hermite polynomials are also important and have applications in quantum mechanics. In fact, discrete q-Hermite polynomials are one of the family of q-classical polynomials in the Hahn sense. The so-called q-polynomials have great applications in several problems on theoretical and mathematical physics, e.g., in the continued fractions, Eulerian series, [16], algebras and quantum groups [21, 22, 31], discrete mathematics, algebraic combinatorics (coding theory, design theory, various theories of group representation) [10], q-Schrödinger equation and q-harmonic oscillators [5, 6, 7, 8, 9, 11, 23, 13].

The Hermite polynomials and their q-analogues are at the bottom of a family of the classical orthogonal polynomials and their q-version, [4, 19, 20]. They contain only

q parameter, and one can get them by using a suitable limit relation from the other orthogonal polynomials.

The classical orthogonal polynomials and their q-analogues have some properties. One of the most important characteristic property of these kind of polynomials is a second-order differential equation of hypergeometric type (1.1) or their discrete version which they satisfy. Actually this important characteristic property is considered by Routh in 1885 [28], and by Bochner in 1929 [12].

Another useful property of these polynomials was developed by Sonine in 1887 and by Hahn in 1939. They claimed the following theorem:

Theorem 1.1.1 (Sonine-Hahn [3, 18, 24]) *A given sequence of orthogonal polynomials $(P_n)_n$ is a classical sequence if and only if the sequence of its derivatives $(P'_n)_n$ is an orthogonal polynomial sequence.*

Rodrigues formula is another characterization for the orthogonal polynomials which is derived by Tricomi [30] and Cryer [15]. The Rodrigues formula provides explicit representation for the classical polynomials.

Three term recurrence relation (TTRR) is another way to introduce the classical orthogonal polynomials and their discrete version. Chihara [14] and Szegö [29] have studied them by using the TTRR.

Another practical characteristics is the generating function which was first introduced by Abraham de Moivre in 1730. It provides a way to get the polynomials as a part of the coefficients in a formal series [19, 20].

An important characteristic property other than the differential equation is the orthogonality property with respect to a suitable inner product [25, 26, 27].

The central idea in this thesis is to study these important properties of the Hermite polynomials and Discrete q-Hermite I polynomials. Their properties will be considered in the same manner.

Discrete q-Hermite I polynomials are the q-analogues of the Hermite polynomials.

They satisfy the following q-difference equation of hypergeometric type:

$$-D_q D_{q^{-1}} y + \frac{x}{1-q} D_{q^{-1}} y + \lambda y = 0, \tag{1.2}$$

where $\lambda = -\frac{q^{1-n}}{1-q} [n]_q$ and D_q is called the q-Jackson derivative [19, 20, 25, 26, 27]. Hermite polynomials are obtained from the discrete q-Hermite I polynomials in the limiting case as $q \to 1$. We are also going to deal with this limit relation between the Hermite polynomials and the discrete q-Hermite I polynomials.

This thesis is organized as follows: In the rest of this chapter, some basic definitions related with q-calculus are established. In Chapter 2, we study some important characteristic properties, such as polynomial solutions of the differential equation, Rodrigues formula, TTRR, generating function and orthogonality relation [3, 19, 20, 25, 26, 27] of the Hermite polynomials. Chapter 3 includes the discrete version of these properties for the discrete q-Hermite I polynomials in the same manner. In Chapter 4, some known limit relations between the identified q-Hermite I polynomials and the classical Hermite polynomials [19, 20] are introduced in detailed.

1.2 Some definitions and notations

In this section, some notations that are used in q-calculus like q-integer, q-factorial, q-binomial coefficient, q-Pochhammer's symbol together with some operators like q-derivative and q-integral will be presented. Also, hypergeometric series and their q-analogues will be stated. The definitions and notations that are given here can be found in [4, 19, 20, 25, 26, 27].

Let $q > 0$. For any $n \in \mathbb{N}_0$, the q-integer $[n]_q$ is defined by

$$[n]_q := 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}, \quad n = 1, 2, \ldots, \quad [0]_q := 0 \tag{1.3}$$

and the q-factorial $[n]_q!$ by

$$[n]_q! = [1]_q [2]_q \cdots [n]_q, \quad n = 1, 2, \ldots, \quad [0]_q! := 1. \tag{1.4}$$

For integers $0 \leqslant k \leqslant n$, the q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}. \tag{1.5}$$

Clearly, when $q = 1$, one has the classical notations. That is,

$$[n]_1 = n, \quad [n]_1! = n!, \quad \begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$$

where $\binom{n}{k}$ is the binomial coefficient defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

For the sake of completeness of the definition, when $k < 0$ or $k > n$ we set $\begin{bmatrix} n \\ k \end{bmatrix}_q = 0$.

Recall the Pochhammer's symbol (or shifted factorial) defined by

$$(a)_k = a(a+1)\cdots(a+k-1) \quad \text{for} \quad k = 1, 2, \ldots, \quad (a)_0 := 1. \tag{1.6}$$

A similar notation is used to define q-Pochhammer's symbol (or q-shifted factorial):

$$(a;q)_0 := 1, \quad (a;q)_k := \prod_{s=0}^{k-1}(1 - aq^s), \quad (a;q)_\infty := \prod_{s=0}^{\infty}(1 - aq^s). \tag{1.7}$$

We first observe the following results:

Observation 1.2.1 *For any $a \in \mathbb{C}$, and nonnegative integer k,*

$$(a;q)_{k+1} = (1-a)(aq;q)_k = (a;q)_k(1 - aq^k). \tag{1.8}$$

Repeated application of this observation gives us:

Observation 1.2.2 *For any $a \in \mathbb{C}$, and integers $0 \leqslant k \leqslant n$,*

$$(a;q)_n = (a;q)_k(aq^k;q)_{n-k}. \tag{1.9}$$

Taking the limit as $n \to \infty$ in the previous result yields:

Observation 1.2.3 *For any $a \in \mathbb{C}$ and $k \in \mathbb{N}_0$, $(a;q)_\infty = (a;q)_k(aq^k;q)_\infty$.*

From the definition, we can see that:

Observation 1.2.4 *If n is a nonnegative integer, then $(q^{-n};q)_k = 0$ for all $k > n$.*

Lemma 1.2.5 *For all $k = 0, 1, \ldots, n$, one has*

$$(q^{-n}; q)_k = \frac{(q; q)_n}{(q; q)_{n-k}} (-1)^k q^{\binom{k}{2} - nk}. \tag{1.10}$$

Proof. For $0 \leqslant k \leqslant n$, we have

$$(q^{-n}; q)_k = \prod_{s=0}^{k-1} (1 - q^{-n+s}) = \prod_{s=0}^{k-1} (-q^{-n+s}) \prod_{s=0}^{k-1} (1 - q^{n-s}) = (-1)^k q^{\binom{k}{2} - nk} \prod_{s=n-k}^{n-1} \left(1 - q^{s+1}\right)$$
$$= (-1)^k q^{\binom{k}{2} - nk} \frac{\prod_{s=0}^{n-1} \left(1 - q^{1+s}\right)}{\prod_{s=0}^{n-k-1} (1 - q^{1+s})} = \frac{(q; q)_n}{(q; q)_{n-k}} (-1)^k q^{\binom{k}{2} - nk}.$$

□

Lemma 1.2.6 *For $a \in \mathbb{C}$ and $k \in \mathbb{N}_0$, we have $(a; q^2)_k (aq; q^2)_k = (a; q)_{2k}$.*

Proof.

$$(a; q^2)_k (aq; q^2)_k = \prod_{s=0}^{k-1} \left(1 - a(q^2)^s\right) \prod_{s=0}^{k-1} \left(1 - aq(q^2)^s\right) = \prod_{s=0}^{2k-1} (1 - aq^s) = (a; q)_{2k}.$$

□

A simple corollary of this result is stated below which can be obtained by taking the limit as $k \to \infty$.

Corollary 1.2.7 *For $a \in \mathbb{C}$, we have $(a; q^2)_\infty (aq; q^2)_\infty = (a; q)_\infty$.*

Lemma 1.2.8 *For $a \in \mathbb{C}$ and $k \in \mathbb{N}_0$, we have $(a; q)_k (-a; q)_k = (a^2; q^2)_k$.*

Proof.

$$(a; q)_k (-a; q)_k = \prod_{s=0}^{k-1} (1 - aq^s) \prod_{s=0}^{k-1} (1 + aq^s) = \prod_{s=0}^{k-1} \left(1 - a^2 q^{2s}\right) = (a^2; q^2)_k.$$

□

Taking the limit as $k \to \infty$, we obtain the following:

Corollary 1.2.9 *For any $a \in \mathbb{C}$, we have $(a; q)_\infty (-a; q)_\infty = (a^2; q^2)_\infty$.*

Observation 1.2.10 *The relation between q- and q^{-1}-integer is*

$$[k]_{q^{-1}} = q^{1-k}[k]_q. \tag{1.11}$$

Lemma 1.2.11 *For any integers $0 \leqslant k \leqslant n$, we have*

$$[n]_q = [k]_q + q^k[n-k]_q. \tag{1.12}$$

Proof. From the definition of $[n]_q$ given in (1.3), we write

$$[k]_q + q^k[n-k]_q = \frac{1-q^k}{1-q} + q^k\frac{1-q^{n-k}}{1-q} = \frac{1-q^n}{1-q} = [n]_q.$$

□

Recall that the binomial coefficients obey the well-known Pascal's rule. That is, for integers k and n, we have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

In what follows, we give similar results for q-binomial coefficients.

Lemma 1.2.12 *For integers k and n,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q. \tag{1.13}$$

Proof. The right hand side of this equality is

$$\begin{aligned} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q &= \frac{[n-1]_q!}{[k-1]_q![n-k]_q!} + q^k \frac{[n-1]_q!}{[k]_q[n-1-k]_q!} \\ &= \frac{[k]_q[n-1]_q! + q^k[n-k]_q[n-1]_q!}{[k]_q![n-k]_q!} \\ &= \frac{[n-1]_q!([k]_q + q^k[n-k]_q)}{[k]_q![n-k]_q!}. \end{aligned}$$

Using (1.12), one arrives at

$$\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q = \frac{[n-1]_q![n]_q}{[k]_q![n-k]_q!} = \frac{[n]_q!}{[k]_q![n-k]_q!} = \begin{bmatrix} n \\ k \end{bmatrix}_q$$

as desired. □

Since, like the classical binomial coefficient, the q-binomial coefficients satisfy the simple property $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$, the q-Pascal rule has another version, namely,

Lemma 1.2.13 *For integers k and n,*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q .$$

The following result gives the relation between the q-binomial coefficients and the q-Pochammer's symbol:

Lemma 1.2.14 *For any integer $0 \leqslant k \leqslant n$, we have*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}. \tag{1.14}$$

Proof. To begin with the proof, we first note that the q-factorial defined by (1.4) can be written in terms of q-Pochammer's symbol given in (1.7) as

$$[k]_q! = \prod_{s=1}^{k} [s]_q = \prod_{s=1}^{k} \frac{1-q^s}{1-q} = (1-q)^{-k} (q;q)_k. \tag{1.15}$$

Now,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(1-q)^{-n} (q;q)_n}{\left[(1-q)^{-k} (q;q)_k\right] \left[(1-q)^{-(n-k)} (q;q)_{n-k}\right]} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}.$$

□

Note that if q^{-1} were used instead of q, then we would have:

Lemma 1.2.15 *For integers k and n*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = (-1)^k q^{k(k+1)/2} \frac{(q^{-n};q)_k}{(q;q)_k}. \tag{1.16}$$

Proof. Using the definition, we write

$$\begin{aligned} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} &= \frac{[n]_{q^{-1}}!}{[k]_{q^{-1}}! [n-k]_{q^{-1}}!} = \prod_{s=0}^{k-1} \frac{[n-s]_{q^{-1}}}{[k-s]_{q^{-1}}} = \prod_{s=0}^{k-1} \frac{1-q^{-n+s}}{1-q^{-k+s}} \\ &= \prod_{s=0}^{k-1} \frac{1-q^{-n+s}}{-q^{-k+s}(1-q^{k-s})} = \frac{\prod_{s=0}^{k-1} \left(-q^{k-s}\right) \prod_{s=0}^{k-1} (1-q^{-n+s})}{\prod_{s=0}^{k-1} (1-q^{k-s})} \end{aligned}$$

which gives us

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = (-1)^k q^{k(k+1)/2} \frac{(q^{-n};q)_k}{(q;q)_k}$$

as desired. $\square$

The Pochammer's symbol is given in general for complex numbers. Accordingly, the definition of q-integer given by (1.3) is extended to the complex numbers and we have, for $q \neq 1$,

$$[\alpha]_q := \frac{1-q^\alpha}{1-q}, \quad \alpha \in \mathbb{C}.$$

The limit relation between the Pochammer's symbol and the q-Pochammer's symbol is given by the following lemma:

Lemma 1.2.16 *For $\alpha \in \mathbb{C}$, and positive integer k, we have*

$$\lim_{q\to 1} \frac{(q^\alpha;q)_k}{(1-q)^k} = (\alpha)_k. \tag{1.17}$$

Proof. First, we note that

$$\frac{(q^\alpha;q)_k}{(1-q)^k} = \prod_{s=0}^{k-1} \frac{1-q^{\alpha+s}}{1-q} = \prod_{s=0}^{k-1} [\alpha+s]_q = [\alpha]_q[\alpha+1]_q \cdots [\alpha+k-1]_q.$$

Now, since $\lim_{q\to 1} [m]_q = m$, taking the limit of both sides as $q \to 1$ in the above expression, we get the stated result. $\square$

Next, we give the definition of the q-gamma function [4]:

Definition 1.2.17 *The q-gamma function is defined by*

$$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty} (1-q)^{1-x}. \tag{1.18}$$

This is a q-analogue of the gamma function since we have

$$\lim_{q\to 1} \Gamma_q(x) = \Gamma(x). \tag{1.19}$$

Note that,

$$\Gamma_q(1) = 1, \qquad \Gamma_q(x+1) = [x]_q \Gamma_q(x).$$

1.3 q-Derivative

Next, we provide the q-derivative of a function when $q \neq 1$.

Definition 1.3.1 (q-Derivative) *The q-derivative of a function f, for $q \neq 1$, is defined as*

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x}, & x \neq 0 \\ f'(0), & x = 0 \end{cases} \tag{1.20}$$

Observe that if $f(x)$ is differentiable, then

$$\lim_{q \to 1} D_q f(x) = \frac{df(x)}{dx}.$$

Example 1.3.2 *As our first example, let us find the q-derivative of a constant function. If $f(x) = c$, then from* (1.20)*,*

$$D_q c = \frac{c - c}{(1-q)x} = 0.$$

That is, the q-derivative of a constant function is identically equal to zero as in the case of classical derivative.

Example 1.3.3 *As another example, let us find the q-derivative of monomials. If $f(x) = x^n$, where n is a positive integer, then, for $x \neq 0$,* (1.20) *gives us*

$$D_q x^n = \frac{x^n - (qx)^n}{(1-q)x} = \frac{1-q^n}{1-q} x^{n-1}.$$

Using the notation (1.3)*, this is written as*

$$D_q x^n = [n]_q x^{n-1}. \tag{1.21}$$

Since $f(x) = x^n$ is differentiable and $f'(x) = nx^{n-1}$, the identity (1.21) is analogous to the classical one.

The simplest property of the q-derivative is its linearity as given in the next lemma.

