EXISTENCE OF SOLUTIONS FOR HIGHER ORDER MULTI-POINT IMPULSIVE BOUNDARY VALUE PROBLEMS ON TIME SCALES

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I certify that this thesis satisfies all the requirements as a thesis for the degree of **Master of Science in Mathematics Department, Atilim University**.

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ABSTRACT

EXISTENCE OF SOLUTIONS FOR HIGHER ORDER MULTI-POINT IMPULSIVE BOUNDARY VALUE PROBLEMS ON TIME SCALES

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In this thesis, we investigate the sufficient conditions for existence of solutions of multi-point higher order impulsive boundary value problems on time scales. In particular, a class of third order impulsive boundary value problem and a class of $2n+1$, $n \geq 1$, order impulsive boundary value problem is studied. In chapter 1, we give the definitions and basic notions on time scales calculus. Then, we present some examples and give the fixed point theorems that are used in the thesis. Chapter 2 is devoted to existence of solutions of third order multi-point dynamic impulsive boundary value problems. In chapter 3, we focus on existence of solutions of multi-point dynamic impulsive boundary value problems of odd order. Finally, we give a short conclusion in Chapter 4. The results in this thesis are published/accepted for publication in the Georgian Mathematical Journal and Miskolc Mathematical Notes, respectively.

Keywords: impulsive dynamic equation, multi-point, boundary value problem, time scales, higher order.

ÖZ

ZAMAN SKALALARINDA YÜKSEK MERTEBEDEN ÇOK NOKTALI İMPALSİF SINIR DEĞER PROBLEMLERİNİN ÇÖZÜMLERİNİN VARLIĞI

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Bu tezde, çok noktalı yüksek mertebeden impalsif sınır değer problemlerinin zaman skalalarında çözümlerinin bulunması için yeterli koşulları araştırdık. Özellikle, üçüncü mertebeden impalsif sınır değer problemlerinin bir sınıfı ve $2n+1,\ n \geq 1$ mertebeden bir impalsif sınır değer problemi sınıfı incelenmiştir. Bölüm 1'de zaman skalası ve bazı ilgili kavramların tanımları ile birlikte örnekler verilmiştir. Sonrasında tezde kullanılan sabit nokta teoremleri verilmiştir. Bölüm 2, üçüncü mertebeden çok noktalı dinamik impalsif sınır değer problemlerinin çözümlerinin varlığına ayrılmıştır. Bölüm 3'de tek sayı mertebeli çok noktalı dinamik impalsif sınır değer problemlerinin çözümlerinin varlığına odaklanılmıştır. Son olarak, Bölüm 4'te kısa bir sonuç verilmiştir. Bu tezdeki sonuçların bir kısmı Georgian Mathematical Journal dergisinde basılmış, bir kısmı da Miskolc Mathematical Notes dergisinde basılmak üzere kabul edilmiştir.

Anahtar Kelimeler: impalsif dinamik denklem, çok nokta, sınır değer problemi, zaman skalası, yüksek mertebe.

To my dearest parents, my wife and my lovely daughter

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LIST OF SYMBOLS

$\mathbb{N}$: the set of natural numbers

$\mathbb{N}_0$: the set of non-negative integers

$\mathbb{Z}$: the set of integers

$\mathbb{R}$: the set of real numbers

$\mathbb{T}$: a time scale

f^{Δ} : delta derivative of f on time scale

Δf : impulse operator applied on f

CHAPTER 1

INTRODUCTION

The discrete-time systems are as necessary as continuous time systems, and as it is well-known that differential equations and difference equations are very adequate tools to model many processes occurring in applied sciences. However, there are several real phenomena encountered in various fields such as logistics, population dynamics, biology, electrical engineering, physics, neural networks [5, 6, 7, 13] which cannot be modeled using only continuous or only discrete dynamical systems. As they contain both continuous and discrete data, such models require simultaneous use of both. The time scales theory projected by Stefan Hilger [16] in 1988 unifies the study of both. Since then, it has been used intensely by many researchers working in different areas to produce solutions for the modeling problems mentioned above. This caused the eye of researchers within the literature. For practical examples and deep knowledge about the qualitative theory of dynamic systems on time scales, we refer the reader to the books [5, 6, 12, 17, 20].

Several processes studied in applied sciences are painted by differential equations. But, it is more likely for many physical phenomena to have sudden changes in their states, and so, ordinary differential equations become inadequate to model them. The appropriate tools for modelling such problems are impulsive differential equations. Recently, the plurality of applications has a fast result on differential equations involving impulse effects, and, in the last decades major developments have been made in the theory of impulsive differential equations since they give a natural description of many evolution phenomena in real life. The well-known books [3, 19, 26] are excellent sources for readers interested in impulsive differential equations.

In this thesis, combining the impulsive differential equations with the time scales, the higher order dynamic impulsive equations on time scales are considered. In particular, the existence of solutions for some types of impulsive boundary value problems (IBVPs) on time scales with multi-point boundary conditions is studied. By applying the well-known fixed point theorems, the existence of solutions is proved. Examples are also provided to show the efficiency of the results.

The thesis is organized as follows. In the present chapter, we introduce notions of basic calculus on time scales, the definition of impulsive differential equations and some theorems related to the fixed point theory. In Chapter 2, we show the existence of solutions to third order multi-point IBVPs on time scales. In Chapter 3, we discuss the existence of solutions to odd order multi-point IBVPs on time scales. Finally, we give a short conclusion in Chapter 4.

1.1 CALCULUS ON TIME SCALES

The definitions and the results in this section can be found in [5, 6, 13] and [7].

1.1.1 Basic Definitions

Definition 1.1.1 ([5]) *A time scale $\mathbb{T}$ is any non-empty closed subset of the real numbers. Some very well known examples are $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{N}_0$ and the Cantor set. The intervals $(1,2)$, $[1,3)$ or $(-1,0]$ are not time scales since they are not closed.*

We define next some basic concepts on time scales.

Definition 1.1.2 ([5]) *Let $\mathbb{T}$ be a time scale.*

i) Forward jump operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ is defined by

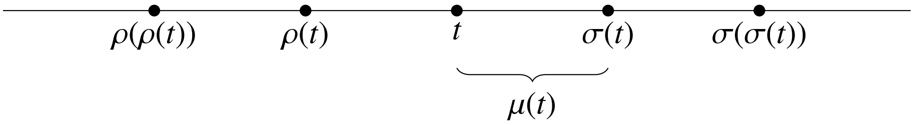
$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad t \in \mathbb{T}.$$

ii) Backward jump operator $\rho\colon \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad t \in \mathbb{T}.$$

iii) The graininess function $\mu\colon \mathbb{T} \to [0,\infty)$ *is defined by*

$$\mu(t) = \sigma(t) - t, \qquad t \in \mathbb{T}.$$



Remark 1.1.3 *We note that* $\sigma(t) \geq t$ *and* $\rho(t) \leq t$*, for any* $t \in \mathbb{T}$.

