On the block functions generating the limit q-Lupaş operator*

Sofiya Ostrovska, Mehmet Turan

Atilim University, Department of Mathematics, Incek 06836, Ankara, Turkey
e-mail: sofia.ostrovska@atilim.edu.tr, mehmet.turan@atilim.edu.tr
Tel: +90 312 586 8211, Fax: +90 312 586 8091

Abstract

The limit q-Lupaş operator emerged in the study of the Lupaş q-analogue of the Bernstein operator. Afterward, it has been found that it is connected to numerical analysis, computer aided geometric design, probability theory and other areas. In this paper, properties of block functions generating the limit q-Lupaş operator have been investigated.

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1 Introduction

The first known q-analogue of the Bernstein operator was defined by A. Lupaş in [13]. Regrettably, the operators proposed by Lupaşs remained unnoticed for a long while due to the very limited availability of his article published in regional conference proceedings. The most popular q-generalization of the Bernstein polynomials which appeared 10 years after Lupaş's paper and independently from Lupaş, belongs to G. M. Phillips who constructed new polynomials known today as the q-Bernstein polynomials. It is worth mentioning that G. M. Phillips in his review [17] writes in regard to this matter: "Alexandru Lupaş (1942-2007) introduced a wonderful generalization of the Bernstein polynomials in a paper [13] published in 1987. The polynomial (7) (= the q-Bernstein polynomial) and rational function (10) (= Lupaş

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q-analogue) have much in common, and it is worth saying that these two operators were discovered independently. I regret not knowing about Alexandru Lupaş's operator until long after I had discovered mine and explored its properties."

The history cannot be reversed, but due to the progress in information technologies the operators defined by Lupaş have rightfully attracted attention of mathematicians and become an area of fruitful research. We refer to articles [1, 14, 15, 18, 19] and references therein. Along with research on the properties of the q-analogue, new applications of this operators been discovered. It turns out that this operator is closely related to numerical analysis, applied mathematics, probability theory and other disciplines, see, for example, [6, 7, 8, 10].

The present paper deals with properties of the functions generating the limit q-Lupaş operator. It was introduced in [15] as the limit for a sequence of the Lupaş q-analogues. It can be noticed that the operator is connected to the Heine q-distribution, q-boson operator calculus, the Valiron method of summation for divergent series and other subjects.

First, we recall the necessary notation which will be employed in the text. Let $q > 0$. For any $a \in \mathbb{C}$, as given in [4, Ch. 10], we denote:

$$(a;q)_0 := 1, \quad (a;q)_k := \prod_{s=0}^{k-1}(1-aq^s), \quad (a;q)_\infty := \prod_{s=0}^{\infty}(1-aq^s).$$

Definition 1.1. [15] Given $q \in (0,1)$ and $f : [0,1] \to \mathbb{C}$, the *limit q-Lupaş operator* of f is defined by

$$(\Lambda_q f)(x) = \sum_{k=0}^{\infty} f(1-q^k)\ell_k(x), \quad x \geqslant 0,$$

where the function

$$\ell_k(x) = \frac{q^{k(k-1)/2}x^k}{(q;q)_k(-x;q)_\infty} \tag{1.1}$$

is called the k-th *block function* generating the limit q-Lupaş operator.

Notice that, the functions (1.1) can be viewed as probabilities of the Heine distributions with parameter $\lambda = x/(1-q)$ which is a q-extension of the classical Poisson distribution, see [5]. It is worth mentioning that this distribution is log-concave, that is, for every $k \in \mathbb{N}$, one has $\ell_k^2(x) \geqslant \ell_{k-1}(x)\ell_{k+1}(x) > 0$. Such distributions have been studied by many authors from different angles and have shown to be of interest for applications. See, for example, [3].