Lemma 1.3.4 *For functions f, g and constants C_1, C_2, we have*

$$D_q[C_1 f(x) + C_2 g(x)] = C_1 D_q f(x) + C_2 D_g(x).$$

We would like to note here that, when α is a nonzero scalar, the q-derivative of $f(x)$ at αx and the q-derivative of $f(\alpha x)$ are not the same. In fact, we have

$$D_q f(\alpha x) = \frac{f(\alpha x) - f(q\alpha x)}{(1-q)x} = \alpha \, \frac{f(\alpha x) - f(q\alpha x)}{(1-q)\alpha x} = \alpha \, D_q f(t)\Big|_{t=\alpha x}.$$

Equivalently,

$$D_q f(t)\Big|_{t=\alpha x} = \alpha^{-1} D_q f(\alpha x). \tag{1.22}$$

Generalizing (1.22), one can see that

$$D_q^n f(t)\Big|_{t=\alpha x} = \alpha^{-n} D_q f(\alpha x). \tag{1.23}$$

Also, the q-derivative of $f(t)$ at $t = q^{-1}x$ and the q-derivative of $f(q^{-1}x)$ are not the same. In fact, we have

$$D_q f(q^{-1}x) = \frac{f(q^{-1}x) - f(x)}{(1-q)x} = \frac{f(q^{-1}x) - f(x)}{q(q^{-1}-1)x} = q^{-1} D_{q^{-1}} f(x). \tag{1.24}$$

Similarly,

$$D_{q-1} f(qx) = \frac{f(qx) - f(x)}{(1-q^{-1})x} = \frac{f(qx) - f(x)}{q^{-1}(q-1)x} = q D_q f(x). \tag{1.25}$$

However, the q-derivative of f at $q^{-1}x$ is the q^{-1}-derivative of f at x. More precisely, for $\alpha = q^{-1}$ in (1.22), we get

$$D_q f(t)\Big|_{t=q^{-1}x} = q D_q f(q^{-1}x) = D_{q^{-1}} f(x). \tag{1.26}$$

Similarly,

$$D_{q^{-1}} f(t)\Big|_{t=qx} = D_q f(x). \tag{1.27}$$

The higher order q-derivative of f is defined in a similar way that the higher order derivative in classical sense is defined. The nth order q-derivative of f is

$$D_q^n f(x) = D_q(D_q^{n-1} f(x)), \quad D_q^0 f(x) := f(x).$$

There is a nice relation between $D_q D_{q^{-1}}$ and $D_{q^{-1}} D_q$ as stated in the following lemma.

Lemma 1.3.5 *For any $q \neq 1$, the following relation holds:*

$$D_q D_{q^{-1}} = q^{-1} D_{q^{-1}} D_q. \tag{1.28}$$

Proof. For any f we have

$$D_q D_{q^{-1}} f(x) = \frac{D_{q^{-1}} f(x) - D_{q^{-1}} f(t)\big|_{t=qx}}{(1-q)x}.$$

Using (1.26) and (1.27) we get

$$D_q D_{q^{-1}} f(x) = \frac{D_q f(t)\big|_{t=q^{-1}x} - D_q f(x)}{(1-q)x} = q^{-1} D_{q^{-1}} D_q f(x)$$

which shows the validity of the claimed identity. □

The next assertion gives chance to express $D_{q^{-1}}$ in terms of D_q and $D_q D_{q^{-1}}$.

Lemma 1.3.6 *The following identity is true:*

$$D_{q^{-1}} = D_q + (1-q)x D_q D_{q^{-1}} \tag{1.29}$$

Proof. As in the proof of the previous theorem, we have

$$D_q D_{q^{-1}} f(x) = \frac{D_{q^{-1}} f(x) - D_{q^{-1}} f(t)\big|_{t=qx}}{(1-q)x} = \frac{D_{q^{-1}} f(x) - D_q f(x)}{(1-q)x}.$$

Therefore, $D_{q^{-1}} = D_q + (1-q)x D_q D_{q^{-1}}$. □

In the next lemma, the q-derivative for the product of two functions is given.

Lemma 1.3.7 (Product rule) *For functions f and g, we have*

$$D_q(f(x)g(x)) = f(x) D_q g(x) + g(qx) D_q f(x). \tag{1.30}$$

Proof. From the definition, one has

$$D_q(f(x)g(x)) = \frac{f(x)g(x) - f(qx)g(qx)}{(1-q)x}$$

Adding and subtracting the term $f(x)g(qx)$ in the numerator, we write

$$\begin{aligned} D_q(f(x)g(x)) &= \frac{f(x)g(x) - f(x)g(qx) + f(x)g(qx) - f(qx)g(qx)}{(1-q)x} \\ &= f(x)\frac{g(x) - g(qx)}{(1-q)x} + g(qx)\frac{f(x) - g(qx)}{(1-q)x} \\ &= f(x) D_q g(x) + g(qx) D_q f(x) \end{aligned}$$

as required. $\square$

Changing the roles of f and g, the rule given in (1.30) can also be written as

$$D_q(f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x).$$

The higher order q-derivative of the product fg, known as the Leibniz rule, is provided next.

Lemma 1.3.8 *For functions f and g, one has*

$$D_q^n(f(x)g(x)) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q D_q^k f(x)\, D_q^{n-k} g(t)\Big|_{t=q^k x}. \tag{1.31}$$

Proof. The proof is based on induction on n. For $n = 1$, the right side of (1.31) reads as

$$\sum_{k=0}^{1} \begin{bmatrix} 1 \\ k \end{bmatrix}_q D_q^k f(x)\, D_q^{1-k} g(t)\Big|_{t=q^k x} = f(x)D_q g(x) + D_q f(x) g(qx)$$

which is nothing but $D_q(f(x)g(x))$. Suppose that the hypothesis is true for some $n-1$, that is,

$$D_q^{n-1}(f(x)g(x)) = \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q D_q^k f(x)\, D_q^{n-1-k} g(t)\Big|_{t=q^k x}.$$

Applying D_q to both sides of this identity, by the linearity of D_q and the product rule, we get

$$\begin{aligned} D_q^n(f(x)g(x)) &= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q D_q \left(D_q^k f(x)\, D_q^{n-1-k} g(t)\Big|_{t=q^k x} \right) \\ &= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left\{ D_q^{k+1} f(x)\, D_q^{n-1-k} g(t)\Big|_{t=q^{k+1} x} + D_q^k f(x)\, D_q \left(D_q^{n-1-k} g(t)\Big|_{t=q^k x} \right) \right\}. \end{aligned}$$

Using (1.22) in the right most term leads to

$$\begin{aligned} D_q^n(f(x)g(x)) &= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \left\{ D_q^{k+1} f(x)\, D_q^{n-1-k} g(t)\Big|_{t=q^{k+1} x} + D_q^k f(x)\, D_q^{n-k} g(t)\Big|_{t=q^k x} q^k \right\} \\ &= \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q D_q^{k+1} f(x)\, D_q^{n-1-k} g(t)\Big|_{t=q^{k+1} x} \\ &\quad + \sum_{k=0}^{n-1} q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q D_q^k f(x) D_q^{n-k} g(t)\Big|_{t=q^k x}. \end{aligned}$$

Replacing k by $k-1$ in the first sum on the right side, we get

$$D_q^n(f(x)g(x)) = \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q D_q^k f(x)\, D_q^{n-k} g(t)\big|_{t=q^k x} + \sum_{k=0}^{n-1} q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q D_q^k f(x) D_q^{n-k} g(t)\big|_{t=q^k x}.$$

Since the first sum vanishes for $k = 0$ and the second one vanishes for $k = n$, we can write both of them as a sum over k from 0 to n. Then combining the two sums we obtain

$$D_q^n(f(x)g(x)) = \sum_{k=0}^{n} \left(\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \right) D_q^k f(x) D_q^{n-k} g(t)\big|_{t=q^k x}.$$

Now, with the help of (1.13) we complete induction. This proves the lemma. □

As a final property of the q-derivatives, we present the q-quotient rule.

Lemma 1.3.9 (Quotient rule) *For functions f and g,*

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)}.$$

Proof. As before, we start with the definition:

$$D_q\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{f(x)}{g(x)} - \frac{f(qx)}{g(qx)}}{(1-q)x} = \frac{f(x)g(qx) - f(qx)g(x)}{(1-q)xg(x)g(qx)}.$$

Now, we add and subtract the term $f(x)g(x)$ in numerator to get

$$\begin{aligned} D_q\left(\frac{f(x)}{g(x)}\right) &= \frac{f(x)g(qx) - f(x)g(x) + f(x)g(x) - f(qx)g(x)}{(1-q)xg(x)g(qx)} \\ &= \frac{-f(x)D_q g(x) + g(x)D_q f(x)}{g(x)g(qx)} \\ &= \frac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)} \end{aligned}$$

as stated. □

We now give and proof the q-binomial theorem (cf. [4, Theorem 10.2.1]).

Theorem 1.3.10 *For $|x| < 1$, $|q| < 1$,*

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k = \frac{(ax;q)_\infty}{(x;q)_\infty}. \tag{1.32}$$

Proof. There are several proofs of this theorem. Here, we present one of them. Let

$$f_a(x) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k.$$

Then, taking the q-derivative of both sides and using (1.8) and (1.21), we get

$$\begin{aligned} D_q f_a(x) &= \sum_{k=1}^{\infty} \frac{(a;q)_k}{(q;q)_k} [k]_q x^{k-1} \\ &= \sum_{k=1}^{\infty} \frac{(a;q)_k}{(q;q)_{k-1}} \frac{x^{k-1}}{1-q} \\ &= \frac{1-a}{1-q} \sum_{k=0}^{\infty} \frac{(aq;q)_k}{(q;q)_k} x^k \\ &= \frac{1-a}{1-q} f_{aq}(x) \end{aligned}$$

which gives us

$$f_a(x) - f_a(qx) = (1-a)x f_{aq}(x). \tag{1.33}$$

On the other hand,

$$f_a(x) - f_{aq}(x) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} x^k - \sum_{k=0}^{\infty} \frac{(aq;q)_k}{(q;q)_k} x^k = \sum_{k=1}^{\infty} \frac{(a;q)_k - (aq;q)_k}{(q;q)_k} x^k$$

By the use of (1.8) again, the last equality becomes

$$\begin{aligned} f_a(x) - f_{aq}(x) &= \sum_{k=0}^{\infty} \frac{(a;q)_{k+1} - (aq;q)_{k+1}}{(q;q)_{k+1}} x^{k+1} \\ &= \sum_{k=0}^{\infty} \frac{(1-a)(aq;q)_k - (aq;q)_k(1-aq^{k+1})}{(q;q)_{k+1}} x^{k+1} \\ &= \sum_{k=0}^{\infty} \frac{(aq;q)_k(-a+aq^{k+1})}{(q;q)_{k+1}} x^{k+1} \\ &= -ax \sum_{k=0}^{\infty} \frac{(aq;q)_k(1-q^{k+1})}{(q;q)_{k+1}} x^k \\ &= -ax \sum_{k=0}^{\infty} \frac{(aq;q)_k}{(q;q)_k} x^k = -ax f_{aq}(x) \end{aligned}$$

or

$$f_a(x) = (1-ax) f_{aq}(x). \tag{1.34}$$

Eliminating $f_{aq}(x)$ from (1.33) and (1.34), we obtain

$$f_a(x) = \frac{1-ax}{1-x} f_a(qx).$$

Now, iterating this n times gives us

$$f_a(x) = \frac{1-ax}{1-x} f_a(qx) = \frac{(1-ax)(1-aqx)}{(1-x)(1-qx)} f_a(q^2 x) = \frac{(ax;q)_n}{(x;q)_n} f_a(q^n x).$$

Taking the limit as $n \to \infty$ and noting that $f_a(0) = 1$, we end up with

$$f_a(x) = \frac{(ax;q)_\infty}{(x;q)_\infty} f_a(0) = \frac{(ax;q)_\infty}{(x;q)_\infty}$$

which completes the proof. □

Some immediate consequences of this theorem are stated below:

Corollary 1.3.11 (Euler) *For* $|x| < 1, |q| < 1$

$$\sum_{k=0}^{\infty} \frac{x^k}{(q;q)_k} = \frac{1}{(x;q)_\infty}. \tag{1.35}$$

Proof. Put $a = 0$ in (1.32). □

Corollary 1.3.12 (Euler) *For* $|q| < 1$

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}} x^k}{(q;q)_k} = (x;q)_\infty. \tag{1.36}$$

Proof. Replace a with $1/a$, and x with ax in (1.32) to get

$$\frac{(x;q)_\infty}{(ax;q)_\infty} = \sum_{k=0}^{\infty} \frac{(1/a;q)_k}{(q;q)_k} (ax)^k = \sum_{k=0}^{\infty} \frac{(a-1)(a-q)\cdots(a-q^{k-1})}{(q;q)_k} x^k.$$

For $a = 0$, this becomes

$$(x;q)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} x^k,$$

as claimed. □

Corollary 1.3.13 (Rothe)

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (-x)^k = (x;q)_n. \tag{1.37}$$

Proof. Let $a = q^{-n}$ and replace x with $q^n x$ in (1.32) to get

$$\sum_{k=0}^{\infty} \frac{(q^{-n};q)_k}{(q;q)_k} (q^n x)^k = \frac{(x;q)_\infty}{(q^n x;q)_\infty}.$$

Using Observations 1.2.3, 1.2.4, relations (1.10) and (1.14), this becomes

$$(x;q)_n = \sum_{k=0}^{n} \frac{(q;q)_n}{(q;q)_{n-k}(q;q)_k} (-1)^k q^{\binom{k}{2}-nk} (q^n x)^k = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} (-x)^k$$

and the conclusion follows. □

Corollary 1.3.14

$$\sum_{k=0}^{n} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k = \frac{1}{(x;q)_n}.$$

Proof. First, note from (1.9) that $(q;q)_{n+k-1} = (q;q)_{n-1}(q^n;q)_k$. Now, let $a = q^n$ in (1.32):

$$\sum_{k=0}^{\infty} \frac{(q^n;q)_k}{(q;q)_k} x^k = \frac{(q^n x;q)_\infty}{(x;q)_\infty}.$$

By the use of the Observation 1.2.3 and (1.14), this becomes

$$\frac{1}{(x;q)_n} = \sum_{k=0}^{\infty} \frac{(q^n;q)_k}{(q;q)_k} x^k = \sum_{k=0}^{\infty} \frac{(q;q)_{n+k-1}}{(q;q)_k (q;q)_{n-1}} x^k = \sum_{k=0}^{n} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q x^k$$

as stated. □

Corollary 1.3.15

$$\lim_{q\to 1} \frac{1}{((1-q)x;q)_\infty} = e^x. \tag{1.38}$$

Proof. Replace x with $(1-q)x$ in (1.35) to get:

$$\frac{1}{((1-q)x;q)_\infty} = \sum_{k=0}^{\infty} \frac{(1-q)^k x^k}{(q;q)_k}.$$

Taking the limit as $q \to 1$ and using (1.17) with $\alpha = 1$, we get

$$\lim_{q\to 1} \frac{1}{((1-q)x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{(1)_k} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

which completes the proof. □

1.4 q-Integral

In this section, the q-integral will be given as an antiderivate. Before defining the q-integral let us observe the following result.

Lemma 1.4.1 *Let f be a function such that*

$$F(x) = (1-q)x\sum_{k=0}^{\infty} q^k f(q^k x)$$

is defined. Then, $D_qF(x) = f(x)$.