Remark 1.1.4 *We assume that* $\sup\emptyset = \inf\mathbb{T}$, $\inf\emptyset = \sup\mathbb{T}$, *where* $\emptyset$ *denotes the empty set.*

Definition 1.1.5 ([5]) *Let* $\mathbb{T}$ *be a time scale with forward jump and backward jump operators* σ *and* ρ *respectively. Then*

- *t is right scattered if* $\sigma(t) > t$,
- *t is right dense if* $t < \sup\mathbb{T}$ *and* $\sigma(t) = t$,
- *t is left scattered if* $\rho(t) < t$,
- *t is left dense if* $t > \inf\mathbb{T}$ *and* $\rho(t) = t$,
- *t is isolated if it is both left and right scattered.*

As illustrated by the figure above :

- t_1 is a dense point;
- t_2 is a left dense and right scattered point;
- t_3 is an isolated point;
- t_4 is a left scattered and right dense point.

The following examples illustrate the above concepts on various time scales :

Example 1.1.6 *Let* $\mathbb{T} = \{\sqrt{n} : n \in \mathbb{N}_0\}$.

For any $t \in \mathbb{T}$, $t = \sqrt{n}$ and we compute

$$\begin{aligned} \sigma(t) = \inf\{s \in \sqrt{\mathbb{N}_0} : s > t\} &= \inf\{\sqrt{s} : \sqrt{s} > \sqrt{n}\} \\ &= \sqrt{n+1} = \sqrt{t^2+1}. \end{aligned}$$

Then, we have $\sigma(t) = \sqrt{t^2+1}$.

For $t = 0$ we compute $\rho(0) = \sup\{\sqrt{k} : \sqrt{k} < 0\} = \sup\emptyset = \inf\mathbb{T} = 0$

For $t > 0$ we have $t = \sqrt{n}$, where $n \geq 1$ and

$$\rho(t) = \sup\{\sqrt{s} : \sqrt{s} < \sqrt{n}\} = \sqrt{n-1} = \sqrt{t^2-1}.$$

Then,

$$\rho(t) = \begin{cases} \sqrt{t^2-1} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

and

$$\mu(t) = \sigma(t) - t = \sqrt{t^2+1} - t.$$

Example 1.1.7 ([5]) *Let $N(t)$ be the number of plants of one particular kind at time t in a certain area. By experiments it is known that N grows exponentially according to the rule $N' = N$ during the months from February to July. At the beginning of August, all plants suddenly die, but the seeds remain in the ground and start growing again at the beginning of February with N now being doubled. This situation can be modeled using the time scale*

$$\mathbb{T} = P_{1,1} = \bigcup_{k=0}^{\infty} [2k, 2k+1],$$

where $t = 0$ is February 1 of the current year, $t = 1$ is August 1 of the current year, $t = 2$ is February 1 of the next year, $t = 3$ is August 1 of the next year, and so on. We have

$$\sigma(t) = \begin{cases} t, & \text{if } 2k \le t < 2k+1 \\ t+1, & \text{if } t = 2k+1, \end{cases}$$

and

$$\mu(t) = \begin{cases} 0, & \text{if } 2k \le t < 2k+1 \\ 1, & \text{if } t = 2k+1. \end{cases}$$

1.1.2 Differentiation

In this section, we give a definition for Hilger derivative and we list some of its properties.

Definition 1.1.8 ([5]) *If $\mathbb{T}$ has a left scattered maximum m, then we define $\mathbb{T}^{\kappa} = \mathbb{T} - \{m\}$. Otherwise, we define $\mathbb{T}^{\kappa} = \mathbb{T}$. In other words,*

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \mathbb{T}. \end{cases}$$

Definition 1.1.9 ([5]) *If $f\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$, we define $f^{\Delta}(t)$ to be the number with the property that given any $\epsilon > 0$ there is a neighborhood U of t such that*

$$\left|[f(\sigma(t)) - f(s)] - f^{\Delta}[\sigma(t) - s]\right| \le \epsilon\, |\sigma(t) - s|$$

for all $s \in U$. Here, $f^{\Delta}(t)$ is called the delta derivative of f at t, and f is called delta differentiable in $\mathbb{T}^{\kappa}$ provided that $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

Theorem 1.1.10 ([5]) *Let $f\colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. Then, we have*

(i) if f is delta differentiable at t, then f is continuous at t,

(ii) if f is continuous at t and t is right-scattered ($\sigma(t) > t$), then f is delta differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)},$$

(iii) if t is right-dense, then f is differentiable at t iff the following limit exists

$$\lim_{s\to t}\frac{f(t)-f(s)}{t-s},$$

(iv) if f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).$$

Example 1.1.11 *Let $q > 1$ and $\mathbb{T} = \{q^k : k \in \mathbb{Z}\}$ and $f(t) = t^2$ Then $f^{\Delta}(t) = qt + t$.*

We compute $f^{\Delta}(t)$ as follows :

$$\begin{aligned} f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} &= \frac{(qt)^2 - t^2}{qt - t} \\ &= \frac{t^2(q^2-1)}{t(q-1)} \\ &= t(q+1) = qt + t. \end{aligned}$$

Theorem 1.1.12 ([5]) *Suppose $f, g \colon \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^{\kappa}$. Then,*

(i) $(f \pm g)^{\Delta}(t) = f^{\Delta}(t) \pm g^{\Delta}(t)$,

(ii) for any constant $a \in \mathbb{R}$, $(af)^{\Delta}(t) = af^{\Delta}(t)$,

(iii) $(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$,

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $\frac{1}{f}$ is differentiable at t with

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))},$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentiable at t with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Example 1.1.13 *Let a be constant and $m \in \mathbb{N}$. For $f(t) = (t-a)^m$ we have*

$$f^{\Delta}(t) = \sum_{\nu=0}^{m-1}(\sigma(t)-a)^{\nu}(t-a)^{m-1-\nu}.$$

For a = 0, we get

$$f^{\Delta}(t)=\begin{cases}\displaystyle\sum_{v=0}^{m-1}(\sigma(t))^{v}(t)^{m-1-v}=1 & \text{if } m=1,\\ \displaystyle\sum_{v=0}^{m-1}(\sigma(t))^{v}(t)^{m-1-v}=\sigma(t)+t & \text{if } m=2\ .\end{cases} \tag{1.1}$$

Example 1.1.14 *Let a be constant and* $m\in\mathbb{N}$, $f(t)=(t-a)^m$ *and*

$$f^{\Delta}(t)=\sum_{v=0}^{m-1}(\sigma(t)-a)^{v}(t-a)^{m-1-v}.$$

$g(t)=\dfrac{1}{f(t)}$, $t\in\mathbb{T}$. *Using Example* 1.1.13, *we compute* $g^{\Delta}(t)$ *as follows:*

$$\begin{aligned} g^{\Delta}(t) &= -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}\\ &= -\frac{\displaystyle\sum_{v=0}^{m-1}(\sigma(t)-a)^{v}(t-a)^{m-1-v}}{(t-a)^{m}(\sigma(t)-a)^{m}}\\ &= -\sum_{v=0}^{m-1}\frac{1}{(\sigma(t)-a)^{m-v}(t-a)^{v+1}},\quad t\in\mathbb{T}. \end{aligned}$$

Example 1.1.15 *Let* $\mathbb{T}=\{n^3 : n\in\mathbb{N}_0\}$, $f(t)=t^2+2t$, $t\in\mathbb{T}$. *We will find* $f^{\Delta}(t)$, $t\in\mathbb{T}$.