In this work, new results concerning the block functions $\ell_k(x)$ will be presented. These results are important in describing the geometric properties of the q-Lupaş operators and estimating the distance between two such operators. By geometric properties of a linear operator in a Banach space or a more general topological vector space we mean such properties of its kernel and range as isometry/isomorphism to one of the classical Banach spaces (see [12]), closeness of the image, existence of a complement for the kernel and closure of the range. Such properties of operators are crucial for the study of the corresponding linear equations and ill-posed problems (see [9, 11]). Since Λ_q can be regarded as a linear operator acting in various function spaces on $[0, 1]$, it is important to study the mentioned properties of this operator. Naturally, the first step in this direction for the limit q-Lupaş operator is a thorough investigation of its building block functions. In the study of the limit q-Bernstein operator, the analysis of the block functions revealed some geometric properties of the uniformly discrete metric space of these operators [16].

On the other hand, the outcomes presented here are related to the probabilistic interpretation of the block functions, giving the most probable value depending on the parameter x and the probabilities that random variable following Heine's distribution takes values $\equiv 0 \pmod{m}, m \in \mathbb{N}$.

2 Results on the general behavior of the block functions

Prior to stating our first result, let us recollect that the q-derivative of a function f at $x \neq 0$ is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x}.$$

We start with the following auxiliary lemma related to a few properties of the block functions.

Lemma 2.1. *For each $k = 1, 2, \ldots,$*

(i) the functions $\ell_k(x)$ and $\ell_{k+1}(x)$ intersect at $x = 0$ and $x = \alpha_k := q^{-k} - q$. In addition, functions $\ell_0(x)$ and $\ell_1(x)$ intersect only at $x = \alpha_0 := 1 - q$;

(ii) the q-derivative of $\ell_k(x)$ vanishes at $x = \beta_k := q^{-k} - 1$;

(iii) the function $\ell_k(x)$ has a unique maximum at a point $\gamma_k \in (q\beta_k, \beta_k)$.

Proof. (i) Obviously, for $k = 1, 2, \ldots,$ one has $\ell_k(0) = 0$. Further, when $x > 0$ it is true that

$$\ell_k(x) = \ell_{k+1}(x) \Leftrightarrow 1 - q^{k+1} = q^k x \Leftrightarrow x = \alpha_k \quad \text{for } k = 0, 1, \ldots.$$

(*ii*) For $x > 0$, there holds

$$D_q \ell_k(x) = \frac{\ell_k(x) - \ell_k(qx)}{(1-q)x}.$$

Thus, $D_q \ell_k(x) = 0$ implies that

$$\frac{q^{\binom{k}{2}} x^k}{(q;q)_k(-x;q)_\infty} = \frac{q^{\binom{k}{2}} (qx)^k}{(q;q)_k(-qx;q)_\infty} \Leftrightarrow \frac{1}{1+x} = q^k \Leftrightarrow x = \beta_k.$$

Notice that $\alpha_k = q\beta_{k+1}$ for all $k = 0, 1, \ldots$.

(*iii*) As $\ell_k(x) = \frac{q^{\binom{k}{2}} x^k}{(q;q)_k} \prod_{j=0}^\infty \frac{1}{1+q^j x}$, we have

$$\frac{d}{dx}\ell_k(x) = \ell_k(x) \left[\frac{k}{x} - \sum_{j=0}^\infty \frac{q^j}{1+q^j x} \right] := \frac{\ell_k(x)}{x}[k - \varphi(x)]. \tag{2.1}$$

Thus, the critical points of ℓ_k are solutions of the equation $\varphi(x) = k$. Since φ is increasing with $\varphi(0) = 0$ and $\lim_{x\to\infty} \varphi(x) = +\infty$, there is a unique number γ_k such that $\varphi(x) = k$ and ℓ_k has a unique local (which is also global) maximum at $x = \gamma_k$. To finish the proof, notice that by part (*ii*) we have

$$\ell_k(q\beta_k) = \ell_k(\beta_k)$$

which implies that $\gamma_k \in (q\beta_k, \beta_k)$. □

As an application of this result, we obtain the following statement related to finding the most probable value of the Heine distribution.