Proof. From the definition of q-derivative, we write

$$D_qF(x) = \frac{F(x)-F(qx)}{(1-q)x} = \sum_{k=0}^{\infty} q^k f(q^k x) - \sum_{k=0}^{\infty} q^{k+1} f(q^{k+1}x).$$

Shifting the index in the second sum, we get

$$D_qF(x) = \sum_{k=0}^{\infty} q^k f(q^k x) - \sum_{k=1}^{\infty} q^k f(q^k x) = f(x)$$

and the proof is complete. □

Since $D_qF(x) = f(x)$, we are now in a position to define the definite q-integral of f over the interval $[0, x]$ to be $F(x)$ by nothing that $F(0) = 0$. In fact, this will be the Fundamental Theorem of q-calculus analogues to the classical case. The definition of the q-integral is given below:

Definition 1.4.2 *The q-integral of f is defined as*

$$\int_0^a f(x)\, d_qx = (1-q)a\sum_{k=0}^{\infty} q^k f(q^k a) \tag{1.39}$$

for $a \geqslant 0$.

In fact, when $f(x) \geqslant 0$ for $x \in [0, a]$, the right side of (1.39) is nothing but the Riemann sum as the sum of the areas under the graph of f on the interval $[0, a]$ with the base on the interval $[q^{k+1}a, q^k a]$ and the height $f(q^k a)$.

When $a < 0$, the q-integral, on the interval $[a, 0]$, is defined as

$$\int_a^0 f(x)\, d_q x = \int_0^{-a} f(-x) d_q x.$$

For $0 < a < b$, we have

$$\int_a^b f(x)\, d_q x = \int_0^b f(x)\, d_q x - \int_0^a f(x)\, d_q x,$$

and for $a < 0 < b$, we have

$$\int_a^b f(x)\, d_q x = \int_a^0 f(x)\, d_q x + \int_0^b f(x)\, d_q x.$$

Lemma 1.4.3 *Let f be a continuous function such that*

$$\frac{f(qx)}{f(x)} = a(x)$$

for some function $a(x)$ and suppose that $\lim_{x\to 0} f(x) = f(0)$ exists. Then,

$$f(x) = f(0) \exp\left[\int_0^x \frac{1}{(q-1)t} \ln(a(t)) d_q t\right]. \tag{1.40}$$

Proof. Observe that

$$\frac{1}{(q-1)t} \ln\left(\frac{f(qt)}{f(t)}\right) = \frac{\ln f(qt) - \ln f(t)}{(q-1)t} = D_q \ln f(t).$$

Therefore,

$$\begin{aligned}\int_0^x \frac{1}{(q-1)t} \ln a(t)\, d_q t &= \int_0^x \frac{1}{(q-1)t} \ln\left(\frac{f(qt)}{f(t)}\right) d_q t \\ &= \int_0^x D_q \ln f(t)\, d_q t \\ &= \ln f(t)\Big|_0^x = \ln f(x) - \ln f(0).\end{aligned}$$

Alternatively, we can use the definition of q-integral (1.39):

$$\begin{aligned}\int_0^x \frac{1}{(q-1)t} \ln[a(t)] d_q t &= \int_0^x \frac{1}{(q-1)t} \ln\left(\frac{f(qt)}{f(t)}\right) d_q t \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \left[(1-q)xq^k \frac{1}{(q-1)q^k x} \ln \frac{f(q^{k+1}x)}{f(q^k x)}\right] \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} [\ln f(q^k x) - \ln f(q^{k+1}x)]\end{aligned}$$

$$= \lim_{n\to\infty} \left(\ln f(x) - \ln f(q^{n+1}x)\right)$$

$$= \ln f(x) - \ln f(0)$$

which yields

$$f(x) = f(0) \exp\left[\int_o^x \frac{1}{(q-1)t} \ln a(t)\, d_q t\right]$$

as desired. □

The next lemma states the integration by parts formula for the q-integrals:

Lemma 1.4.4 (Integration By Parts) *For functions f and g on an interval $[a, b]$ we have*

$$\int_a^b f(x) D_q g(x) d_q x = f(x)g(x)\Big|_a^b - \int_a^b g(qx) D_q f(x) d_q x. \tag{1.41}$$

Proof. The relation (1.41) follows immediately by taking the q-integral on both sides of (1.30) over the interval $[a, b]$. □

As we are going to deal with the orthogonality of polynomials in the later chapters, an inner product is going to be needed. So, for continuous functions f and g defined on an interval $[a, b]$ (not necessarily bounded and closed), the inner product of f and g with respect to a weight function $w(x)$ is given as

$$\langle f, g\rangle = \int_a^b w(x) f(x) g(x)\, dx = 0. \tag{1.42}$$

For the discrete case, we have the inner product

$$\langle f, g\rangle_q = \int_a^b w(x) f(x) g(x)\, d_q x = 0. \tag{1.43}$$

1.5 q-Hypergeometric functions

Recall that the hypergeometric series ${}_rF_s$ is defined by

$${}_rF_s\left(\begin{matrix} a_1, a_2, \ldots a_r \\ b_1, b_2, \ldots b_s \end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r)_k}{(b_1, b_2, \ldots, b_s)_k} \frac{z^k}{k!}$$

where $(a_1, a_2, \ldots, a_r)_k = (a_1)_k(a_2)_k \cdots (a_r)_k$, $(a_i)_k$ is the Pochammer's symbol given in (1.6). Here, $a_1, \ldots, a_r$ are parameters of the first kind (appearing in the numerator), $b_1, \ldots, b_s$ are parameters of the second kind (appearing in denominator) such that $(b_i)_k$ is never zero, and z is the independent variable.

As it can be observed from the definition of Pochammer's symbol, when n is a nonnegative integer, we have $(-n)_k = 0$ for $k > n$. Therefore, none of the parameters of the second kind is allowed to be a nonnegative integer. Moreover, if one of the parameters of the first kind is a nonnegative integer $-n$, then the hypergeometric series terminates and becomes a polynomial of degree n. In all other cases, it can be shown that the radius of convergence R of the series is

$$R = \begin{cases} \infty, & r < s + 1 \\ 1, & r = s + 1 \\ 0, & r > s + 1. \end{cases}$$

When the hypergeometric series converges, it converges to a function called hypergeometric function.

In a similar fashion, a q-hypergeometric function ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}\middle| q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_s; q)_k} \left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r} \frac{z^k}{(q;q)_k} \tag{1.44}$$

where $(a_1, a_2, \ldots, a_r; q)_k = (a_1; q)_k \cdots (a_r; q)_k$ and $(a_i; q)_k$ is the q-Pochammer's symbol given in (1.7). Again the parameters of the second argument are taken in such a way that the series is well defined. Observing the q-Pochammer's symbol, one can see that when n is a positive integer, we have $(q^{-n}; q)_k = 0$ for $k > n$. Thus, none of the arguments of the second kind is allowed to be of the form q^{-n} for any positive integer n. On the other hand, if one of the parameters of the first kind equals q^{-n} for some positive integer n, then the q-hypergeometric series terminates and becomes a polynomial of degree n in z. For any other case, the radius of convergence R of the series is

$$R = \begin{cases} \infty, & r < s + 1 \\ 1, & r = s + 1 \\ 0, & r > s + 1, \end{cases}$$

as for the hypergeometric series. Note that when $r = s + 1$, the series is

$$ {}_r\phi_s\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}\middle| q; z\right) = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_k}{(b_1, b_2, \ldots, b_s; q)_k} \frac{z^k}{(q; q)_k}. $$

CHAPTER 2

HERMITE POLYNOMIALS

2.1 Introduction

A second order linear differential equation of the form

$$\sigma(z)y'' + \tau(z)y' + \lambda y = 0, \tag{2.1}$$

where $\sigma(z)$ and $\tau(z)$ are polynomials of degree at most two and one respectively, and λ is a constant, is referred to as *an equation of hypergeometric type*. A solution of the equation (2.1) is called *a function of hypergeometric type*. This equation and its solutions are classified according to the degree of $\sigma(z)$.

In this thesis, we are going to take $\sigma(z) = 1$ in which case we have

$$y'' + (az + b)y' + \lambda y = 0, \tag{2.2}$$

where a, b and λ are constants. Note that when $a = 0$, the equation (2.3) becomes a constant coefficient differential equation whose solutions are already known in terms of elementary functions. For this reason, we shall omit this case, and focus on the case when $a \neq 0$. In the latter case, it can be shown that under some suitable transformation of the independent variable, it is possible to reduce (2.2) to the so-called normal form

$$y'' - 2zy' + \lambda y = 0, \quad \lambda \in \mathbb{C} \tag{2.3}$$

which is known as the Hermite differential equation. Our first observation is that all derivatives of a solution of a Hermite differential equation are also solutions of a Hermite differential equation.

Theorem 2.1.1 *If y is a solution of* (2.3)*, then all derivatives of y are also solutions of a differential equation of the form* (2.3).

Proof. Take the derivative on both sides of (2.3) with respect to z, to get

$$y''' - 2y' - 2zy'' + \lambda y' = 0.$$

Let $v_1 := y'$ and $\mu_1 := \lambda - 2$. Then,

$$v_1'' - 2v_1' + \mu_1 v_1 = 0$$

which means that v_1 is also a solution of a Hermite differential equation. For the purpose of induction, assume that $v_k := y^{(k)}$ satisfies

$$v_k'' - 2zv_k'(z) + \mu_k v_k = 0$$

for some non-negative integer k. As before, taking the derivative of both sides yields

$$v_k''' - 2zv_k'' + (\mu_k - 2)v_k = 0.$$

Thus, $v_{k+1} = y^{(k+1)} = v_k'$ satisfies the equation

$$v_{k+1}'' - 2zv_{k+1}' + \mu_{k+1}v_{k+1} = 0$$

where $\mu_{k+1} = \mu_k - 2$. So, by mathematical induction, $v_n(z) := y^{(n)}(z)$ satisfies the equation

$$v_n''(z) - 2zv_n'(z) + \mu_n v_n = 0, \tag{2.4}$$

for all $n = 0, 1, \ldots$ where $v_0(z) := y(z)$, $\mu_0 := \lambda$, and

$$\mu_n = \mu_{n-1} - 2 = \lambda - 2n. \tag{2.5}$$

This completes the proof. □

Theorem 2.1.2 *If $\lambda = \lambda_n := 2n$, then the Hermite differential equation has a polynomial solution of degree n for all $n = 0, 1, \ldots$*

Proof. If $\lambda = 2n$, then from (2.5) we get $\mu_n = 0$, and (2.4) becomes

$$v_n''(z) - 2zv_n'(z) = 0.$$

Multiplying this equation by e^{-z^2}, we obtain

$$\frac{d}{dz}\left(e^{-z^2} v_n'(z)\right) = 0$$

which gives us

$$v_n'(z) = c_1 e^{z^2}.$$

Integrating both sides

$$v_n(z) = c_1 \int_0^z e^{t^2} dt + c_2.$$

Therefore, one solution of $v_n'' - 2zv_n' = 0$ is constant and the other solution is $\int_0^z e^{t^2}\, dt$. Since $v_n(z) = y^{(n)}(z)$, for the constant solution $y^{(n)}(z) =$ constant, we see that y is a polynomial of degree n. □

Definition 2.1.3 *For each $n = 0, 1, \ldots,$ the polynomial solution of $y'' - 2zy' + 2ny = 0$ with leading coefficient 2^n is called Hermite polynomial of degree n and it is denoted by $H_n(z)$. That is,*

$$H_n(z) = 2^n z^n + \cdots \tag{2.6}$$

where the dots denote the lower degree terms, and they satisfy

$$H_n''(z) - 2zH_n'(z) + 2nH_n(z) = 0. \tag{2.7}$$

Remark 2.1.4 *Taking (2.4) into account, it can be seen that $v_{kn} := H_n^{(k)}(z)$ is a polynomial solution of the equation*

$$v_{kn}'' - 2zv_{kn}' + \mu_{kn} v_{kn} = 0 \tag{2.8}$$

where

$$\mu_{kn} = 2n - 2k \tag{2.9}$$

for all $k = 0, 1, \ldots, n,$ and $n = 0, 1, \ldots$

Lemma 2.1.5 *For any $n \in \mathbb{N}$, we have*

$$H_n'(z) = 2nH_{n-1}(z). \tag{2.10}$$

Proof. We know that $H_n(z)$ is a polynomial satisfying

$$H_n''(z) - 2zH_n'(z) + 2nH_n(z) = 0.$$

By taking the derivative, we see that

$$H_n'''(z) - 2zH_n''(z) + 2(n-1)H_n'(z) = 0.$$

That is, H_n' satisfies the equation

$$u'' - 2zu' + 2(n-1)u = 0.$$

Since a polynomial solution of this equation is H_{n-1}, we get $H_n' = c_n H_{n-1}$ for some constant c_n. Comparing the leading coefficients, we get $n2^n = c_n 2^{n-1}$, or $c_n = 2n$. This completes the proof. □

2.2 Weight function

Let us consider the Sturm-Liouville or formal self-adjoint form of (2.7)

$$\frac{d}{dz}\left[\rho(z)H_n'(z)\right] + 2n\rho(z)H_n(z) = 0$$

where $\rho(z)$ satisfies the separable ordinary differential equation

$$\rho'(z) = -2z\rho(z). \tag{2.11}$$

Solving (2.11), we get $\rho(z) = e^{-z^2}$, up to a constant multiple. Here, the function

$$\rho(z) = e^{-z^2} \tag{2.12}$$

is called the weight function which will be used later to show the orthogonality.

2.3 The Rodrigues formula

The Hermite polynomials, like any other polynomial of hypergeometric type, can be constructed explicitly by a celebrated formula due to Rodrigues. In this section, we shall obtain the Rodrigues formula for the Hermite polynomials. Recall from the previous section that the self-adjoint form of the Hermite differential equation is

$$\frac{d}{dz}\left[e^{-z^2}\frac{dH_n(z)}{dz}\right] + 2ne^{-z^2}H_n(z) = 0. \tag{2.13}$$

By (2.10), this equation is written as

$$\frac{d}{dz}\left[e^{-z^2}2nH_{n-1}(z)\right] + 2ne^{-z^2}H_n(z) = 0$$

or, after canceling the terms $2n$,

$$e^{-z^2}H_n(z) = -\frac{d}{dz}\left[e^{-z^2}H_{n-1}(z)\right].$$

By repeated application of the last equation, we obtain

$$\begin{aligned} e^{-z^2}H_n(z) &= (-1)^2\frac{d^2}{dz^2}\left[e^{-z^2}H_{n-2}(z)\right] \\ &= (-1)^3\frac{d^3}{dz^3}\left[e^{-z^2}H_{n-3}(z)\right] \\ &\vdots \\ &= (-1)^k\frac{d^k}{dz^k}\left[e^{-z^2}H_{n-k}(z)\right]. \end{aligned}$$

For $k = n$, we receive

$$e^{-z^2}H_n(z) = (-1)^n\frac{d^n}{dz^n}\left[e^{-z^2}H_0(z)\right].$$

By (2.6), it is known that $H_0(z) = 1$. Hence,

$$e^{-z^2}H_n(z) = (-1)^n\frac{d^n}{dz^n}e^{-z^2}$$

or

$$H_n(z) = (-1)^n e^{z^2}\frac{d^n}{dz^n}e^{-z^2}. \tag{2.14}$$

The formula (2.14) is known as the Rodrigues formula which gives representation of Hermite polynomials. Using (2.14), we can obtain the Hermite polynomials of degree 0, 1, 2 and 3, as

$$\begin{aligned} H_0(z) &= 1, \\ H_1(z) &= 2z, \\ H_2(z) &= 4z^2 - 2, \\ H_3(z) &= 8z^3 - 12z, \end{aligned}$$

respectively.

2.4 Three term recurrence relation

Using the Rodrigues formula (2.14), it is not so easy to find the Hermite polynomial $H_n(z)$ for an arbitrary value of n. In this section, we are going to find a three term recurrence relation satisfied by $H_n(z)$ which will make it easier to find $H_n(z)$ for any $n \in \mathbb{N}$.