For $t\in\mathbb{T}$, $t=n^3$, $n\in\mathbb{N}_0$, $n=\sqrt[3]{t}$ we have

$$\sigma(t)=\inf\{l^3 : l^3>n^3\}=(n+1)^3=(\sqrt[3]{t}+1)^3.$$

Therefore, all points of $\mathbb{T}$ are right scattered. Moreover, we have

$$f^{\Delta}(t)=(t^2+2t)^{\Delta}=(t^2)^{\Delta}+(2t)^{\Delta}=\sigma(t)+t+2=(\sqrt[3]{t}+1)^3+t+2,\quad t\in\mathbb{T}.$$

Definition 1.1.16 *For a function* $f:\mathbb{T}\to\mathbb{R}$, *we define* $f^{\sigma}:\mathbb{T}\to\mathbb{R}$ *by* $f^{\sigma}(t)=f(\sigma(t))$.

1.1.3 Integration

In this section, we introduce Hilger integral on arbitrary time scale and provide some of its properties.

Definition 1.1.17 ([5]) *A function* $f\colon \mathbb{T} \to \mathbb{R}$ *is called regulated provided that its right-sided limits exist at all right-dense points in* $\mathbb{T}$ *and its left-sided limits exist at all left-dense points in* $\mathbb{T}$*.*

Definition 1.1.18 ([5]) *A function* $f\colon \mathbb{T} \to \mathbb{R}$ *is called rd-continuous if it is continuous at right-dense points of* $\mathbb{T}$ *and its left limits exist (finite) at left-dense points in* $\mathbb{T}$*. The set of rd-continuous functions is denoted as*

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

Definition 1.1.19 ([5]) *A continuous function* $f\colon \mathbb{T} \to \mathbb{R}$ *is called pre-differentiable with (region of differentiation) D, provided* $D \subset \mathbb{T}^{\kappa}$*,* $\mathbb{T}^{\kappa} \backslash D$ *is countable and contains no right-scattered elements of* $\mathbb{T}$*, and* f *is differentiable at each* $t \in D$*.*

Theorem 1.1.20 ([5]) *Suppose* $f\colon \mathbb{T} \to \mathbb{R}$ *is regulated function. Then there is a function F which is pre-differentiable with region of differentiation D such that*

$$F^{\Delta}(t) = f(t)$$

for all $t \in D$*.*

The function F in Theorem 1.1.20 is called pre-antiderivative of f.

Definition 1.1.21 ([5]) *The indefinite integral of a regulated function* f *is defined as*

$$\int f(t)\Delta t = F(t) + C,$$

where C is an arbirary constant.

Below we will list some of the properties of the Cauchy integral.

Theorem 1.1.22 ([5]) *Let* $a, b, c \in \mathbb{T}$*,* $\alpha \in \mathbb{R}$ *and* $f, g \in C_{rd}$*. Then, we have the following*

(i) $\displaystyle\int_a^b [f(t) \pm g(t)]\Delta t = \int_a^b f(t)\Delta t \pm \int_a^b g(t)\Delta t,$

(ii) $\displaystyle\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t,$

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t,$

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t,$

(v) $\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t,$

(vi) $\int_a^b f^{\Delta}(t)g(t)\Delta t = (fg)(b) - (fg)(a) - \int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t,$

(vii) $\int_a^a f(t)\Delta(t) = 0,$

(viii) *if* $|f(t)| \leq g(t)$ *on* $[a,b)$, *then* $\left|\int_a^b f(t)\Delta t\right| \leq \int_a^b g(t)\Delta t,$

(ix) *if* $f(t) \geq 0$ *for all* $a \leq t < b$, *then* $\int_a^b f(t)\Delta t \geq 0,$

(x) *if* $t \in \mathbb{T}^K$, *then* $\int_t^{\sigma(t)} f(t)\Delta(t) = \mu(t)f(t).$

1.1.4 Exponential function on time scales

In this section, we introduce the Hilger exponential function and we give some of its properties.

Definition 1.1.23 ([5]) *If we define the "circle plus" addition* $\oplus$ *on* $\mathbb{C}_h$ *by*

$$z \oplus w = z + w + zwh,$$

then $(\mathbb{C}_h, \oplus)$ *is an Abelian group.*

Definition 1.1.24 ([5]) *The additive inverse of z under the operation* $\oplus$ *is*

$$\ominus z = \frac{-z}{1+zh}$$

Definition 1.1.25 ([5]) *We say that a function* $p\colon \mathbb{T} \to \mathbb{R}$ *is regressive provided*

$$1 + \mu(t)p(t) \neq 0 \quad for\, all \quad t \in \mathbb{T}^{\kappa}$$

holds. The set of all regressive and rd-continuous functions $f\colon \mathbb{T} \to \mathbb{R}$ *will be denoted by* $\mathcal{R}$.

Definition 1.1.26 ([5]) *If $p \in \mathcal{R}$, then we define the exponential function by*

$$e_p(t,s) = exp\left(\int_s^t \xi_{\mu(\mathcal{T})}(p(\mathcal{T}))\Delta\mathcal{T}\right),$$

where

$$\xi_h(z) = \begin{cases} \dfrac{\log(1+hz)}{h} & if\, h \neq 0 \\ z & if\, h = 0 \end{cases}$$

is the cylinder transformation for $s,t \in \mathbb{T}$.