Theorem 2.2. *For each $k = 0, 1, \ldots$ and $x \in [\alpha_{k-1}, \alpha_k]$, the following statement is true:*

$$\max_{j\geqslant 0} \ell_j(x) = \ell_k(x),$$

where by tacit agreement $\alpha_{-1} = 0$.

Proof. From the asymptotic behavior as $x \to 0^+$ and the definition of α_k, it can be derived that

$$\ell_{k+1}(x) < \ell_k(x) \quad \text{for all} \quad x \in (0, \alpha_k).$$

Since $\ell_{k-1}(x)$ and $\ell_k(x)$ intersect only at 0 and α_{k-1}, it follows from the asymptotic behavior as $x \to \infty$ that

$$\ell_{k-1}(x) < \ell_k(x) \quad \text{for all} \quad x \in (\alpha_{k-1}, \infty).$$

Therefore, whenever $j > k$,

$$\ell_j(x) < \ell_k(x) \quad \text{for all} \quad x \in (0, \alpha_k),$$

meanwhile, for $j < k$,

$$\ell_j(x) < \ell_k(x) \quad \text{for all} \quad x \in (\alpha_{k-1}, \infty).$$

Combining the last two inequalities yields that, as long as $j \neq k$,

$$\ell_j(x) < \ell_k(x) \quad \text{for all} \quad x \in (\alpha_{k-1}, \alpha_k).$$

By continuity argument, we obtain the statement. □

Corollary 2.3. *Let X be a random variable possessing the Heine distribution with parameter λ. If $\lambda = \alpha_k/(1-q)$, $k \in \mathbb{N}_0$, then there are two most probable values, namely, k and $k+1$. Otherwise, if $\alpha_{k-1}/(1-q) < \lambda < \alpha_k/(1-q)$, then there is just one most probable value, k.*

Theorem 2.4. *Consider the sequence $\{\ell_k(\gamma_k)\}_{k=0}^{\infty}$, where as described above γ_k is the point at which ℓ_k attains its maximum. Then, $\{\ell_k(\gamma_k)\}_{k=0}^{\infty}$ is strictly decreasing.*

Proof. The proof is divided into several steps which are of interest for their own sake.

Step 1. For all $x \in (0, \beta_k)$, there holds $\ell_k(x) < \ell_{k-1}(qx)$.

Indeed,

$$\frac{\ell_k(x)}{\ell_{k-1}(qx)} = \frac{q^{\binom{k}{2}} x^k}{(q;q)_k(-x;q)_\infty} \frac{(q;q)_{k-1}(-qx;q)_\infty}{q^{\binom{k-1}{2}}(qx)^{k-1}} = \frac{1}{1-q^k}\frac{x}{1+x}.$$

As the function $x \mapsto x/(1+x)$ is increasing on $(0, \infty)$, for $x \in (0, \beta_k)$, we get

$$\frac{\ell_k(x)}{\ell_{k-1}(qx)} < \frac{1}{1-q^k}\frac{\beta_k}{1+\beta_k} = \frac{q^{-k}-1}{(1-q^k)q^{-k}} = 1.$$

Step 2. Let γ_k be the critical point of ℓ_k. Then $q\gamma_{k+1} > \gamma_k$.

Recall that by virtue of (2.1), one has $\varphi(\gamma_k) = k$. Now,

$$\begin{aligned}\varphi(q\gamma_{k+1}) &= \sum_{j=0}^{\infty} \frac{q^{j+1}\gamma_{k+1}}{1+q^{j+1}\gamma_{k+1}} = \sum_{j=1}^{\infty} \frac{q^j\gamma_{k+1}}{1+q^j\gamma_{k+1}} \\ &= \varphi(\gamma_{k+1}) - \frac{\gamma_{k+1}}{1+\gamma_{k+1}} = k+1-\frac{\gamma_{k+1}}{1+\gamma_{k+1}}\end{aligned}$$

$$= \varphi(\gamma_k) + \frac{1}{1+\gamma_{k+1}} > \varphi(\gamma_k).$$

As φ is an increasing function, we get $q\gamma_{k+1} > \gamma_k$ as desired.