Theorem 2.4.1 *The Hermite polynomials $H_n(z)$ satisfy the three term recurrence relation*

$$H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0, \quad n = 0, 1, \ldots \tag{2.15}$$

where $H_{-1}(z) := 0$ and $H_0(z) = 1$.

Proof. Using (2.10) in (2.7), we get

$$2nH'_{n-1}(z) - 4nzH_{n-1}(z) + 2nH_n(z) = 0,$$

and hence,

$$H'_{n-1}(z) - 2zH_{n-1}(z) + H_n(z) = 0.$$

Replacing n by $n+1$ leads to

$$H'_n(z) - 2zH_n(z) + H_{n+1}(z) = 0.$$

Using (2.10) once again will give us

$$2nH_{n-1}(z) - 2zH_n(z) + H_{n+1}(z) = 0$$

which is recurrence relation that we want. Since this recurrence relation consists of three terms, knowing two term will give the third term. $H_0(z) = 1$ is already known from (2.6) and the setting $H_{-1}(z) := 0$ is put only for convenience. □

2.5 Generating function

This section is devoted to the generating function of the form $\sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n$, for Hermite polynomials $H_n(z)$.

Theorem 2.5.1 *The Hermite polynomials $H_n(z)$ can be obtained from the generating function relation*

$$e^{2zt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n. \tag{2.16}$$

Proof. Let $G(z,t) := \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n$. Then, for $t = 0$, we get $G(z,0) = H_0(z) = 1$.

Now, taking the derivative with respect to t, we get,

$$\frac{\partial G}{\partial t} = \sum_{n=1}^{\infty} \frac{nH_n(z)}{n!} t^{n-1} = \sum_{n=1}^{\infty} \frac{H_n(z)}{(n-1)!} t^{n-1} = \sum_{n=0}^{\infty} \frac{H_{n+1}(z)}{(n)!} t^n.$$

Using (2.15), we obtain

$$\begin{aligned}
\frac{\partial G}{\partial t} &= \sum_{n=0}^{\infty} \frac{2zH_n(z) - 2nH_{n-1}(z)}{n!} t^n \\
&= 2z \sum_{n=0}^{\infty} \frac{H_n(z)}{(n)!} t^n - 2 \sum_{n=1}^{\infty} \frac{H_{n-1}(z)}{(n-1)!} t^n \\
&= 2z\, G(z,t) - 2t \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n \\
&= (2z - 2t)\, G(z,t).
\end{aligned}$$

Thus, for each fixed z, this equation can be considered as an ordinary differential with the general solution

$$G(z,t) = \exp\left(2zt - t^2\right) \cdot \phi(z)$$

for some function ϕ of z. Since $G(z,0) = 1$, we see that $\phi(z) = 1$, and hence,

$$G(z,t) = \exp\left(2zt - t^2\right)$$

which completes the proof. □

Observation 2.5.2 *For $z = 0$ in (2.16), we get*

$$\sum_{n=0}^{\infty} \frac{H_n(0)}{n!} t^n = e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}.$$

Comparing the coefficients of like powers of t, we see that

$$H_{2n+1}(0) = 0 \quad \textit{and} \quad H_{2n}(0) = \frac{(-1)^n (2n)!}{n!} \quad \textit{for all} \quad n = 0, 1, \ldots$$

Observation 2.5.3 *In* (2.16)*, replacing z and t with −z and −t, repectively, we get*

$$\sum_{n=0}^{\infty} \frac{H_n(-z)}{n!} (-t)^n = e^{2zt-t^2} = \sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n.$$

Comparing the coefficients of like powers of t, we see that

$$H_n(-z) = (-1)^n H_n(z), \quad \textit{for all} \quad n = 0, 1, 2 \ldots$$

That is, $H_n(z)$ *is an odd function if n is odd, and* $H_n(z)$ *is an even function if n is even.*

2.6 Hypergeometric representation

In this section, using the generating function for the Hermite polynomials, a hypergeometric representaion will be obtained. From (2.16) we can write

$$\sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n = e^{2zt} e^{-t^2} = \sum_{n=0}^{\infty} \frac{(2zt)^n}{n!} \sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (2z)^n}{n!\, k!} t^{n+2k}.$$

On the right side we replace n by $n - 2k$ to obtain

$$\sum_{n=0}^{\infty} \frac{H_n(z)}{n!} t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (2z)^{n-2k}}{(n-2k)!\, k!} t^n,$$

where $\lfloor x \rfloor$ denotes the floor function and equals the largest integer not exceeding x. Comparing the two sides, we get

$$H_n(z) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k n!}{(n-2k)!\, k!} (2z)^{n-2k} = (2z)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!(-z^{-2})^k}{2^{2k}(n-2k)!\, k!}. \tag{2.17}$$

To write this sum in the hypergeometric form, we note that

$$\begin{aligned}
\frac{n!}{2^{2k}(n-2k)!} &= \frac{n(n-1)\cdots(n-2k+1)}{2^{2k}} \\
&= \frac{(-n)(-n+1)\cdots(-n+2k-1)}{2^{2k}} \\
&= \left(\frac{-n}{2}\right)\left(\frac{1-n}{2}\right)\left(\frac{-n}{2}+1\right)\left(\frac{1-n}{2}+1\right)\cdots\left(\frac{-n}{2}+k-1\right)\left(\frac{1-n}{2}+k-1\right) \\
&= \left(\frac{-n}{2}\right)_k \left(\frac{1-n}{2}\right)_k.
\end{aligned}$$

Therefore, (2.17) becomes

$$H_n(z) = (2z)^n \sum_{k=0}^{\lfloor n/2 \rfloor} \left(\frac{-n}{2}\right)_k \left(\frac{1-n}{2}\right)_k \frac{(-z^{-2})^k}{k!} = (2z)^n \, {}_2F_0\left(\left.\begin{matrix} \frac{-n}{2}, \frac{1-n}{2} \\ - \end{matrix}\right| -\frac{1}{z^2}\right) \tag{2.18}$$

which is the hypergeometric representation of the Hermite polynomial $H_n(z)$ as a ${}_2F_0$.

2.7 Orthogonality

2.7.1 Orthogonality of $H_n(z)$

Recall that the formal self-adjoint form of (2.7) is

$$\frac{d}{dz}\left[e^{-z^2}H_n'(z)\right] + 2ne^{-z^2}H_n(z) = 0. \tag{2.19}$$

The same form for $H_m(z)$ would be

$$\frac{d}{dz}\left[e^{-z^2}H_m'(z)\right] + 2me^{-z^2}H_m(z) = 0. \tag{2.20}$$

Multiplying (2.19) by $H_m(z)$ and (2.20) by $H_n(z)$, and then subtracting one of the resulting equation from the other one, we get

$$2(n-m)e^{-z^2}H_n(z)H_m(z) + H_m(z)\left[e^{-z^2}H_n'(z)\right]' - H_n(z)\left[e^{-z^2}H_m'(z)\right]' = 0$$

which can be written as

$$2(n-m)e^{-z^2}H_n(z)H_m(z) = \frac{d}{dz}\left\{e^{-z^2}\left[H_n(z)H_m'(z) - H_m(z)H_n'(z)\right]\right\}. \tag{2.21}$$

Integrating (2.21) with respect to z on $(-\infty, \infty)$ leads to

$$2(n-m)\int_{-\infty}^{\infty} e^{-z^2}H_n(z)H_m(z)dz = e^{-z^2}W[H_n, H_m]\Big|_{-\infty}^{\infty} \tag{2.22}$$

where $W[H_n, H_m] = H_n(z)H_m'(z) - H_m(z)H_n'(z)$ denotes the Wronskian of H_n and H_m, and is a polynomial of degree $n+m-1$. Since

$$\lim_{z\to\pm\infty} e^{-z^2}P(z) = 0, \tag{2.23}$$

for any polynomial $P(z)$, (2.22) becomes

$$2(n-m)\int_{-\infty}^{\infty} e^{-z^2}H_n(z)H_m(z)dz = 0. \tag{2.24}$$

It is clear now from (2.24) that

$$\int_{-\infty}^{\infty} e^{-z^2}H_n(z)H_m(z)dz = 0 \quad \text{for all} \quad n \neq m. \tag{2.25}$$

When $n = m$, the left side of (2.25) will be a positive number. Thus, we have shown that the following result holds:

Theorem 2.7.1 *The set of Hermite polynomials* $\{H_n(z)\}_{n=0}^{\infty}$ *is orthogonal on the interval* $(-\infty, \infty)$ *with respect to the inner product* (1.42) *with the weight function* $\rho(z) = e^{-z^2}$. *More precisely, the Hermite polynomials* $H_n(z)$ *satisfy*

$$\int_{-\infty}^{\infty} e^{-z^2} H_n(z)H_m(z)dz = \mathcal{N}_n^2\delta_{nm},$$

where δ_{nm} *is the Kronecker's delta given by*

$$\delta_{nm} = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$

and $\mathcal{N}_n$ *is the norm of* $H_n(z)$.

The discussion given before this theorem does not allow us to find the norm $\mathcal{N}_n$. Most of the existing results use the generating function relation (2.16) to find ${\mathcal{N}_n}^2$. Here, a less known approach will be provided, because it will be easier to follow similar procedures in the case of discrete Hermite polynomials that will be given in the next chapter.

2.7.2 Orthogonality of $H_n^{(k)}(z)$

Consider the self-adjoint form of (2.8)

$$\frac{d}{dz}\left[e^{-z^2} v'_{kn}(z)\right] + \mu_{kn} e^{-z^2} v_{kn}(z) = 0 \tag{2.26}$$

in which $v_{kn}(z) = H_n^{(k)}(z)$, $k = 0, 1, \ldots, n$ and μ_{kn} is given by (2.9). The same form for $v_{km}(z)$ would be

$$\frac{d}{dz}\left[e^{-z^2} v'_{km}(z)\right] + \mu_{km} e^{-z^2} v_{km}(z) = 0. \tag{2.27}$$

As it was done before, multiply (2.26) by $v_{km}(z)$, (2.27) by $v_{kn}(z)$ and subtract the resulting equations from each other to obtain

$$(\mu_{kn} - \mu_{km})\, e^{-z^2} v_{kn}(z)v_{km}(z) + v_{km}(z)\left[e^{-z^2} v'_{kn}(z)\right]' - v_{kn}(z)\left[e^{-z^2} v'_{km}(z)\right]' = 0$$

or

$$(\mu_{kn} - \mu_{km})\, e^{-z^2} v_{kn}(z)v_{km}(z) = \frac{d}{dz}\left\{e^{-z^2}\left[v_{kn}(z)v'_{km}(z) - v_{km}(z)v'_{kn}(z)\right]\right\}. \tag{2.28}$$

Integrate both sides of (2.28) with respect to z on $(-\infty, \infty)$ to get

$$(\mu_{kn} - \mu_{km}) \int_{-\infty}^{\infty} e^{-z^2} v_{kn}(z) v_{km}(z) dz = e^{-z^2} W[v_{kn}, v_{km}] \Big|_{-\infty}^{\infty}, \tag{2.29}$$

where $W[v_{kn}, v_{km}]$ is the Wronskian of v_{kn} and v_{km}, and is a polynomial. Using (2.9) and (2.23) in (2.29) gives us

$$2(n-m) \int_{-\infty}^{\infty} e^{-z^2} v_{kn}(z) v_{km}(z) dz = 0. \tag{2.30}$$

Hence, we have from (2.30) that

$$\int_{-\infty}^{\infty} e^{-z^2} v_{kn}(z) v_{km}(z) dz = 0 \quad \text{for all} \quad n \neq m. \tag{2.31}$$

When $n = m \geqslant k$, the left side of (2.31) will be a positive number, and hence, we have shown that the following result holds:

Theorem 2.7.2 *For each $k = 0, 1, \ldots,$ the set $\left\{H_n^{(k)}(z)\right\}_{n=0}^{\infty}$ is orthogonal on the interval $(-\infty, \infty)$ with respect to the weight function $\rho(z) = e^{-z^2}$. More precisely,*

$$\int_{-\infty}^{\infty} e^{-z^2} H_n^{(k)}(z) H_m^{(k)}(z) dz = \mathcal{N}_{kn}^2 \, \delta_{nm}, \tag{2.32}$$

where δ_{nm} is the Kronecker's delta and $\mathcal{N}_{kn}$ is the norm of $H_n^{(k)}(z)$.

Remark 2.7.3 *When $k > n$ the kth derivative of $H_n(z)$ vanishes. For this reason, in* (2.32)*, it is clear that $\mathcal{N}_{kn} = 0$ if $k > n$. Nevertheless, we need more to evaluate $\mathcal{N}_{kn}$ for $k = 0, 1, \ldots, n$.*

2.7.3 Evaluation of the norms

In the previous two subsections, we could not evaluate the norms $\mathcal{N}_n$ and $\mathcal{N}_{kn}$. Here, we present a method to evaluate both. First, we note that $\mathcal{N}_n = \mathcal{N}_{0n}$ for all $n = 0, 1, \ldots$ As observed before, we have $\mathcal{N}_{kn} = 0$ for $k > n$.

Lemma 2.7.4 *Let $\mathcal{N}_{kn}$ denote the norm of $H_n^{(k)}(z)$. That is,*

$$\mathcal{N}_{kn}^2 = \int_{-\infty}^{\infty} e^{-z^2} v_{kn}^2(z) \, dz. \tag{2.33}$$

Then, for all $k = 0, 1, \ldots, n-1$,

$$\mathcal{N}_{kn}^2 = \frac{1}{\mu_{kn}} \mathcal{N}_{k+1,n}^2. \tag{2.34}$$

Proof. Multiply the self-adjoint form (2.26) by $v_{kn}(z)$ and integrate both sides with respect to z on $(-\infty, \infty)$:

$$\mu_{kn} \int_{-\infty}^{\infty} e^{-z^2} v_{kn}^2(z) dz + \int_{-\infty}^{\infty} v_{kn}(z) \left[e^{-z^2} v_{kn}'(z) \right]' dz = 0.$$

Using the integration by parts in the second integral, we get

$$\mu_{kn} \mathcal{N}_{kn}^2 + e^{-z^2} v_{kn}(z) v_{kn}'(z) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-z^2} \left[v_{kn}'(z) \right]^2 dz = 0.$$

Using (2.23) and the fact that $v_{kn}'(z) = v_{k+1,n}(z)$, we obtain

$$\mu_{kn} \mathcal{N}_{kn}^2 - \int_{-\infty}^{\infty} e^{-z^2} v_{k+1,n}^2(z) dz = 0,$$

or, equivalently, $\mu_{kn} \mathcal{N}_{kn}^2 - \mathcal{N}_{k+1,n}^2 = 0$ which completes the proof. □

Lemma 2.7.5 $\mathcal{N}_{nn}^2 = 2^{2n} (n!)^2 \sqrt{\pi}$ *for all* $n = 0, 1, \ldots$

Proof. Since $v_{nn}(z)$ is the nth order derivative of $H_n(z)$ and $H_n(z)$ is a polynomial of degree n with the leading coefficient 2^n, we have $v_{nn} = H_n^{(n)} = 2^n n!$ for all n. Thus, for $k = n$ in (2.33), we get

$$\mathcal{N}_{nn}^2 = \int_{-\infty}^{\infty} e^{-z^2} v_{nn}^2 \, dz = 2^{2n} (n!)^2 \int_{-\infty}^{\infty} e^{-z^2} dz = 2^{2n} (n!)^2 \sqrt{\pi}$$

as desired. □

Lemma 2.7.6 *For all* $n = 0, 1, \ldots,$ *and* $k = 0, 1, \ldots, n,$ *we have*

$$\mathcal{N}_{kn}^2 = \frac{2^{n+k} (n!)^2}{(n-k)!} \sqrt{\pi}.$$

Proof. Repeated application of (2.34) gives us

$$\mathcal{N}_{kn}^2 = \frac{1}{\mu_{kn}} \mathcal{N}_{k+1,n}^2 = \frac{1}{\mu_{kn} \mu_{k+1,n}} \mathcal{N}_{k+2,n}^2 = \frac{1}{\mu_{kn} \mu_{k+1,n} \cdots \mu_{n-1,n}} \mathcal{N}_{nn}^2.$$

Using (2.9), we obtain

$$\mu_{kn} \mu_{k+1,n} \cdots \mu_{n-1,n} = (2n - 2k)(2n - 2k - 2) \cdots (2) = 2^{n-k} (n-k)!.$$

Now, with the help of Lemma 2.7.5, we get

$$\mathcal{N}_{kn}^2 = \frac{2^{2n} (n!)^2}{2^{n-k} (n-k)!} \sqrt{\pi} = \frac{2^{n+k} (n!)^2}{(n-k)!} \sqrt{\pi}$$

as claimed. □

Thus, we have proved the following orthogonality relation for $H_n^{(k)}(z)$.