Example 1.1.27 *Let $\mathbb{T} = q^k$, $q > 1$. Every point of $\mathbb{T}$ is isolated and $\mu(t) = (q-1)t$ for all $t \in \mathbb{T}$. Then, for $t, t_0 \in \mathbb{T}$, $t > t_0$, we get*

$$\begin{aligned} e_\alpha(t,t_0) &= exp\left(\int_{t_0}^t \frac{1}{\mu(\tau)} Log(1+\alpha(\tau)\mu(\tau))\Delta\mathcal{T}\right) \\ &= exp\left(\sum_{s\in[t_0,t)} \frac{1}{\mu(s)} Log(1+\alpha(s)\mu(s))\mu(s)\right) \\ &= exp\left(\sum_{s\in[t_0,t)} Log(1+\alpha(s)\mu(s))\right) \\ &= exp\left(\sum_{s\in[t_0,t)} Log(1+(q-1)s\alpha(s))\right) \\ &= \prod_{s\in[t_0,t)} \Big(1+(q-1)s\alpha(s)\Big). \end{aligned}$$

Theorem 1.1.28 ([5]) *If $p, q \in \mathcal{R}$, then*

(i) $e_0(t,s) = 1$ *and* $e_p(t,t) = 1$,

(ii) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(t,s)$,

(iii) $e_p(t,u)e_p(u,s) = e_p(t,s)$,

(iv) $e_p^{\Delta}(t,t_0) = p(t)e_p(t,t_0)$ *for* $t \in \mathbb{T}^\kappa$ *and* $t_0 \in \mathbb{T}$,

(v) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$.

Example 1.1.29 ([5]) *Let us consider the model introduced in Example 1.1.7.*

We know that $N(2k+2) = 2N(2k+1)$, $N^{\Delta} = N$ *at* $2k+1$. *Thus, if* $N(0) = 1$ *is given,* N *is exactly* $e_1(\cdot,0)$ *on the time scale* $\mathbb{T}$. *We can calculate* N *as follows: if* $k \in \mathbb{N}_0$ *and* $t \in [2k, 2k+1]$, *then* N *satisfies* $N' = N$ *so that* $N(t) = \alpha_k e^t$ *for some* $\alpha_k \in \mathbb{R}$.

Since N(0)=1, we have

$$1 = N(0) = \alpha_0 e^0 \quad and \quad N(t) = \alpha_0 e^t = e^t \quad for \quad 0 \leq t \leq 1.$$

Thus, $N(1) = e$ *and* $N(2) = 2N(1) = 2e$. *Now,*

$$2e = N(2) = \alpha_1 e^2 \quad and \quad N(t) = \alpha_1 e^t = \frac{2}{e} e^t = 2e^{t-1} \quad for \quad 2 \leq t \leq 3.$$

Hence, $N(3) = 2e^2$ *and* $N(4) = 2N(3) = 4e^2$. *Next,*

$$4e^2 = N(4) = \alpha_2 e^4 \quad and \quad N(t) = \alpha_2 e^t = \frac{4}{e^2} e^t = 4e^{t-2} \quad for \quad 4 \leq t \leq 5.$$

We now use mathematical induction to show that

$$N(t) = \left(\frac{2}{e}\right)^k e^t \quad for \quad t \in [2k, 2k+1].$$

The statement is already shown for $k = 0$. *Assume it is true for* $k = m$, $m \in \mathbb{N}_0$. *Then* $N(t) = (2/e)^m e^t$ *for* $t \in [2m, 2m+1]$ *so that*

$$N(2m+1) = \left(\frac{2}{e}\right)^m e^{m+1} = 2^m e^{m+1}$$

and

$$N(2m+2) = 2N(2m+1) = 2 \cdot 2^m e^{m+1} = (2e)^{m+1}.$$

Therefore

$$(2e)^{m+1} = N(2m+2) = \alpha_{m+1} e^{2m+2}$$

and

$$N(t) = \alpha_{m+1} e^t = \left(\frac{2}{e}\right)^{m+1} e^t \quad for \quad 2m+2 \leq t \leq 2m+3.$$

Definition 1.1.30 ([5]) *If* $p \in \mathcal{R}$, *then the first order linear dynamic equation*

$$z^{\Delta} = p(t)z \tag{1.2}$$

is called regressive.

Theorem 1.1.31 ([5]) *Suppose (1.2) is regressive and fix* $t_0 \in \mathbb{T}$*. The solution of the initial value problem*

$$z^{\Delta} = p(t)z, \quad z(t_0) = z_0$$

is given by

$$z(t) = e_p(t, t_0)z_0.$$

1.2 Impulsive Differential Equations

Consider the initial value problem

$$\begin{cases} z'(t) = f(t, z) \\ z(0) = z_0 \end{cases} \tag{1.3}$$

subject to impulse effects

$$\Delta z|_{t=t_k} = I_k(z(t_k^-)), \quad k = 1, \ldots, m \tag{1.4}$$

with $f\colon ([a,b] \setminus \{t_1, \ldots, t_m\}) \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $\Delta z|_{t=t_k} = z(t_k^+) - z(t_k^-)$ is the impulse operator, where $z(t_k^+) = \lim_{h \to 0^+} z(t_k + h)$ and $z(t_k^-) = \lim_{h \to 0^-} z(t_k + h)$ represent the right and left limits of $z(t)$ at $t = t_k$, respectively, $k = 1, \ldots, m$.

Theorem 1.2.1 ([5]) *An integral representation of the IVP (1.3) with the impulse effects (1.4) is*

$$z(t) = z_0 + \int_0^t f(z(s))\,\mathrm{d}s + \sum_{0 < t_k < t} I_k(z(t_k^-)).$$

1.3 Auxiliary Fixed Point Theorems

We provide some necessary definitions and theorems derived from the book [27].

Definition 1.3.1 ([27]) *A subset* S *of a metric space* X *is relatively compact if its closure* $\bar{S}$ *is compact.*

Definition 1.3.2 ([27]) *Let X and Y be two normed linear spaces and $T : X \to Y$ a linear map between X and Y. Then, T is called a compact operator if for all bounded sets $E \subseteq X, T(E)$ is relatively compact in Y.*

Definition 1.3.3 ([27]) *Let X, Y be Banach spaces and $T : D \subset X \to Y$. The operator T is said to be completely continuous if it is continuous and maps any bounded subset of D into a relatively compact subset of Y.*

Theorem 1.3.4 (Arzela–Ascoli Theorem) *[[27]] A set $X \subset C([a,b])$ is relatively compact if and only if the following two conditions are satisfied.*

(i) The set X is bounded in $C([a,b])$, that is $\|y\| \leq c$ for all $y \in X$, for some $c > 0$.

(ii) For any given $\epsilon \geq 0$, there exists $\lambda \geq 0$ depending only on ϵ such that $|y(t_1) - y(t_2)| \leq \epsilon$ for any $y \in X$ and $t_1, t_2 \in [a,b]$ with $|t_1 - t_2| \leq \lambda$.