Step 3. By Lemma 2.1 (iii), $\gamma_k \in (q\beta_k, \beta_k)$. Using $x = \gamma_k$ in Step 1, we obtain

$$\ell_k(\gamma_k) < \ell_{k-1}(q\gamma_k) < \ell_{k-1}(\gamma_{k-1}).$$

The second inequality is strict due to the assertion of Step 2 and the selection of γ_{k-1}. □

3 Results on the sums of the block functions

It can be readily seen that $\ell(x) \geqslant 0$ for $x \geqslant 0$ and, by virtue of Euler's identity

$$\sum_{k=0}^{\infty} \ell_k(x) = 1.$$

For this identity we refer to [4, Ch.10, Section 10.2, Corollary 10.2.2]:

$$(-x;q)_\infty = \sum_{k=0}^{\infty} \frac{q^{k(k-1)/2}}{(q;q)_k} x^k, \quad |q| < 1.$$

In this section, we study some properties of the sums

$$S_m(x) := \sum_{k=0}^{\infty} \ell_{mk}(x). \tag{3.1}$$

As it has been noticed, $S_1(x) \equiv 1$ for all $x \geqslant 0$. Following [20], we say that a function $f(x)$ is q-periodic with period q^a if $f(q^a x) = f(x)$ for all $x > 0$. The next statement demonstrates asymptotic q-periodicity of the sums $S_m(x)$.

Theorem 3.1. *Let $S_m(x)$ be given by (3.1). Then, as $x \to +\infty$, $S_m(x) \sim h_m(x)$, where $h_m(x)$ is q-periodic with period q^m.*

First, we prove two auxiliary lemmas.

Lemma 3.2. *Let $f(x) = \sum_{k=0}^{\infty} a_k x^k$ be a transcendental entire function with non-negative cofficients and $\{\rho_k\} \to \rho$. Then*

$$\sum_{k=0}^{\infty} \rho_k a_k x^k \sim \rho f(x) \ \text{ as } \ x \to +\infty.$$

Proof. Since $\rho_k = \rho + \delta_k$, where $\delta_k = o(1)$, $k \to \infty$, it suffices to show that

$$\sum_{k=0}^{\infty} \delta_k a_k x^k = o(f(x)),\ x \to \infty.$$

Given $\varepsilon > 0$, opt for $N = N(\varepsilon)$ such that $|\delta_k| < \varepsilon$ whenever $k > N$. Consequently,

$$\left| \sum_{k=0}^{\infty} \delta_k a_k x^k \right| \leqslant \sum_{k=0}^{N} \delta_k a_k x^k + \varepsilon \sum_{k=N+1}^{\infty} a_k x^k =: P_N(x) + R_N(x),$$

whence

$$\left| \frac{\sum_{k=0}^{\infty} \delta_k a_k x^k}{f(x)} \right| \leqslant \frac{P_N(x)}{f(x)} + \varepsilon \frac{\sum_{k=N+1}^{\infty} a_k x^k}{f(x)}. \tag{3.2}$$

Since $f(x)$ is transcendental and $\max_{|z| \leqslant x} |f(z)| = f(x)$, by the Liouville Theorem, $P(x) = o(f(x))$ as $x \to +\infty$ for any polynomial P and, as such, for $P_N(x)$. Hence, for $x > 0$ large enough, the first term in (3.2) is $< \varepsilon$. Meanwhile, the ratio $\left(\sum_{k=N+1}^{\infty} a_k x^k\right) / f(x) \leqslant 1$ for all $x > 0$, and, therefore, (3.2) yields:

$$\left| \frac{\sum_{k=0}^{\infty} \delta_k a_k x^k}{f(x)} \right| \leqslant 2\varepsilon \quad \text{for} \quad x > 0 \quad \text{large enough}.$$

As a result, one has $\sum_{k=0}^{\infty} \delta_k a_k x^k = o(f(x)), x \to +\infty$, as required.