Theorem 2.7.7 *Let $H_n^{(k)}(z)$ be the kth order derivative of $H_n(z)$ and $p = \min\{k, n\}$. Then, for all $k, m, n \in \mathbb{N}_0$, we have*

$$\int_{-\infty}^{\infty} e^{-z^2} H_n^{(k)}(z)\, H_m^{(k)}(z)\, dz = \frac{2^{n+k}(n!)^2}{(n-k)!} \sqrt{\pi}\, \delta_{kp}\, \delta_{mn}. \tag{2.35}$$

For the special case $k = 0$, in (2.35), one derives the orthogonality of $H_n(z)$ which is stated as follows:

Corollary 2.7.8 *For all $m, n \in \mathbb{N}_0$, we have*

$$\int_{-\infty}^{\infty} e^{-z^2} H_n(z) H_m(z)\, dz = 2^n\, n!\, \sqrt{\pi}\, \delta_{mn}. \tag{2.36}$$

CHAPTER 3

q-HERMITE I POLYNOMIALS

3.1 Introduction

In the sequel, it is assumed that $0 < q < 1$. This chapter aims to present some properties of q-Hermite I polynomials. The results given here are already known. However, some of the results are going to be proven using approaches different from the existing ones. The equation

$$-D_q D_{q^{-1}} y + \frac{x}{1-q} D_{q^{-1}} y + \lambda y = 0 \tag{3.1}$$

is known as the q-Hermite difference equation.

Theorem 3.1.1 *All q-derivatives of a solution of* (3.1) *are also solutions of an equation of the same kind. More precisely, if $v_k = D_q^k y(x)$ with $v_0 = y(x)$, then for any $k = 0, 1, \ldots,$ the function v_k is a solution of*

$$-D_q D_{q^{-1}} v_k + \frac{x}{1-q} D_{q^{-1}} v_k + \mu_k v_k = 0, \tag{3.2}$$

where

$$\mu_k = q^k \lambda + \frac{q[k]_q}{1-q}. \tag{3.3}$$

Proof. Taking the q-derivative of (3.1), and using the product rule for the q-derivatives given in (1.30) with $f(x) = x/(1-q)$ and $g(x) = D_{q^{-1}} y$, we get

$$D_q[-D_q D_{q^{-1}} y] + \frac{x}{1-q} D_q D_{q^{-1}} y + D_q \left(\frac{x}{1-q} \right) D_{q^{-1}} y(t) \Big|_{t=qx} + D_q \lambda y = 0. \tag{3.4}$$

Using (1.27) and (1.28), (3.4) becomes

$$-q^{-1}D_qD_{q^{-1}}D_qy(x) + \frac{q^{-1}x}{1-q}D_{q^{-1}}D_qy(x) + \frac{1}{1-q}\,D_qy(x) + \lambda D_qy(x) = 0.$$

Multiplying this equation by q and letting $v_1 = D_qy$, we obtain

$$-D_qD_{q^{-1}}v_1 + \frac{x}{1-q}D_{q^{-1}}v_1 + \mu_1v_1 = 0,$$

where

$$\mu_1 = q\lambda + \frac{q}{1-q}.$$

Suppose now that $v_k = D_q^ky$ is a solution of

$$-D_qD_{q^{-1}}v_k + \frac{x}{1-q}D_{q^{-1}}v_k + \mu_kv_k = 0,$$

for some $k \in \mathbb{N}_0$. Taking the q-derivative of both sides, and using (1.27) and (1.28) yields

$$-q^{-1}D_qD_{q^{-1}}D_qv_k + \frac{q^{-1}x}{1-q}D_{q^{-1}}D_qv_k + \frac{1}{1-q}\,D_qv_k(x) + \mu_kD_qv_k = 0.$$

Thus, $v_{k+1} = D_q^{k+1}y = D_qv_k$ satisfies the equation

$$-D_qD_{q^{-1}}v_{k+1} + \frac{x}{1-q}D_{q^{-1}}v_{k+1} + \mu_{k+1}v_{k+1} = 0,$$

where $\mu_{k+1} = q\mu_k + q/(1-q)$. So, by mathematical induction, $v_k(z) := D_q^ky(z)$ satisfies the equation

$$-D_qD_{q^{-1}}v_k + \frac{x}{1-q}D_{q^{-1}}v_k + \mu_kv_k = 0,$$

for all $k = 0, 1, \ldots$ where $v_0(z) := y(z)$, $\mu_0 := \lambda$, and

$$\mu_k = q\mu_{k-1} + \frac{q}{1-q} = q^k\mu_0 + \frac{q^k + q^{k-1} + \cdots + q}{1-q} = q^k\lambda + \frac{q[k]_q}{1-q},$$

which completes the proof. □

Theorem 3.1.2 *If* $\lambda = \lambda_n := -\dfrac{q^{1-n}[n]_q}{1-q}$, *then the equation* (3.1) *has a polynomial solution of degree* n.

Proof. If $\lambda = \lambda_n$, from (3.3) we see that

$$\mu_n = q^n\lambda + \frac{q[n]_q}{1-q} = q^n\left(-\frac{q^{1-n}[n]_q}{1-q}\right) + \frac{q}{1-q}[n]_q = -\frac{q[n]_q}{1-q} + \frac{q[n]_q}{1-q} = 0.$$

Hence, for $k = n$, (3.2) becomes

$$-D_q D_{q^{-1}} v_n + \frac{x}{1-q} D_{q^{-1}} v_n = 0.$$

Clearly, this equation has a constant solution, say $v_n = c$. Since $v_n = D_q^n y(x)$, we have $D_q^n y(x) = c$. After q-integrating n times, we see that y is a polynomial of degree n. □

Definition 3.1.3 *For each $n \in \mathbb{N}_0$, the monic polynomial solution of q-Hermite difference equation is called q-Hermite I polynomial and is denoted by $h_{n,q}(x)$. That is,*

$$h_{n,q}(x) = x^n + \cdots \tag{3.5}$$

where the dots denote the lower degree terms, and it satisfies

$$-D_q D_{q^{-1}} y + \frac{x}{1-q} D_{q^{-1}} y - \frac{q^{1-n}[n]_q}{1-q} y = 0. \tag{3.6}$$

Remark 3.1.4 *Taking* (3.2) *into account, one can see that $v_{kn} := D_q^k h_{n,q}(x)$ is a polynomial solution of the equation*

$$-D_q D_{q^{-1}} v_{kn} + \frac{x}{1-q} D_{q^{-1}} v_{kn} + \mu_{kn} v_{kn} = 0 \tag{3.7}$$

where

$$\mu_{kn} = q^k \lambda_n + \frac{q[k]_q}{1-q} = -\frac{q^{1-n+k}[n-k]_q}{1-q} = \lambda_{n-k} \tag{3.8}$$

for all $n \in \mathbb{N}_0$ and $k = 0, 1, \ldots, n$.

Lemma 3.1.5 *For all $n = 1, 2, \ldots,$ we have*

$$D_q h_{n,q}(x) = [n]_q h_{n-1,q}(x) \tag{3.9}$$

Proof. We know that $h_{n,q}(x)$ is a polynomial solution of

$$-D_q D_{q^{-1}} y + \frac{x}{1-q} D_{q^{-1}} y + \lambda_n y = 0,$$

and $v_{1n} = D_q h_{n,q}(x)$ is a solution of (3.7) with $k = 1$. From (3.8) we get $\mu_{1n} = \lambda_{n-1}$. Hence, $D_q h_{n,q}(x)$ is a solution of

$$-D_q D_{q^{-1}} y + \frac{x}{1-q} D_{q^{-1}} y + \lambda_{n-1} y = 0.$$

As a polynomial solution of this equation is $h_{n-1,q}(x)$, we get $D_q h_{n,q}(x) = c_n h_{n-1,q}(x)$, for some constant c_n. Comparing the leading coefficients, we get $c_n = [n]_q$ which completes the proof. □

3.2 The q-weight function

Let us consider the Sturm-Liouville or formal self-adjoint form of (3.6)

$$D_{q^{-1}}[-\rho_q(x)D_q y] + q\rho_q(x)\lambda_n y = 0, \tag{3.10}$$

where $\rho_q(x)$ satisfies the so-called Pearson equation

$$D_{q^{-1}}[-\rho_q(x)] = \frac{qx}{1-q}\rho_q(x). \tag{3.11}$$

From (3.11), one obtains $\rho_q(q^{-1}x) = (1-x^2)\rho_q(x)$, or

$$\frac{\rho_q(qx)}{\rho_q(x)} = \frac{1}{1-q^2x^2}.$$

Using Lemma 1.4.3, we get

$$\begin{aligned}
\rho_q(x) &= \rho_q(0)\exp\left[\int_0^x \frac{1}{(q-1)t}\ \ln\frac{1}{1-q^2t^2}\right]d_qt \\
&= \rho_q(0)\exp\left[(1-q)x\lim_{n\to\infty}\sum_{k=0}^{n} q^k\frac{1}{(q-1)q^kx}\ \ln\frac{1}{1-q^{2k+2}x^2}\right] \\
&= \rho_q(0)\exp\left[\lim_{n\to\infty}\sum_{k=0}^{n}\ \ln\left(1-q^{2k+2}x^2\right)\right] \\
&= \rho_q(0)\lim_{n\to\infty}\prod_{k=0}^{n}\left(1-q^{2k+2}x^2\right) \\
&= \rho_q(0)\lim_{n\to\infty}\left(q^2x^2;q^2\right)_{n+1} \\
&= \rho_q(0)\left(q^2x^2;q^2\right)_\infty .
\end{aligned}$$

Alternatively, we could use repeated application of $\rho_q(x) = (1-q^2x^2)\rho_q(qx)$ to obtain

$$\rho_q(x) = (1-q^2x^2)(1-q^4x^2)\rho_q(q^2x) = (q^2x^2;q^2)_n\rho_q(q^nx).$$

Taking the limit as $n\to\infty$, we find $\rho_q(x) = (q^2x^2;q^2)_\infty\rho_q(0)$ as before.

Setting $\rho_q(0) := 1$ and using Corollary 1.2.9 we obtain

$$\rho_q(x) = (qx, -qx; q)_\infty, \tag{3.12}$$

which is called the q-weight function.

3.3 The Rodrigues formula

Like in the classical case, we are going to present a formula which gives the q-Hermite I polynomials as the n-th order q-derivative of the q-weight function. The self-adjoint form of (3.7) is

$$D_{q^{-1}}[-\rho_q(x)D_q v_{kn}(x)] + q\mu_{kn}\rho_q(x)v_{kn}(x) = 0. \tag{3.13}$$

Since, $v_{kn}(x) = D_q^k h_{n,q}(x)$, we get

$$\rho_q(x)v_{kn}(x) = \frac{1}{q\mu_{kn}} D_{q^{-1}}[\rho_q(x)v_{k+1,n}(x)].$$

Application of this recursion $n-k$ times leads to

$$\rho_q(x)v_{kn}(x) = \frac{1}{q^{n-k}\mu_{kn}\mu_{k+1,n}\cdots\mu_{nn}} D_{q^{-1}}^{n-k}[\rho_q(x)v_{nn}(x)],$$

which, for $k=0$, becomes

$$\rho_q(x)v_{0n}(x) = \frac{q^{-n}}{\mu_{0n}\mu_{1n}\cdots\mu_{nn}} D_{q^{-1}}^{n}[\rho_q(x)v_{nn}(x)].$$

Now, since $v_{0n}(x) = h_{n,q}(x)$ and $v_{nn} = D_q^n h_{n,q}(x) = [n]_q!$, we obtain

$$\rho_q(x)h_{n,q}(x) = \frac{q^{-n}[n]_q!}{\mu_{0n}\mu_{1n}\cdots\mu_{nn}} D_{q^{-1}}^{n}[\rho_q(x)].$$

Using (3.8), we see that the coefficient on the right side of the last equation is

$$\frac{q^{-n}[n]_q!}{\mu_{0n}\mu_{1n}\cdots\mu_{nn}} = \prod_{k=0}^{n-1}\frac{q^{-1}[n-k]_q}{\mu_{kn}} = \prod_{k=0}^{n-1}(1-q^{-1})q^{n-k-1} = (1-q^{-1})^n q^{\binom{n}{2}}.$$

Therefore, we have

$$h_{n,q}(x) = (1-q^{-1})^n q^{\binom{n}{2}} \frac{D_{q^{-1}}^{n}[\rho_q(x)]}{\rho_q(x)} \tag{3.14}$$

for all $n = 0, 1, \ldots$ Formula (3.14) is known as the Rodrigues formula for the q-Hermite I polynomials. Using (3.14), one can see for $n=0$ that $h_{0,q}(x) = 1$ and for $n=1$ that

$$h_{1,q}(x) = (1-q^{-1})\frac{D_{q^{-1}}\rho_q(x)}{\rho_q(x)} = \frac{\rho_q(x)-\rho_q(q^{-1}x)}{x\rho_q(x)} = \frac{\rho_q(x)-(1-x^2)\rho_q(x)}{x\rho_q(x)} = x.$$

For $n=2$ one gets

$$\begin{aligned} h_{2,q}(x) &= (1-q^{-1})^2 q \frac{D_{q^{-1}}^2\rho_q(x)}{\rho_q(x)} = \frac{(1-q^{-1})^2 q}{\rho_q(x)} D_{q^{-1}}\left[\frac{\rho_q(x)-\rho_q(q^{-1}x)}{(1-q^{-1})x}\right] \\ &= \frac{(1-q^{-1})q}{\rho_q(x)} D_{q^{-1}}[x\rho_q(x)] = \frac{(1-q^{-1})q}{\rho_q(x)} \frac{x\rho_q(x) - q^{-1}x\rho_q(q^{-1}x)}{(1-q^{-1})x} \\ &= \frac{q\rho_q(x)-\rho_q(q^{-1}x)}{\rho_q(x)} = \frac{q\rho_q(x)-(1-x^2)\rho_q(x)}{\rho_q(x)} = x^2-1+q. \end{aligned}$$

3.4 Three term recurrence relation

As it can be seen, the Rodrigues formula (3.14) is not very practical to use even for $n = 2$. For larger values of n, it will be more difficult to compute $h_{n,q}(x)$. For this reason, in the present chapter, we shall obtain the three term recurrence relation for the q-Hermite I polynomials.