Theorem 1.3.5 (Schauder's Fixed Point Theorem) *[[27]]Let Ω be a nonempty, bounded, closed, and convex subset of a Banach space $\mathfrak{B}$. Then each continuous and compact map $T : \Omega \to \Omega$ has at least one fixed point in Ω.*

Theorem 1.3.6 (Schaefer Fixed Point Theorem) *[[27]] Assume that S is a normed linear space and the operator $F : S \to S$ is compact. Define*

$$H(F) = \{y \in S : y = \lambda F y, \quad \lambda \in (0,1)\}.$$

Then, either

1. *the set $H(F)$ is unbounded, or*
2. *F has a fixed point in S.*

CHAPTER 2

THIRD ORDER MULTI-POINT IMPULSIVE IBVPS ON TIME SCALES

2.1 Introduction

In this chapter, we focus on multi-point dynamic IBVPs of third order. We give new results on existence of solutions of third order m-point IBVPs, and we provide some examples too. The multi-point IBVPs are more adequate than the classical two point IBVPs for modelling many real-world phenomena as it is more likely for a dynamical system to have multi-points of freedom. We may refer the reader to the book [1] which offers an excellent review and several examples of applications modeled by IBVPs. Many good papers in literature deal with third-order m-point IBVPs, however most of them have focused on differential IBVPs on the real line $\mathbb{R}$, see for example [8, 18]. Regarding to dynamic IBVPs on arbitrary time scales, there are very few studies some of which are mentioned below. The results in this chapter have been accepted for publication in Miskolc Mathematical Notes journal [15].

The nonlinear third order IBVP on time scales:

$$\begin{cases} -z^{\Delta^3}(t) = f(t, z(t), z^{\Delta}(t), z^{\Delta^2}(t)), \quad t \in [0,T]_{\mathbb{T}} \backslash \Omega, \\ \Delta z(t_k) = I_k, \Delta z^{\Delta}(t_k) = J_k, \Delta z^{\Delta^2}(t_k) = L_k \quad k = 1, 2, \dots, m, \\ z(0) = \lambda z(\sigma(T)), \quad z^{\Delta}(0) = \lambda z^{\Delta}(\sigma(T)), \quad z^{\Delta^2}(0) = \lambda z^{\Delta^2}(\sigma(T)) \end{cases}$$

is studied by Li and Li in [22]. They used Schauder's fixed point theorem to prove that the IBVP above has at least one solution.

Liang and Zhang obtained sufficient conditions for the existence of three positive solutions of an IBVP of the form

$$\begin{cases} (\varphi(-z''(t)))' + a(t)f(z(t)) = 0, \quad t \neq t_k, \quad 0 < t < 1; \\ \Delta z|_{t=t_k} = I_k(z(t_k)), \quad k = 1,2,\ldots,N, \\ z(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i z(\tau_i), \\ z'(1) = 0, \qquad z''(0) = 0, \end{cases}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is an homomorphism with $\varphi(0) = 0$ and it is both positive and increasing. They employed the five-functionals fixed point theorem in their work [23].

In [11], Karaca and Fen considered the following IBVP for the third order nonlinear m-point system on time scales:

$$\begin{cases} (\phi_p(z^{\Delta\Delta}(t)))^{\Delta} + b(t)f(t,z(t),z^{\Delta}(t),z^{\Delta^2}(t)) = 0, \quad t \in J = [0,1]_{\mathbb{T}} \quad t \neq t_k, \quad 0 < t < 1; \\ \Delta z(t_k) = I_k(z(t_k)), \quad k = 1,2,\ldots,n, \\ \Delta z^{\Delta}(t_k) = -J_k(z(t_k), z^{\Delta}(t_k)), \quad k = 1,2,\ldots,n, \\ az(0) - bz^{\Delta}(0) = \displaystyle\sum_{i=1}^{m-2} \alpha_i z(\tau_i), \\ cz(1) + dz^{\Delta}(1) = \displaystyle\sum_{i=1}^{m-2} \beta_i z(\tau_i), \\ z^{\Delta^2}(0) = 0, \end{cases}$$

where $\mathbb{T}$ is an arbitrary time scale, and $\phi_p(s)$ is the ϕ-Laplacian operator i.e $\phi_p(s) = |s|^{p-2}s$ for $p > 1$, $(\phi_p)^{-1}(s) = \phi_q(s)$ where $\dfrac{1}{p} + \dfrac{1}{q} = 1$. They proved the existence of solutions in view of four functionals fixed point theorem to reach the result.

Motivated by the studies mentioned above, in this chapter we deal with the following third order multi-point IBVP for dynamic impulsive equations on time scales

$$\begin{cases} z^{\Delta^3}(t) = f(t,z(t),z^{\Delta}(t),z^{\Delta^2}(t)), \quad t \in J_0, \\ z^{\Delta^2}(t_k^+) = z^{\Delta^2}(t_k) + I_k(z(t_k), z^{\Delta}(t_k), z^{\Delta^2}(t_k)), \quad k \in \{1,\ldots,m\}, \\ z^{\Delta^2}(0) = \lambda z^{\Delta^2}(T), \\ z^{\Delta}(0) = \sum\limits_{j=1}^{n} \alpha_j z^{\Delta}(\tau_j), \\ z(0) = 0, \end{cases} \tag{2.1}$$

where $\mathbb{T}$ is an arbitrary time scale.

Throughout the chapter, we assume that

$$0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T,$$

t_k, $k \in \{1, \ldots, m\}$, are left dense. We define $J = [0, T]_{\mathbb{T}}$, $J_0 = J \backslash \{t_k\}_{k=1}^{m}$ and assume the following:

(A1) $f \in C([0, T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$,

(A2) $I_k \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, $k \in \{1, \ldots, m\}$,

(A3) $0 \le \tau_1 < \tau_2 < \ldots < \tau_n \le T$,

(A4) there exist nonnegative constants $\alpha \ge 0$ and $\gamma_k \ge 0$, $k \in \{1, \ldots, n\}$, so that

$$\alpha = \limsup_{|x|+|y|+|z| \to \infty} \left(\max_{t \in [0,T]} \frac{f(t, x, y, z)}{|x| + |y| + |z|} \right),$$

$$\gamma_k = \limsup_{|x|+|y|+|z| \to \infty} \frac{I_k(x, y, z)}{|x| + |y| + |z|}, \quad k \in \{1, \ldots, n\},$$

(A5) λ, $\alpha_j \in \mathbb{R}$, $j \in \{1, \ldots, n\}$, $\lambda \ne 1$, $1 - \sum_{j=1}^{n} \alpha_j \ne 0$.

Using an integral representation of the IBVP (2.1), sufficient conditions for the existence of at least one positive solution will be obtained. Examples that support the main results will also be provided.

2.2 Integral Representation

In this section, an integral representation of the solutions is obtained. To find an integral equation for the IBVP (2.1), we begin with the following IBVP for the sake of brevity. Consider

$$\begin{cases} z^{\Delta^3}(t) & = \eta(t), \quad t \in J_0, \\ z^{\Delta^2}(t_k^+) & = z^{\Delta^2}(t_k) + I_k(z(t_k), z^{\Delta}(t_k), z^{\Delta^2}(t_k)), \quad k \in \{1, \ldots, m\}, \\ z^{\Delta^2}(0) & = \lambda z^{\Delta^2}(T), \\ z^{\Delta}(0) & = \sum\limits_{j=1}^{n} \alpha_j z^{\Delta}(\tau_j), \\ z(0) & = 0, \end{cases} \tag{2.2}$$

where

(A6) $\eta \in C([0,T]_{\mathbb{T}})$.