□

Lemma 3.3. *For S_m defined by (3.1) and $C_{m,q} = (q^{m^2}; q^{m^2})_\infty / (q;q)_\infty$, the following asymptotic relation holds:*

$$S_m(x) \sim C_{m,q} \cdot \frac{\left(-q^{m(m-1)/2} x^m; q^{m^2}\right)_\infty}{(-x;q)_\infty}, \quad x \to +\infty.$$

Proof. Writing

$$\begin{aligned} S_m(x) &= \frac{1}{(-x;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{mk(mk-1)/2} x^{mk}}{(q;q)_{mk}} \\ &= \frac{1}{(-x;q)_\infty} \sum_{k=0}^{\infty} \frac{q^{m^2 k(k-1)/2} \left(q^{m(m-1)/2} x^m\right)^k}{(q^{m^2}; q^{m^2})_k} \cdot \frac{(q^{m^2}; q^{m^2})_k}{(q;q)_{mk}} \end{aligned}$$

and bearing in mind that

$$\left\{ \frac{(q^{m^2}; q^{m^2})_k}{(q;q)_{mk}} \right\} \to \frac{(q^{m^2}; q^{m^2})_\infty}{(q;q)_\infty} = C_m,$$

it can be derived from Lemma 3.2 that

$$\sum_{k=0}^{\infty} \frac{q^{mk(mk-1)/2} x^{mk}}{(q;q)_{mk}} \sim C_m \left(-q^{m(m-1)/2} x^m; q^{m^2}\right)_\infty .$$

The statement now follows. □

Proof of Theorem 3.1. Consider the function

$$f(x;q) = \prod_{j=0}^{\infty} \left(1 + q^j x\right) \cdot \prod_{j=1}^{\infty} \left(1 + q^j/x\right).$$

It is known that $f(x;q)$ satisfies the following functional equation:

$$f(x/q;q) = (x/q) f(x;q), \tag{3.3}$$

see, e.g. [2, Ch. 4, §18].

Replacing x by x/q^{m-1} in (3.3) yields

$$f(x/q^m;q) = (x/q^m) f(x/q^{m-1};q), \quad m \in \mathbb{N}.$$

Applying the last equality recursively, it can be readily seen that

$$f\left(\frac{x}{q^m};q\right) = \frac{x^m}{q^{m(m+1)/2}} f(x;q), \quad m \in \mathbb{N}. \tag{3.4}$$

Set $g_m(x;q) := f\left(q^{m(m-1)/2} x^m; q^{m^2}\right)$. Then,

$$g_m\left(\frac{x}{q^m};q\right) = f\left(\frac{q^{m(m-1)/2} x^m}{q^{m^2}}; q^{m^2}\right) = \frac{x^m}{q^{m(m+1)/2}} \cdot g_m(x;q). \tag{3.5}$$

In the last step, we use equation (3.3) with $q \mapsto q^{m^2},\ x \mapsto q^{m(m-1)/2} x$.

Now, for

$$h_m(x;q) := C_{m,q} \frac{g_m(x;q)}{f(x;q)},$$

one has, using (3.4) and (3.5):

$$h_m(x/q^m;q) = C_{m,q} \cdot \frac{(x^m/q^{m(m+1)/2}) g_m(x;q)}{(x^m/q^{m(m+1)/2}) f(x;q)} = h_m(x;q).$$

This shows that $h_m(x;q)$ is q-periodic with a q-period q^m. Obviously, $f(x;q) \sim (-x;q)_\infty$ as $x \to +\infty$ because $(-q/x;q)_\infty \to 1$ as $x \to +\infty$. Likewise, $g_m(x;q) \sim (-q^{m(m-1)/2} x^m; q^{m^2})_\infty$, $x \to +\infty$. Consequently, $S_m(x) \sim h_m(x)$ as $x \to +\infty$ and the theorem is proved. □

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