Theorem 3.4.1 *The q-Hermite I polynomials $h_{n,q}(x)$ satisfy the three term recurrence relation*

$$h_{n+1,q}(x) - xh_{n,q}(x) + q^{n-1}(1-q^n)h_{n-1,q}(x) = 0, \quad n = 0, 1, \dots \tag{3.15}$$

where $h_{-1,q}(x) := 0$ and $h_{0,q}(x) = 1$.

Proof. Using (1.26) and (3.9), we obtain

$$D_{q^{-1}}h_{n,q}(x) = D_q h_{n,q}(t)\Big|_{t=q^{-1}x} = [n]_q h_{n-1,q}(q^{-1}x),$$

and

$$D_q D_{q^{-1}}h_{n,q}(x) = q^{-1}[n]_q[n-1]_q h_{n-2,q}(q^{-1}x).$$

Thus, (3.6) becomes

$$-q^{-1}[n-1]_q h_{n-2,q}(q^{-1}x) + \frac{x}{1-q}h_{n-1,q}(q^{-1}x) - \frac{q^{1-n}}{1-q}h_{n,q}(x) = 0.$$

Replacing n with $n+1$, x with qx in the above equation, we get

$$-q^{-1}[n]_q h_{n-1,q}(x) + \frac{qx}{1-q}h_{n,q}(x) - \frac{q^{-n}}{1-q}h_{n+1,q}(qx) = 0. \tag{3.16}$$

From (3.9) we write

$$\frac{h_{n+1,q}(x) - h_{n+1,q}(qx)}{(1-q)x} = [n+1]_q h_{n,q}(x)$$

or

$$h_{n+1,q}(qx) = h_{n+1,q}(x) - x(1-q^{n+1})h_{n,q}(x). \tag{3.17}$$

Using (3.17) in (3.16), and rearranging, we get

$$-q^{n-1}(1-q^n)h_{n-1,q}(x) + xh_{n,q}(x) - h_{n+1,q}(x) = 0,$$

which is the relation that we want. Since this recurrence relation consists of three terms, knowing two term will give all the other terms. $h_{0,q}(x) = 1$ is already known from (3.5) and we set $h_{-1,q}(x) := 0$ for convenience. □

3.5 Generating function

In this section we are going to find a generating function of the form $\sum_{n=0}^{\infty} \frac{h_{n,q}(x)}{(q;q)_n} t^n$ for the q-Hermite I polynomials.

Theorem 3.5.1 *The q-Hermite I polynomials have the following generating function relation*

$$\frac{(t^2;q^2)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{h_{n,q}(x)}{(q;q)_n} t^n. \tag{3.18}$$

Proof. Let

$$G(x,t) = \sum_{n=0}^{\infty} \frac{h_{n,q}(x)}{(q;q)_n} t^n.$$

For each fixed x, taking the q-derivative of both sides with respect to t, we obtain

$$\frac{G(x,t) - G(x,qt)}{(1-q)t} = \sum_{n=1}^{\infty} \frac{h_{n,q}(x)}{(q;q)_n} [n]_q\, t^{n-1} = \sum_{n=0}^{\infty} \frac{h_{n+1,q}(x)}{(1-q)(q;q)_n} t^n,$$

which gives us

$$G(x,t) - G(x,qt) = \sum_{n=0}^{\infty} \frac{h_{n+1,q}(x)}{(q;q)_n} t^{n+1}.$$

Using three term recurrence relation (3.15), we get

$$\begin{aligned}
G(x,t) - G(x,qt) &= \sum_{n=0}^{\infty} \frac{x h_{n,q}(x) - q^{n-1}(1-q^n) h_{n-1,q}(x)}{(q;q)_n} t^{n+1} \\
&= xt \sum_{n=0}^{\infty} \frac{h_{n,q}(x)}{(q;q)_n} t^n - \sum_{n=1}^{\infty} \frac{q^{n-1}(1-q^n) h_{n-1,q}(x)}{(q;q)_n} t^{n+1} \\
&= xtG(x,t) - \sum_{n=1}^{\infty} \frac{q^{n-1} h_{n-1,q}(x)}{(q;q)_{n-1}} t^{n+1} \\
&= xtG(x,t) - t^2 \sum_{n=0}^{\infty} \frac{h_{n,q}(x)}{(q;q)_n} (qt)^n \\
&= xtG(x,t) - t^2 G(x,qt)
\end{aligned}$$

which can be written as

$$G(x,t) = \frac{1-t^2}{1-xt} G(x,qt).$$

Repeated application of this relation gives us

$$G(x,t) = \frac{1-t^2}{1-xt}\frac{1-q^2t^2}{1-qxt}G(x,q^2t) = \frac{(t^2;q^2)_k}{(xt;q)_k}G(x,q^kt).$$

Taking the limit as $k \to \infty$ and nothing that $G(x,0) = 1$, we obtain

$$G(x,t) = \frac{(t^2;q^2)_\infty}{(xt;q)_\infty}G(x,0) = \frac{(t^2;q^2)_\infty}{(xt;q)_\infty}$$

which completes the proof. □

Observation 3.5.2 *For $x = 0$ in* (3.18)*, and using* (1.36) *with $x = t^2$ and q replaced by q^2, we get*

$$\sum_{n=0}^{\infty} \frac{h_{n,q}(0)}{(q;q)_n} t^n = (t^2;q^2)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)}}{(q^2;q^2)_k} t^{2k}.$$

Comparing the coefficients of like powers of t and using Lemma 1.2.6, we see that

$$h_{2n+1,q}(0) = 0 \quad \textit{and} \quad h_{2n,q}(0) = (-1)^n q^{n(n-1)} \frac{(q;q)_{2n}}{(q^2;q^2)_n} = (-1)^n q^{n(n-1)}(q;q^2)_n,$$

for all $n = 0, 1, \ldots$

Observation 3.5.3 *In* (3.18)*, replacing x and t with $-x$ and $-t$, respectively, we get*

$$\sum_{n=0}^{\infty} \frac{h_{n,q}(-x)}{(q;q)_n}(-t)^n = \frac{\left(t^2;q^2\right)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{h_{n,q}(x)}{(q;q)_n} t^n.$$

Comparing the coefficients of like powers of t and using Lemma 1.2.6, we see that

$$h_{n,q}(-x) = (-1)^n h_{n,q}(x), \quad for\ all \quad n = 0, 1, \ldots$$

That is, $h_{n,q}(x)$ is an odd function if n is odd, and $h_{n,q}(x)$ is an even function if n is even.

3.6 Hypergeometric representation

In this section, a hypergeometric representation $h_{n,q}(x)$ will be derived. Before this, we need some auxiliary results.

Lemma 3.6.1 *If $f(x) = (-qx;q)_\infty$, then $D^k_{q^{-1}} f(x) = \frac{(-1)^k}{(1-q^{-1})^k} f(x)$ for all $k = 0, 1, \ldots$*

Proof. From the definition of q^{-1}-derivative we have

$$D_{q^{-1}}f(x) = \frac{(-qx;q)_\infty - (-x;q)_\infty}{(1-q^{-1})x} = \frac{(-qx;q)_\infty - (1+x)(-qx;q)_\infty}{(1-q^{-1})x} = \frac{-f(x)}{1-q^{-1}},$$

from which one can see that

$$D^k_{q^{-1}}f(x) = \frac{(-1)^k}{(1-q^{-1})^k}\, f(x)$$

for all $k = 0, 1, \ldots$ □

Lemma 3.6.2 *If $g(x) = (qx;q)_\infty$, then $D^k_{q^{-1}}g(x) = \frac{1}{(1-q^{-1})^k}\, g(x)$ for all $k = 0, 1, \ldots$*

Proof. By the definition, we have

$$D_{q^{-1}}g(x) = \frac{(qx;q)_\infty - (x;q)_\infty}{(1-q^{-1})x} = \frac{(qx;q)_\infty - (1-x)(qx;q)_\infty}{(1-q^{-1})x} = \frac{g(x)}{1-q^{-1}}.$$

Therefore,

$$D^k_{q^{-1}}g(x) = \frac{1}{(1-q^{-1})^k}\, g(x)$$

for all $k = 0, 1, \ldots$ □

Lemma 3.6.3 *For all $n \in \mathbb{N}_0$,*

$$D^n_{q^{-1}}\rho_q(x) = \frac{\rho_q(x)}{(1-q^{-1})^n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix}\middle| q; -qx\right) \tag{3.19}$$

Proof. Take $f(x) = (-qx;q)_\infty$ and $g(x) = (qx;q)_\infty$ so that $\rho_q(x) = f(x)g(x)$. Using the Leibniz rule for q-derivative (see (1.31)), and Lemmas 3.6.1 and 3.6.2, we get

$$D^n_{q^{-1}}(\rho_q(x)) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} D^k_{q^{-1}}f(x) D^{n-k}_{q^{-1}}g(t)\Big|_{t=q^{-k}x} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} \frac{(-1)^k}{(1-q^{-1})^n}\, f(x)g(q^{-k}x).$$

Using Observation 1.2.3 with $a = q^{1-k}x$ gives us

$$g(q^{-k}x) = (q^{1-k}x;q)_\infty = (q^{1-k}x;q)_k(qx;q)_\infty = (q^{1-k}x;q)_k g(x).$$

Since

$$(q^{1-k}x;q)_k = \prod_{s=0}^{k-1}\left(1 - q^{1-k+s}x\right) = \prod_{s=0}^{k-1}\left(-q^{1-k+s}x\right)\prod_{s=0}^{k-1}(1 - x^{-1}q^{k-s-1})$$

$$= (-1)^k q^{-\binom{k}{2}} x^k \prod_{s=0}^{k-1} (1 - x^{-1}q^s) = (-1)^k q^{-\binom{k}{2}} x^k (x^{-1};q)_k,$$

we get

$$g(q^{-k}x) = (-1)^k q^{-\binom{k}{2}} x^k (x^{-1};q)_k g(x),$$

and hence,

$$D^n_{q^{-1}}(\rho_q(x)) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} \frac{q^{-\binom{k}{2}}}{(1-q^{-1})^n} x^k (x^{-1};q)_k \rho_q(x).$$

Using (1.16) in this equality, we obtain

$$\begin{aligned} D^n_{q^{-1}}(\rho_q(x)) &= \sum_{k=0}^{n} \frac{(q^{-n};q)_k}{(q;q)_k (1-q^{-1})^n} (-qx)^k (x^{-1};q)_k \rho_q(x) \\ &= \frac{\rho_q(x)}{(1-q^{-1})^n} \sum_{k=0}^{n} \frac{(q^{-n};q)_k (x^{-1};q)_k}{(q;q)_k} (-qx)^k \\ &= \frac{\rho_q(x)}{(1-q^{-1})^n} \, {}_2\phi_1 \left(\left. \begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \right| q; -qx \right) \end{aligned}$$

as stated. □

The next theorem enables us to write $h_{n,q}(x)$ as a q-hypergeometric series.

Theorem 3.6.4 *For any $n = 0, 1, \ldots,$ we have*

$$h_{n,q}(x) = q^{\binom{n}{2}} \, {}_2\phi_1 \left(\left. \begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} \right| q; -qx \right). \tag{3.20}$$

Proof. Using (3.19) in the Rodrigues formula (3.14) gives the required result. □

In the next Chapter, when the limit relation between the hypergeometric forms is presented, (3.20) will not be so easy to use. For this reason, we shall derive another hypergeometric representation of the q-Hermite I polynomials. The two formulas are given together in many books, see [4].

Theorem 3.6.5 *For any $n = 0, 1, \ldots,$ we have*

$$h_{n,q}(x) = x^n \, {}_2\phi_0 \left(\left. \begin{matrix} q^{-n}, q^{-n+1} \\ - \end{matrix} \right| q^2; \frac{q^{2n-1}}{x^2} \right). \tag{3.21}$$

Proof. From (1.35) with x replaced by xt, we get

$$\frac{1}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n t^n}{(q;q)_n}.$$

Also, from (1.36) with x replaced by t^2 and q by q^2, we have

$$(t^2;q^2)_\infty = \sum_{k=0}^{\infty} \frac{(-1)^k q^{2\binom{k}{2}} t^{2k}}{(q^2;q^2)_k}.$$

Thus, (3.18) gives us

$$\sum_{n=0}^{\infty} \frac{h_{n,q}(x)}{(q;q)_n} t^n = \frac{(t^2;q^2)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k q^{2\binom{k}{2}} x^n}{(q^2;q^2)_k (q;q)_n} t^{n+2k}.$$

On the right side, replacing n by $n-2k$ results in

$$\sum_{n=0}^{\infty} \frac{h_{n,q}(x)}{(q;q)_n} t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k q^{2\binom{k}{2}} x^{n-2k}}{(q^2;q^2)_k (q;q)_{n-2k}} t^n.$$

Therefore,

$$h_{n,q}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(q;q)_n (-1)^k q^{2\binom{k}{2}}}{(q^2;q^2)_k (q;q)_{n-2k}} x^{n-2k}. \tag{3.22}$$

From (1.10) with k replaced by $2k$, we obtain

$$\frac{(q;q)_n}{(q;q)_{n-2k}} = (q^{-n};q)_{2k} q^{2nk-\binom{2k}{2}},$$

and hence, (3.22) becomes

$$h_{n,q}(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k (q^{-n};q)_{2k}\, q^{2nk-k^2}}{(q^2;q^2)_k} x^{n-2k}. \tag{3.23}$$

Considering Observation 1.2.4 and using Lemma 1.2.6 for $a = q^{-n}$ in (3.23) gives

$$\begin{aligned} h_{n,q}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k (q^{-n};q^2)_k (q^{-n+1};q^2)_k\, q^{2nk-k^2}}{(q^2;q^2)_k} x^{n-2k} \\ &= x^n \sum_{k=0}^{\infty} \frac{(q^{-n};q^2)_k (q^{-n+1};q^2)_k}{(q^2;q^2)_k} (-1)^k q^{-2\binom{k}{2}} \left(\frac{q^{2n-1}}{x^2}\right)^k \\ &= x^n\, {}_2\phi_0 \left(\left. \begin{matrix} q^{-n}, q^{-n+1} \\ - \end{matrix} \right| q^2; \frac{q^{2n-1}}{x^2} \right). \end{aligned}$$

This completes the proof. □

3.7 Orthogonality

3.7.1 Orthogonality of $h_{n,q}(x)$

Lemma 3.7.1 *If a and b are real numbers such that*

$$\rho(q^{-1}x)x^s\Big|_{x=a,b} = 0, \tag{3.24}$$

then the set of q-Hermite I polynomials $\left\{h_{n,q}(x)\right\}_{n=0}^{\infty}$ *is orthogonal on the interval* (a, b) *with respect to the inner product* $\langle\cdot,\cdot\rangle_q$ *defined by* (1.43) *with the weight function* $\rho_q(x)$ *given in* (3.12).