Lemma 2.2.1 *Suppose* (*A*2)*,* (*A*3)*,* (*A*5) *and* (*A*6)*. Then, z is a solution of the IBVP* (2.2) *if and only if it is a solution of the integral equation*

$$\begin{aligned} z(t) = & \int_0^T G_1(t,\sigma(s))\eta(s)\Delta s + \sum_{0<t_j<T} G_1(t,t_j) I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ & + \int_0^{\tau_l} G_2(t,s)\eta(s)\Delta s + \sum_{0<t_j<\tau_l} G_2(t,t_j) I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)), \end{aligned} \tag{2.3}$$

where

$$G_1(t,s) = \frac{\lambda}{1-\lambda} \begin{cases} \dfrac{\sum\limits_{l=1}^{n} \alpha_l \tau_l}{1 - \sum\limits_{l=1}^{n} \alpha_l} t + a(t) + \left(\dfrac{1-\lambda}{\lambda}\right) b(t,\sigma(s)), & \textit{if } 0 < t < s < T \\ \dfrac{\sum\limits_{l=1}^{n} \alpha_l \tau_l}{1 - \sum\limits_{l=1}^{n} \alpha_l} t + a(t), & \textit{if } 0 < s < t < T, \end{cases} \tag{2.4}$$

$$G_2(t,s) = \frac{t}{1 - \sum\limits_{l=1}^{n} \alpha_l} \sum_{l=1}^{n} \alpha_l(\tau_l - s),$$

and

$$a(t) = \int_0^t s\Delta s, \quad b(t,s) = a(t) - a(s) - (t-s)s.$$

The integral equation (2.3) can be written in the following explicit form:

$$
\begin{aligned}
z(t) &= \left(t\left(\frac{\sum_{l=1}^{n}\alpha_l\tau_l}{1-\sum_{l=1}^{n}\alpha_l}\right)+a(t)\right)\left(\frac{\lambda}{1-\lambda}\right)\int_0^T \eta(s)\Delta s \\
&+\left(a(t)+t\left(\frac{\sum_{l=1}^{n}\alpha_l\tau_l}{1-\sum_{l=1}^{n}\alpha_l}\right)\right)\left(\frac{\lambda}{1-\lambda}\right)\sum_{j=1}^{m} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j)) \\
&+\frac{t}{1-\sum_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s \qquad (2.5) \\
&+\frac{t}{1-\sum_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j) \\
&+\int_0^t b(t,\sigma(s))\eta(s)\Delta s+\sum_{0<t_j<t} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\int_{t_j}^{t}(s-t_j)\Delta s,
\end{aligned}
$$

$t\in[0,T]_{\mathbb{T}}$.

Proof.

1. We first show that if z is a solution of the IBVP (2.2), then it satisfies the integral equation (2.3).

   Let z be a solution of the IBVP (2.2). If we integrate both sides of the first equation of (2.2) step by step from 0 to t,we have

   $$
   \begin{aligned}
   z^{\Delta^2}(t) &= z^{\Delta^2}(0)+\int_0^t \eta(s)\Delta s+\sum_{0<t_j<t}(z^{\Delta^2}(t_j^+)-z^{\Delta^2}(t_j)) \\
   &= z^{\Delta^2}(0)+\int_0^t \eta(s)\Delta s+\sum_{0<t_j<t} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j)), \quad t\in[0,T]_{\mathbb{T}},
   \end{aligned}
   $$

   whereupon

   $$z^{\Delta^2}(T)=z^{\Delta^2}(0)+\int_0^T \eta(s)\Delta s+\sum_{j=1}^{m} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j)),$$

   or

   $$\lambda z^{\Delta^2}(T)=\lambda z^{\Delta^2}(0)+\lambda\int_0^T \eta(s)\Delta s+\lambda\sum_{j=1}^{m} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j)).$$

   Using the boundary condition

   $$z^{\Delta^2}(0)=\lambda z^{\Delta^2}(T),$$

in the equation above, we find

$$z^{\Delta^2}(0) = \lambda z^{\Delta^2}(0) + \lambda \int_0^T \eta(s)\Delta s + \lambda \sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))$$

or

$$z^{\Delta^2}(0) = \frac{\lambda}{1-\lambda} \int_0^T \eta(s)\Delta s + \frac{\lambda}{1-\lambda} \sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)),$$

and so

$$\begin{aligned} z^{\Delta^2}(t) &= \frac{\lambda}{1-\lambda} \int_0^T \eta(s)\Delta s + \frac{\lambda}{1-\lambda} \sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ &\quad + \int_0^t \eta(s)\Delta s + \sum_{0<t_j<t} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)), \quad t \in [0,T]_{\mathbb{T}}. \end{aligned}$$

Now, integrating the last equation from 0 to t leads to

$$\begin{aligned} z^{\Delta}(t) &= z^{\Delta}(0) + t\frac{\lambda}{1-\lambda} \int_0^T \eta(s)\Delta s + t\frac{\lambda}{1-\lambda} \sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ &\quad + \int_0^t \int_0^{s_1} \eta(s)\Delta s \Delta s_1 + \int_0^t \sum_{0<t_j<s} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))\Delta s \\ &= z^{\Delta}(0) + t\frac{\lambda}{1-\lambda} \int_0^T \eta(s)\Delta s + t\frac{\lambda}{1-\lambda} \sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ &\quad + \int_0^t (t-\sigma(s))\eta(s)\Delta s \\ &\quad + \sum_{0<t_j<t} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))(t-t_j), \quad t \in [0,T]_{\mathbb{T}}. \end{aligned}$$

This clearly implies

$$\begin{aligned} z^{\Delta}(\tau_l) &= z^{\Delta}(0) + \tau_l\frac{\lambda}{1-\lambda} \int_0^T \eta(s)\Delta s + \tau_l\frac{\lambda}{1-\lambda} \sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ &\quad + \int_0^{\tau_l} (\tau_l-\sigma(s))\eta(s)\Delta s + \sum_{0<t_j<\tau_l} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))(\tau_l-t_j), \end{aligned}$$

and

$$\begin{aligned} \alpha_l z^{\Delta}(\tau_l) &= \alpha_l z^{\Delta}(0) + \alpha_l\tau_l\frac{\lambda}{1-\lambda} \int_0^T \eta(s)\Delta + \alpha_l\tau_l\frac{\lambda}{1-\lambda} \sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ &\quad +\alpha_l \int_0^{\tau_l} (\tau_l-\sigma(s))\eta(s)\Delta s + \alpha_l \sum_{0<t_j<\tau_l} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))(\tau_l-t_j). \end{aligned}$$

Thus,

$$\begin{aligned}\sum_{l=1}^{n}\alpha_l z^{\Delta}(\tau_l) = z^{\Delta}(0)\sum_{l=1}^{n}\alpha_l &+ \left(\sum_{l=1}^{n}\alpha_l\tau_l\right)\frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s\\ &+\left(\sum_{l=1}^{n}\alpha_l\tau_l\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right)\\ &+\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s\\ &+\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j).\end{aligned}$$

Now, from

$$z^{\Delta}(0) = \sum_{l=1}^{n}\alpha_l z^{\Delta}(\tau_l)$$

we have

$$\begin{aligned}z^{\Delta}(0) = z^{\Delta}(0)\sum_{l=1}^{n}\alpha_l &+ \left(\sum_{l=1}^{n}\alpha_l\tau_l\right)\frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s\\ &+\left(\sum_{l=1}^{n}\alpha_l\tau_l\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right)\\ &+\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s\\ &+\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j)\end{aligned}$$

or

$$\begin{aligned}\left(1-\sum_{l=1}^{n}\alpha_l\right)z^{\Delta}(0) = \left(\sum_{l=1}^{n}\alpha_l\tau_l\right)&\frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s\\ &+\left(\sum_{l=1}^{n}\alpha_l\tau_l\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right)\\ &+\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s\\ &+\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j),\end{aligned}$$

which implies that

$$z^{\Delta}(0) = \left(\frac{\sum_{l=1}^{n}\alpha_l\tau_l}{1-\sum_{l=1}^{n}\alpha_l}\right)\frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s$$

$$
\begin{aligned}
&+\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right)\\
&+\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s\\
&+\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
z^{\Delta}(t)=&\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\frac{\lambda}{1-\lambda}\int_0^T\eta(s)\Delta s\\
&+\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right)\\
&+\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s\\
&+\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j)\\
&+t\frac{\lambda}{1-\lambda}\int_0^T\eta(s)\Delta s+t\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\\
&+\int_0^t(t-\sigma(s))\eta(s)\Delta s\\
&+\sum_{0<t_j<t}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(t-t_j),\quad t\in[0,T]_{\mathbb{T}}.
\end{aligned}
$$

Finally, if we integrate the last equation from 0 to t we obtain

$$
\begin{aligned}
z(t)=t&\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\frac{\lambda}{1-\lambda}\int_0^T\eta(s)\Delta s\\
&+t\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{t}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s\\
&+\frac{t}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j)\\
&+a(t)\frac{\lambda}{1-\lambda}\int_0^T\eta(s)\Delta s+a(t)\frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\\
&+\int_0^t b(t,\sigma(s))\eta(s)\Delta s+\sum_{0<t_j<t} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\int_0^t(s-t_j)\Delta s\\
=&\left(t\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)+a(t)\right)\frac{\lambda}{1-\lambda}\int_0^T\eta(s)\Delta s\\
&+\left(a(t)+t\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right)\\
&+\frac{t}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s\\
&+\frac{t}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j)\\
&+\int_0^t b(t,\sigma(s))\eta(s)\Delta s+\sum_{0<t_j<t} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\int_0^t(s-t_j)\Delta s,\\
=&\int_0^T G_1(t,\sigma(s))\eta(s)\Delta s+\sum_{0<t_j<T} G_1(t,t_j)I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\\
&+\int_0^{\tau_l} G_2(t,s)\eta(s)\Delta s+\sum_{0<t_j<\tau_l} G_2(t,t_j)I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j)),\quad t\in[0,T]_{\mathbb{T}}.
\end{aligned}
$$

From $\sigma(t_j)=t_j$, we observe that

$$
b(t,\sigma(t_j))=(a(t)-a(t_j)-(t-t_j)t_j=\int\limits_{t_j}^{t} s\Delta s-\int\limits_{t_j}^{t} t_j\Delta s=\int_{t_j}^{t}(s-t_j)\Delta s.
$$

Hence, using the functions $G_1(t,s)$ and $G_2(t,s)$, defined in (2.4), $z(t)$ can be rewritten in a more compact form as in (2.3).

2. Now, let z be a solution to the integral equation (2.5). Firstly, we aim to show

that z is a solution of the IBVP (2.2).

From $a(0) = 0$, we have

$$z(0) = 0.$$

So,

$$\begin{aligned} z^{\Delta}(t) &= \left(\frac{\sum\limits_{l=1}^{n} \alpha_l \tau_l}{1 - \sum\limits_{l=1}^{n} \alpha_l} \right) \frac{\lambda}{1-\lambda} \int_0^T \eta(s) \Delta s \\ &\quad + \left(\frac{\sum\limits_{l=1}^{n} \alpha_l \tau_l}{1 - \sum\limits_{l=1}^{n} \alpha_l} \right) \left(\frac{\lambda}{1-\lambda} \sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \right) \\ &\quad + \frac{1}{1 - \sum\limits_{l=1}^{n} \alpha_l} \sum_{l=1}^{n} \alpha_l \int_0^{\tau_l} (\tau_l - \sigma(s)) \eta(s) \Delta s \\ &\quad + \frac{1}{1 - \sum\limits_{l=1}^{n} \alpha_l} \sum_{l=1}^{n} \alpha_l \sum_{0 < t_j < \tau_l} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))(\tau_l - t_j) \\ &\quad + t \frac{\lambda}{1-\lambda} \int_0^T \eta(s) \Delta s + t \frac{\lambda}{1-\lambda} \sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ &\quad + \int_0^t (t - \sigma(s)) \eta(s) \Delta s \\ &\quad + \sum_{0 < t_j < t} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))(t - t_j), \quad t \in [0, T]_{\mathbb{T}}. \end{aligned}$$

Setting $t = 0$ and $t = \tau_l$, respectively, clearly we find

$$\begin{aligned} z^{\Delta}(0) &= \left(\frac{\sum\limits_{l=1}^{n} \alpha_l \tau_l}{1 - \sum\limits_{l=1}^{n} \alpha_l} \right) \frac{\lambda}{1-\lambda} \int_0^T \eta(s) \Delta s \\ &\quad + \left(\frac{\sum\limits_{l=1}^{n} \alpha_l \tau_l}{1 - \sum\limits_{l=1}^{n} \alpha_l} \right) \left(\frac{\lambda}{1-\lambda} \sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \right) \\ &\quad + \frac{1}{1 - \sum\limits_{l=1}^{n} \alpha_l} \sum_{l=1}^{n} \alpha_l \int_0^{\tau_l} (\tau_l - \sigma(s)) \eta(s) \Delta s \\ &\quad + \frac{1}{1 - \sum\limits_{l=1}^{n} \alpha_l} \sum_{l=1}^{n} \alpha_l \sum_{0 < t_j < \tau_l} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))(\tau_l - t_j) \end{aligned}$$