Proof. Recall that the formal self-adjoint form of (3.6) is

$$-D_{q^{-1}}[\rho_q(x)D_q h_{n,q}(x)] + q\rho_q(x)\lambda_n h_{n,q}(x) = 0. \tag{3.25}$$

The same form for $h_{m,q}(x)$ would be

$$-D_{q^{-1}}[\rho_q(x)D_q h_{m,q}(x)] + q\rho_q(x)\lambda_m h_{m,q}(x) = 0. \tag{3.26}$$

Multiplying (3.25) by $h_{m,q}(x)$ and (3.26) by $h_{n,q}(x)$ and then subtracting one of the resulting equation from the other one, we get

$$\begin{aligned}- h_{m,q}(x)D_{q^{-1}}\left(\rho_q(x)D_q h_{n,q}(x)\right) + h_{n,q}(x)D_{q^{-1}}\left(\rho_q(x)D_q h_{m,q}(x)\right)&\\ + q(\lambda_n - \lambda_m)\rho_q(x)h_{n,q}(x)h_{m,q}(x) &= 0.\end{aligned}$$

Take the q-integral of both sides over the interval (a, b), and use (1.24):

$$\begin{aligned}(\lambda_n - \lambda_m)\int_a^b \rho_q(x)h_{n,q}(x)h_{m,q}(x)\,d_qx &= \int_a^b h_{m,q}(x)D_q\left(\rho_q(q^{-1}x)D_q h_{n,q}(t)\Big|_{t=q^{-1}x}\right)d_qx\\ &\quad - \int_a^b h_{n,q}(x)D_q\left(\rho_q(q^{-1}x)D_q h_{m,q}(t)\Big|_{t=q^{-1}x}\right)d_qx.\end{aligned}$$

Using integration by parts and taking (1.26) into account, the first integral on the right side becomes

$$\begin{aligned}&\int_a^b h_{m,q}(x)D_q\left(\rho_q(q^{-1}x)D_q h_{n,q}(t)\Big|_{t=q^{-1}x}\right)d_qx =\\ &\qquad \left(\rho_q(q^{-1}x)h_{m,q}(x)D_{q^{-1}}h_{n,q}(x)\right)\Big|_a^b - \int_a^b \rho_q(x)D_q h_{n,q}(x)D_q h_{m,q}(x)d_qx,\end{aligned}$$

Similarly,

$$\int_a^b h_{n,q}(x) D_q \left(\rho_q(q^{-1}x) D_q h_{m,q}(t) \Big|_{t=q^{-1}x} \right) d_q x =$$

$$\left(\rho_q(q^{-1}x) h_{n,q}(x) D_{q^{-1}} h_{m,q}(x) \right) \Big|_a^b - \int_a^b \rho_q(x) D_q h_{m,q}(x) D_q h_{n,q}(x) d_q x.$$

Therefore,

$$(\lambda_n - \lambda_m) \int_a^b \rho_q(x) h_{n,q}(x) h_{m,q}(x)\, d_q x = \rho_q(q^{-1}x) W_{q^{-1}}[h_{m,q}(x), h_{n,q}(x)] \Big|_a^b, \qquad (3.27)$$

where $W_{q^{-1}}[h_{m,q}(x), h_{n,q}(x)] = h_{m,q}(x) D_{q^{-1}} h_{n,q}(x) - h_{n,q}(x) D_{q^{-1}} h_{m,q}(x)$ is a polynomial of degree $n + m - 1$. Because of the condition (3.24), the right side of (3.27) vanishes and we obtain

$$(\lambda_n - \lambda_m) \int_a^b \rho_q(x) h_{n,q}(x) h_{m,q}(x)\, d_q x = 0. \qquad (3.28)$$

Now, $n \neq m$ if and only if $\lambda_n - \lambda_m = (q^{1-m} - q^{1-n})/(1-q)^2 \neq 0$. Therefore, (3.28) gives us

$$\int_a^b \rho_q(x) h_{n,q}(x) h_{m,q}(x)\, d_q x = 0, \quad \text{for all} \quad n \neq m.$$

When $n = m$, the integral in (3.28) becomes an arbitrary positive number. □

Remark 3.7.2 *Since $\rho_q(q^{-1}x) = (x, -x; q)_\infty$, it is easy to see that* (3.24) *is true if one chooses $a = -1$, $b = 1$. So, in fact we have proved the following theorem:*

Theorem 3.7.3 *The set $\{h_{n,q}(x)\}_{n=0}^{\infty}$ of q-Hermite I polynomials is an orthogonal set on the interval $(-1, 1)$ with respect to the q-weight function $\rho(x) = (qx, -qx; q)_\infty$. More precisely, the q-Hermite I polynomials $h_{n,q}(x)$ satisfy*

$$\int_{-1}^{1} \rho_q(x)\, h_{n,q}(x)\, h_{m,q}(x)\, d_q x = \mathcal{M}_n^2 \delta_{nm}, \qquad (3.29)$$

where $\mathcal{M}_n$ is the norm of $h_{n,q}(x)$.

3.7.2 Orthogonality of $D_q^k h_{n,q}(x)$

Consider the self-adjoint form of (3.7)

$$-D_{q^{-1}}[\rho_q(x) D_q v_{kn}(x)] + q\mu_{kn} \rho_q(x) v_{kn}(x) = 0, \qquad (3.30)$$

in which $v_{kn}(x) = D_q^k h_{n,q}(x)$, $k = 0, 1, \ldots, n$ and μ_{kn} is given by (3.8). The same form for $v_{km}(x)$ is

$$-D_{q^{-1}}[\rho_q(x)D_q v_{km}(x)] + q\mu_{km}\rho_q(x)v_{km}(x) = 0. \tag{3.31}$$

As it was done before, multiply (3.30) by $v_{km}(x)$, (3.31) by $v_{kn}(x)$ and subtract the resulting equations from each other to obtain

$$q(\mu_{kn} - \mu_{km})\rho_q(x)v_{kn}(x)v_{km}(x) = v_{km}(x)D_{q^{-1}}\left(\rho_q(x)D_q v_{kn}(x)\right) - v_{kn}(x)D_{q^{-1}}\left(\rho_q(x)D_q v_{km}(x)\right),$$

or, using (1.24),

$$(\mu_{kn} - \mu_{km})\rho_q(x)v_{kn}(x)v_{km}(x) = v_{km}(x)D_q\left(\rho_q(q^{-1}x)D_q v_{kn}(t)\Big|_{t=q^{-1}x}\right) - v_{kn}(x)D_q\left(\rho_q(q^{-1}x)D_q v_{km}(t)\Big|_{t=q^{-1}x}\right).$$

Take the q-integral of both sides over the interval $(-1, 1)$:

$$(\mu_{kn} - \mu_{km})\int_{-1}^{1}\rho_q(x)v_{kn}(x)v_{km}(x)\,d_qx = \int_{-1}^{1} v_{km}(x)D_q\left(\rho_q(q^{-1}x)D_q v_{kn}(t)\Big|_{t=q^{-1}x}\right)d_qx - \int_{-1}^{1} v_{kn}(x)D_q\left(\rho_q(q^{-1}x)D_q v_{km}(t)\Big|_{t=q^{-1}x}\right)d_qx.$$

Using integration by parts and taking (3.24) into account, the first integral on the right side becomes

$$\int_{-1}^{1} v_{km}(x)D_q\left(\rho_q(q^{-1}x)D_q v_{kn}(t)\Big|_{t=q^{-1}x}\right)d_qx = -\int_{-1}^{1}\rho_q(x)D_q v_{km}(x)D_q v_{kn}(x)\,d_qx.$$

Similarly,

$$\int_{-1}^{1} v_{kn}(x)D_q\left(\rho_q(q^{-1}x)D_q v_{km}(t)\Big|_{t=q^{-1}x}\right)d_qx = -\int_{-1}^{1}\rho_q(x)D_q v_{kn}(x)D_q v_{km}(x)\,d_qx.$$

Therefore,

$$(\mu_{kn} - \mu_{km})\int_{-1}^{1}\rho_q(x)v_{kn}(x)v_{km}(x)\,d_qx = 0. \tag{3.32}$$

Now, if $n \neq m$, then from (3.8) one sees that $\mu_{kn} - \mu_{km} = q^{1-k}(q^{-m} - q^{-n})/(1-q)^2 \neq 0$. Therefore, (3.32) yields

$$\int_{-1}^{1}\rho_q(x)v_{kn}(x)v_{km}(x)\,d_qx = 0 \quad \text{for all} \quad n \neq m.$$

When $n = m \geqslant k$, the left side of (3.32) will be a positive number, and hence, we have shown that the following result holds:

Theorem 3.7.4 *For all $k = 0, 1, \ldots,$ the set $\left\{D_q^k h_{n,q}(x)\right\}_{n=0}^{\infty}$ is orthogonal with respect to the inner product defined by* (1.43) *over the interval* $(-1, 1)$ *with the q-weight function $\rho_q(x)$. More precisely,*

$$\int_{-1}^{1} \rho_q(x)\, D_q^k h_{n,q}(x)\, D_q^k h_{m,q}(x)\, d_q x = \mathcal{M}_{kn}^2\, \delta_{nm} \tag{3.33}$$

where δ_{nm} is the Kronecker's delta and $\mathcal{M}_{kn}$ is the norm of $D_q^k h_{n,q}(x)$.

Remark 3.7.5 *When $k > n$ we have $D_q^k h_{n,q}(x) = 0$. Thus, it is clear from* (3.33) *that $\mathcal{M}_{kn} = 0$ for $k > n$.*

3.7.3 Evaluation of the norms

First, we note that $\mathcal{M}_n = \mathcal{M}_{0n}$ for all $n = 0, 1, \ldots$ Also, as observed before, we have $\mathcal{M}_{kn} = 0$ for $k > n$.

Lemma 3.7.6 *Let $\mathcal{M}_{kn}$ denote the norm of $D_q^k h_{n,q}(x)$. That is,*

$$\mathcal{M}_{kn}^2 = \int_{-1}^{1} \rho_q(x)\, v_{kn}^2(x)\, d_q x. \tag{3.34}$$

Then, for all $k = 0, 1, \ldots, n-1$, we have

$$\mathcal{M}_{kn}^2 = \frac{-1}{\mu_{kn}} \mathcal{M}_{k+1,n}^2. \tag{3.35}$$

Proof. Multiply the self-adjoint form (3.30) by $v_{kn}(x)$ and use (1.24):

$$\mu_{kn}\rho_q(x) v_{kn}^2(x) = v_{kn}(x) D_q\left(\rho_q(q^{-1}x) v_{k+1,n}(q^{-1}x)\right).$$

Then, q-integrate both sides over the interval $(-1, 1)$ and use the integration by parts on the right side:

$$\mu_{kn} \int_{-1}^{1} \rho_q(x) v_{kn}^2(x)\, d_q x = \rho_q(q^{-1}x) v_{kn}(x) v_{k+1,n}(q^{-1}x)\Big|_{-1}^{1} - \int_{-1}^{1} \rho_q(x) v_{k+1,n}(x) D_q v_{k,n}(x) d_q x.$$

Using (3.24) and the fact that $D_q v_{k,n}(x) = v_{k+1,n}(x)$, we arrive at

$$\mu_{kn}\mathcal{M}_{kn}^2 = -\int_{-1}^{1} \rho_q(x) v_{k+1,n}^2(x) d_q x = -\mathcal{M}_{k+1,n}^2,$$

which completes the proof. □

To continue evaluating the norms $\mathcal{M}_{kn}$ we shall need the following auxiliary results:

Lemma 3.7.7 $\sum_{k=0}^{\infty} \frac{q^k}{(q^2;q^2)_k} = (-q;q)_\infty.$

Proof. In (1.35), replace q with q^2 and take $x = q$:

$$\sum_{k=0}^{\infty} \frac{q^k}{(q^2;q^2)_k} = \frac{1}{(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty (q^2;q^2)_\infty}.$$

Using Lemma 1.2.7 with $a = q$, this becomes

$$\sum_{k=0}^{\infty} \frac{q^k}{(q^2;q^2)_k} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty}.$$

Now, using Corollary 1.2.9 with $a = q$ yields the result. □

Lemma 3.7.8

$$\int_{-1}^{1} \rho_q(x) d_q x = (1-q)(q,-1,-q;q)_\infty.$$

Proof. First of all, note that using (1.9) with $a = q$ and with $a = -q$, we obtain

$$\rho_q(1) = (q;q)_\infty (-q;q)_\infty = (q;q)_k (q^{k+1};q)_\infty (-q;q)_k (-q^{k+1};q)_\infty = (q^2;q^2)_k \, \rho_q(q^k) \tag{3.36}$$

for all $k = 0, 1, \ldots$ Thus, since $\rho_q(-x) = \rho_q(x)$,

$$\int_{-1}^{1} \rho_q(x) d_q x = 2 \int_{0}^{1} \rho_q(x) d_q x = 2(1-q) \sum_{k=0}^{\infty} q^k \rho_q(q^k).$$

Using (3.36) and Lemma 3.7.7, the above equality becomes

$$\int_{-1}^{1} \rho_q(x) d_q x = 2(1-q)\rho_q(1) \sum_{k=0}^{\infty} \frac{q^k}{(q^2;q^2)_k} = 2(1-q)\rho_q(1)(-q;q)_\infty.$$

As $2(-q;q)_\infty = (-1;q)_\infty$, we get

$$\int_{-1}^{1} \rho_q(x) d_q x = (1-q)\rho_q(1)(-1;q)_\infty = (1-q)(q,-1,-q;q)_\infty$$

which completes the proof. □

Lemma 3.7.9 $\mathcal{M}_{nn} = (1-q)(q,-1,-q;q)_\infty \left([n]_q!\right)^2$ *for all* $n = 0, 1, \ldots$

Proof. Since $v_{nn}(x) = D_q^n h_{n,q}(x)$ and $h_{n,q}(x)$ is a monic polynomial of degree n, we have $v_{nn} = [n]_q!$. Thus, for $k = n$, in (3.34), we get

$$\mathcal{M}_{nn}^2 = \int_{-1}^{1} \rho_q(x) v_{nn}^2 d_q x = \left([n]_q!\right)^2 \int_{-1}^{1} \rho_q(x) d_q x = (1-q)(q,-1,-q;q)_\infty \left([n]_q!\right)^2.$$

□

Lemma 3.7.10 *For* $n \in \mathbb{N}_0$ *and* $k = 0, 1, \ldots, n-1$ *we have*

$$\mathcal{M}_{kn}^2 = (1-q)^{n-k+1} q^{\binom{n-k}{2}} (q,-1,-q;q)_\infty \frac{\left([n]_q!\right)^2}{[n-k]_q!}.$$

Proof. Repeated application of (3.35) gives us

$$\mathcal{M}_{kn}^2 = \frac{-1}{\mu_{kn}} \mathcal{M}_{k+1,n}^2 = \frac{(-1)^2}{\mu_{kn}\mu_{k+1,n}} \mathcal{M}_{k+2,n}^2 = \frac{(-1)^{n-k}}{\mu_{kn}\mu_{k+1,n}\cdots\mu_{n-1,n}} \mathcal{M}_{nn}^2.$$

Since

$$\prod_{s=k}^{n-1} \mu_{sn} = \prod_{s=k}^{n-1} \frac{-q^{1-n+s}[n-s]_q}{1-q} = \prod_{s=0}^{n-k-1} \frac{-q^{-s}[s+1]_q}{1-q} = \frac{(-1)^{n-k}[n-k]_q!}{(1-q)^{n-k} q^{\binom{n-k}{2}}},$$

using Lemma 3.7.9, we arrive at the stated result. □

Thus, we proved the following orthogonality relation for $D_q^k h_{n,q}(x)$.

Theorem 3.7.11 *Let* $v_{kn}(x)$ *be the kth order q-derivative of* $h_{n,q}(x)$*. and* $p = \min\{k, n\}$*. Then, for all* $k, m, n \in \mathbb{N}_0$*, we have*

$$\int_{-1}^{1} \rho_q(x) v_{kn}(x) v_{km}(x)\, d_q x = (1-q)^{n-k+1} q^{\binom{n-k}{2}} (q,-1,-q;q)_\infty \frac{\left([n]_q!\right)^2}{[n-k]_q!} \delta_{kp}\, \delta_{mn}. \quad (3.37)$$

Corollary 3.7.12 *For all* $m, n \in \mathbb{N}_0$*, we have*

$$\int_{-1}^{1} \rho_q(x) h_{n,q}(x) h_{m,q}(x)\, d_q x = \mathcal{M}_n^2 \delta_{nm}, \quad (3.38)$$

where

$$\mathcal{M}_n^2 = (1-q)(q,q)_n\, q^{\binom{n}{2}} (q,-1,-q;q)_\infty. \quad (3.39)$$

Proof. Let $k = 0$ in (3.37) and use the relation $[n]_q! = (1-q)^{-n}(q;q)_n$. □

CHAPTER 4

LIMIT RELATIONS

In this section, we are going to establish the limit relations between results stated in Chapters 2 and 3.

4.1 Limit relation between the differential equations

Consider the equation (3.6). Let $x = \sqrt{1-q^2}z$ and set $u(z) = y\left(\sqrt{1-q^2}z\right)$. Using (1.22) with $\alpha = \sqrt{1-q^2}$, we see that

$$D_{q^{-1}}y(x)\big|_{x=\sqrt{1-q^2}z} = \frac{1}{\sqrt{1-q^2}}D_{q^{-1}}y\left(\sqrt{1-q^2}z\right) = \frac{1}{\sqrt{1-q^2}}D_{q^{-1}}u(z),$$

and,

$$D_qD_{q^{-1}}y(x)\big|_{x=\sqrt{1-q^2}z} = \frac{1}{1-q^2}D_qD_{q^{-1}}y\left(\sqrt{1-q^2}z\right) = \frac{1}{1-q^2}D_qD_{q^{-1}}u(z).$$

Thus, (3.6) becomes

$$-\frac{1}{1-q^2}D_qD_{q^{-1}}u(z) + \frac{z}{1-q}D_{q^{-1}}u(z) - \frac{q^{1-n}}{1-q}[n]_qu(z) = 0. \tag{4.1}$$

Multiplying this equation by $-(1-q^2)$, gives us

$$D_qD_{q^{-1}}u(z) - (1+q)zD_{q^{-1}}u(z) + q^{1-n}(1+q)[n]_qu(z) = 0.$$

Finally, taking the limit as $q \to 1$ yields

$$u''(z) - 2zu'(z) + 2nu(z) = 0, \tag{4.2}$$

which is the Hermite differential equation. Since monic polynomial solutions of (4.1) and (4.2) are

$$\frac{h_{n,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{n/2}} \qquad \text{and} \qquad \frac{H_n(z)}{2^n},$$

respectively, in fact, we have shown that the following result is true:

Theorem 4.1.1 *For all $n \in \mathbb{N}_0$,*

$$\lim_{q\to 1} \frac{h_{n,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{n/2}} = \frac{H_n(z)}{2^n}. \tag{4.3}$$

4.2 Limit relation between the weight functions

In q-weight function $\rho_q(x) = (q^2x^2; q^2)_\infty$, let $x = \sqrt{1-q^2}z$:

$$\rho_q\left(\sqrt{1-q^2}z\right) = \left(q^2(1-q^2)z^2; q^2\right)_\infty = \frac{\left((1-q^2)z^2; q^2\right)_\infty}{1-(1-q^2)z^2}.$$

Now, take the limit as $q \to 1$ and use the relation (1.38) with q replaced by q^2 :

$$\lim_{q\to 1} \rho_q\left(\sqrt{1-q^2}z\right) = \lim_{q\to 1} \frac{\left((1-q^2)z^2; q^2\right)_\infty}{1-(1-q^2)z^2} = e^{-z^2}.$$

This tells us that

$$\lim_{q\to 1} \rho_q\left(\sqrt{1-q^2}z\right) = \rho(z) \tag{4.4}$$

where $\rho_q(x)$ is the q-weight function for the q-Hermite I polynomials and $\rho(z)$ is the weight function for the Hermite polynomials given by (2.12).

4.3 Limit relation between the Rodrigues formulae

Recall the Rodrigues formula (3.14) for q-Hermite I polynomials:

$$h_{n,q}(x) = (1-q^{-1})^n q^{\binom{n}{2}} \frac{D^n_{q^{-1}}[\rho_q(x)]}{\rho_q(x)}.$$

Let $x = \sqrt{1-q^2}z$. Then,

$$h_{n,q}\left(\sqrt{1-q^2}z\right) = (1-q^{-1})^n q^{\binom{n}{2}} \frac{D^n_{q^{-1}}[\rho_q(x)]\Big|_{x=\sqrt{1-q^2}z}}{\rho_q\left(\sqrt{1-q^2}z\right)}.$$

Using (1.23) with $\alpha = \sqrt{1-q^2}$ in this equality, and dividing both sides by $(1-q^2)^{n/2}$, we obtain

$$\frac{h_{n,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{n/2}} = \frac{(1-q^{-1})^n q^{\binom{n}{2}}}{(1-q^2)^n} \frac{D^n_{q^{-1}}\rho_q\left(\sqrt{1-q^2}z\right)}{\rho_q\left(\sqrt{1-q^2}z\right)} = \frac{(-1)^n q^{\binom{n}{2}-n}}{(1+q)^n} \frac{D^n_{q^{-1}}\rho_q\left(\sqrt{1-q^2}z\right)}{\rho_q\left(\sqrt{1-q^2}z\right)}.$$

Taking the limit of both sides as $q \to 1$, and using (4.3) and (4.4), we see that

$$\frac{H_n(z)}{2^n} = \frac{(-1)^n}{2^n} \frac{D^n e^{-z^2}}{e^{-z^2}}$$

or

$$H_n(z) = (-1)^n e^{z^2} \frac{d^n}{dz^n} \left(e^{-z^2} \right)$$

which is the Rodrigues formula for Hermite polynomials given in (2.14).

4.4 Limit relation between three term recurrence relations

Let $x = \sqrt{1-q^2}z$ in the three term recurrence relation for q-Hermite I polynomials given by (3.15):

$$h_{n+1,q}\left(\sqrt{1-q^2}z\right) - z\sqrt{1-q^2}h_{n,q}\left(\sqrt{1-q^2}z\right) + q^{n-1}(1-q^n)h_{n-1,q}\left(\sqrt{1-q^2}z\right) = 0.$$

Divide both sides by $(1-q^2)^{\frac{n+1}{2}}$ to get

$$\frac{h_{n+1,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{\frac{n+1}{2}}} - \frac{z\sqrt{1-q^2}h_{n,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{\frac{n+1}{2}}} + \frac{q^{n-1}(1-q^n)h_{n-1,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{\frac{n+1}{2}}} = 0,$$

or, equivalently,

$$\frac{h_{n+1,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{\frac{n+1}{2}}} - z\frac{h_{n,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{\frac{n}{2}}} + \frac{q^{n-1}[n]_q}{1+q}\frac{h_{n-1,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{\frac{n-1}{2}}} = 0.$$

Taking the limit as $q \to 1$ and using (4.3), we get

$$\frac{H_{n+1}(z)}{2^{n+1}} - z\frac{H_n(z)}{2^n} + \frac{n}{2}\frac{H_{n-1}(z)}{2^{n-1}} = 0.$$

Multiplying the last equation by 2^{n+1}, we obtain

$$H_{n+1}(z) - 2zH_n(z) + 2nH_{n-1}(z) = 0,$$

which is the three term recurrence relation for Hermite polynomials.

4.5 Limit relation between generating functions

In (1.35), replacing x by $(1-q^2)zt$ and taking the limit as $q \to 1$, gives us

$$\lim_{q\to 1} \frac{1}{((1-q^2)zt;q)_\infty} = \lim_{q\to 1} e^{(1+q)zt} = e^{2zt}.$$

Also, changing the variable from q to q^2, (1.38) gives us

$$\lim_{q\to 1}\left((1-q^2)t^2;q^2\right)_\infty = e^{-t^2}.$$

Now, let $x = \sqrt{1-q^2}z$ and replace t by $\sqrt{1-q^2}t$ in generating function relation (3.18) for q-Hermite I polynomials. Then, we get

$$\begin{aligned}\frac{\left((1-q^2)t^2;q^2\right)_\infty}{((1-q^2)zt;q)_\infty} &= \sum_{n=0}^{\infty}\frac{h_{n,q}\left(\sqrt{1-q^2}z\right)}{(q;q)_n}\left(\sqrt{1-q^2}t\right)^n\\ &= \sum_{n=0}^{\infty}\frac{h_{n,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{n/2}}\frac{(1-q)^n}{(q;q)_n}[(1+q)t]^n.\end{aligned}$$

Taking the limit of both sides as $q\to 1$, using (1.15) and (4.3), we obtain

$$e^{2zt-t^2} = \sum_{n=0}^{\infty}\frac{H_n(z)}{2^n}\frac{(2t)^n}{n!} = \sum_{n=0}^{\infty}\frac{H_n(z)}{n!}t^n$$

which is the generating function relation (2.16) for Hermite polynomials.

4.6 Limit relation between hypergeometric representations

Recall the hypergeometric representation (3.21) of the q-Hermite I polynomials:

$$h_{n,q}(x) = x^n\sum_{k=0}^{\infty}\frac{\left(q^{-n};q^2\right)_k\left(q^{-n+1};q^2\right)_k}{\left(q^2;q^2\right)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{-1}\left(\frac{q^{2n-1}}{x^2}\right)^k.$$

Let $x = \sqrt{1-q^2}z$ and divide both sides by $(1-q^2)^{n/2}$ to obtain

$$\begin{aligned}\frac{h_{n,q}\left(\sqrt{1-q^2}z\right)}{(1-q^2)^{n/2}} &= z^n\sum_{k=0}^{\infty}\frac{\left(q^{-n};q^2\right)_k\left(q^{-n+1};q^2\right)_k}{\left(q^2;q^2\right)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{-1}\left(\frac{q^{2n-1}}{(1-q^2)z^2}\right)^k\\ &= z^n\sum_{k=0}^{\infty}\frac{(q^{-n};q^2)_k}{(1-q^2)^k}\frac{(q^{-n+1};q^2)_k}{(1-q^2)^k}\frac{(1-q^2)^k}{(q^2;q^2)_k}q^{(2n-1)k-\binom{k}{2}}\left(\frac{-1}{z^2}\right)^k.\end{aligned}$$

Taking the limit as $q\to 1$, using (4.3) and (1.17) with q replaced with q^2, we derive from the above equality that

$$\frac{H_n(z)}{2^n} = z^n\sum_{k=0}^{\infty}\frac{\left(\frac{-n}{2}\right)_k\left(\frac{-n+1}{2}\right)_k}{(1)_k}\left(\frac{-1}{z^2}\right)^k$$

or

$$H_n(z) = (2z)^n\sum_{k=0}^{\infty}\left(\frac{-n}{2}\right)_k\left(\frac{-n+1}{2}\right)_k\frac{\left(-z^{-2}\right)^k}{k!} = (2z)^n\,{}_2F_0\left(\left.\begin{matrix}\frac{-n}{2},\frac{1-n}{2}\\ -\end{matrix}\right| -\frac{1}{z^2}\right)$$

which is the hypergeometric representation of the Hermite polynomials given by (2.18).

4.7 Limit relation between the orthogonalities

Use the substitution $x = z\sqrt{1-q^2}$ in the orthogonality relation (3.38):

$$\int_{-\frac{1}{\sqrt{1-q^2}}}^{\frac{1}{\sqrt{1-q^2}}} \rho_q\left(\sqrt{1-q^2}z\right) h_{n,q}\left(\sqrt{1-q^2}z\right) h_{m,q}\left(\sqrt{1-q^2}z\right)\sqrt{1-q^2}d_qx = \mathcal{M}_n^2\,\delta_{nm},$$

where $\mathcal{M}_n$ is the norm of $h_{n,q}(x)$ given in (3.39). Divide both sides by $(1-q^2)^{\frac{n+m+1}{2}}$:

$$\int_{-\frac{1}{\sqrt{1-q^2}}}^{\frac{1}{\sqrt{1-q^2}}} \rho_q\left(z\sqrt{1-q^2}\right)\frac{h_{n,q}\left(z\sqrt{1-q^2}\right)}{(1-q^2)^{\frac{n}{2}}}\frac{h_{m,q}\left(z\sqrt{1-q^2}\right)}{(1-q^2)^{\frac{m}{2}}}\,d_qz = \frac{\mathcal{M}_n^2}{(1-q^2)^{\frac{n+m+1}{2}}}\,\delta_{nm}.$$

Take the limit as $q \to 1$

$$\int_{-\infty}^{\infty} e^{-z^2}\frac{H_n(z)}{2^n}\frac{H_m(z)}{2^m}\,dz = \lim_{q\to 1}\frac{\mathcal{M}_n^2}{(1-q^2)^{\frac{n+m+1}{2}}}\,\delta_{nm}.$$

Since $\delta_{mn} = 0$ for $m \neq n$, the above relation can be written as

$$\int_{-\infty}^{\infty} e^{-z^2} H_n(z)H_m(z)\,dz = \lim_{q\to 1}\frac{2^{2n}\mathcal{M}_n^2}{(1-q^2)^{\frac{2n+1}{2}}}\,\delta_{nm}. \tag{4.5}$$

Now,

$$\begin{aligned}
\lim_{q\to 1}\frac{\mathcal{M}_n^2}{(1-q^2)^{\frac{2n+1}{2}}} &= \lim_{q\to 1}\frac{(1-q)(q;q)_n(q,-1,-q;q)_\infty}{(1-q^2)^{\frac{2n+1}{2}}}\\
&= \lim_{q\to 1}\frac{(1-q)(q;q)_n(q^2;q^2)_\infty(-1,q)_\infty}{(1-q^2)^{\frac{2n+1}{2}}}\\
&= \lim_{q\to 1}(1-q)\frac{(q;q)_n}{(1-q^n)}\frac{(q^2;q^2)_\infty(-1;q)_\infty}{(1+q)^n\sqrt{1-q^2}}\\
&= \lim_{q\to 1}(1-q)[n]_q!\,\frac{(q^2;q^2)_\infty(-1;q)_\infty}{(1+q)^n\sqrt{1-q^2}}.
\end{aligned}$$

From (1.18), one can see that

$$\Gamma_{q^2}\left(\frac{1}{2}\right) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sqrt{1-q^2}.$$

Therefore,

$$\begin{aligned}
\lim_{q\to 1}\frac{\mathcal{M}_n^2}{(1-q^2)^{\frac{2n+1}{2}}} &= \lim_{q\to 1}\,[n]_q!\,\frac{\Gamma_{q^2}\left(\frac{1}{2}\right)(q;q^2)_\infty(-1;q)_\infty}{(1+q)^{n+1}}\\
&= \frac{n!}{2^n}\lim_{q\to 1}\Gamma_{q^2}\left(\frac{1}{2}\right)(q;q^2)_\infty(-q;q)_\infty
\end{aligned}$$

where we have used the fact that $(-1;q)_\infty = 2(-q;q)_\infty$. Using Lemma 1.2.7 and Corollary 1.2.9 with $a = q$, one obtains $(q;q^2)_\infty(-q;q)_\infty = 1$. Also, from (1.19) one has

$$\lim_{q\to 1} \Gamma_{q^2}\left(\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Hence,

$$\lim_{q\to 1} \frac{\mathcal{M}_n^2}{(1-q^2)^{\frac{2n+1}{2}}} = \frac{n!\,\sqrt{\pi}}{2^n}$$

As a result, (4.5) gives us

$$\int_{-\infty}^{\infty} e^{-z^2} H_n(z)H_m(z)\,dz = 2^n\,n!\,\sqrt{\pi}\,\delta_{nm}$$

which is the orthogonality relation for the Hermite polynomials given by (2.36).

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