and

$$
\begin{aligned}
z^{\Delta}(\tau_l) = {} & \left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s \\
& + \left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))\right) \\
& + \frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s \\
& + \frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))(\tau_l-t_j) \\
& + \tau_l\frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s + \tau_l\frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\
& + \int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s \\
& + \sum_{0<t_j<\tau_l} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))(\tau_l-t_j),
\end{aligned}
$$

and so,

$$
\begin{aligned}
\sum_{l=1}^{n}\alpha_l z^{\Delta}(\tau_l) = {} & \sum_{l=1}^{n}\alpha_l\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s \\
& + \sum_{l=1}^{n}\alpha_l\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))\right) \\
& + \sum_{l=1}^{n}\alpha_l\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s \\
& + \sum_{l=1}^{n}\alpha_l\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j))(\tau_l-t_j) \\
& + \sum_{l=1}^{n}\alpha_l\tau_l\frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s
\end{aligned}
$$

$$\begin{aligned}
&+\sum_{l=1}^{n}\alpha_l\tau_l\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\\
&+\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s\\
&+\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j)\\
={}&\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\frac{\lambda}{1-\lambda}\int_0^T\eta(s)\Delta s\\
&+\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right)\\
&+\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))\eta(s)\Delta s\\
&+\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j)\\
={}&z^{\Delta}(0).
\end{aligned}$$

Thus, it is seen that

$$z^{\Delta}(0)=\sum_{l=1}^{n}\alpha_l z^{\Delta}(y_l).$$

Moreover, since

$$\begin{aligned}
z^{\Delta^2}(t)={}&\frac{\lambda}{1-\lambda}\int_0^T\eta(s)\Delta s+\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\\
&+\int_0^t\eta(s)\Delta s+\sum_{0<t_j<t}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j)),\quad t\in[0,T]_{\mathbb{T}},
\end{aligned}$$

for $t=t_k$ one has

$$\begin{aligned}
z^{\Delta^2}(t_k^+)={}&\frac{\lambda}{1-\lambda}\int_0^T\eta(s)\Delta s+\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\\
&+\int_0^{t_k}\eta(s)\Delta s+\sum_{0<t_j<t_k^+}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j)),
\end{aligned}$$

and

$$\begin{aligned} z^{\Delta^2}(t_k) = {} & \frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s + \frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ & + \int_0^{t_k} \eta(s)\Delta s + \sum_{0<t_j<t_k} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)), \end{aligned}$$

$k \in \{1, \ldots, m\}$. Hence,

$$\begin{aligned} z^{\Delta^2}(t_k^+) - z^{\Delta^2}(t_k) = {} & \frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s + \frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ & + \int_0^{t_k} \eta(s)\Delta s + \sum_{0<t_j<t_k^+} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ & - \frac{\lambda}{1-\lambda}\int_0^T \eta(s)\Delta s - \frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ & - \int_0^{t_k} \eta(s)\Delta s - \sum_{0<t_j<t_k} I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\ = {} & I_k(z(t_k), z^{\Delta}(t_k), z^{\Delta^2}(t_k)), \quad k \in \{1, \ldots, m\}. \end{aligned}$$

Finally, note that

$$z^{\Delta^3}(t) = \eta(t), \quad t \in [0, T]_{\mathbb{T}}.$$

Therefore, z is a solution to the BVP (2.2). This completes the proof.

□

2.3 Existence of Solutions

In this section, by an application of the Schauder fixed point theorem, the existence of the solutions is proved.

Lemma 2.3.1 *Suppose* $(A1)$-$(A4)$ *hold. Then there exist positive constants* Q *and* R *such that*

$$\begin{aligned} |f(t, x, y, z)| &\le (|x| + |y| + |z|)Q + R, \\ |I_k(x, y, z)| &\le (|x| + |y| + |z|)Q + R, \quad k \in \{1, \ldots, m\}, \end{aligned}$$

where $t \in [0, T]_{\mathbb{T}}$ *and* $x, y, z \in \mathbb{R}$.

Proof. In view the first condition of $(A4)$, it follows that there exists a constant $N_1 \geq 0$ such that

$$|f(t,x,y,z)| \leq (|x|+|y|+|z|)\alpha, \quad t \in [0,1]_{\mathbb{T}}, \quad |x|+|y|+|z| > N_1.$$

Since $f \in C([0,T]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R})$, there exists a positive constant Q_1 such that

$$|f(t,x,y,z)| \leq Q_1, \quad t \in [0,1]_{\mathbb{T}}, \quad |x|+|y|+|z| \leq N_1.$$

Therefore

$$|f(t,x,y,z)| \leq (|x|+|y|+|z|)\alpha + Q_1, \quad t \in [0,T]_{\mathbb{T}}, \quad x,y,z \in \mathbb{R}.$$

Now, from the second condition of $(A4)$, there exist positive constants N_{2j}, $j \in \{1,\ldots,m\}$ such that

$$|I_j(x,y,z)| \leq (|x|+|y|+|z|)\delta_j, \quad |x|+|y|+|z| > N_{2j},$$

$j \in \{1,\ldots,m\}$. Since $I_j \in C(\mathbb{R} \times \mathbb{R} \times \mathbb{R})$, $j \in \{1,\ldots,m\}$, there exist positive constants Q_{2j}, $j \in \{1,\ldots,m\}$, such that

$$|I_j(x,y,z)| \leq Q_{2j}, \quad |x|+|y|+|z| \leq N_{2j}, \quad j \in \{1,\ldots,m\}.$$

Consequently

$$|I_j(x,y,z)| \leq (|x|+|y|+|z|)\delta_j + Q_{2j}, \quad x,y,z \in \mathbb{R},$$

$j \in \{1,\ldots,m\}$. Let

$$\begin{aligned} Q &= \max\{\alpha, \delta_1, \delta_2, \ldots \delta_m\}, \\ R &= \max\{Q_1, Q_{21}, Q_{22}, \ldots, Q_{2m}\}. \end{aligned}$$

Then, we get the desired inequalities. □

Now, define the function spaces

$$PC([0,T]_{\mathbb{T}}) = \left\{ z \in C(J_0), \quad \lim_{t \to t_j^+} z(t) \text{ and } \lim_{t \to t_j^-} z(t) \text{ exist} \quad z(t_j^-) = z(t_j), \quad j \in \{1,\ldots,m\} \right\},$$

$$PC^1([0,T]_{\mathbb{T}}) = \left\{ z \in PC([0,T]_{\mathbb{T}}) : z^{\Delta} \in C(J_0), \quad \lim_{t \to t_j^+} z^{\Delta}(t) \text{ and } \lim_{t \to t_j^-} z^{\Delta}(t) \text{ exist} , \right.$$
$$\left. z^{\Delta}(t_j^-) = z^{\Delta}(t_j), \quad j \in \{1,\ldots,m\} \right\}$$

and

$$PC^2([0,T]_{\mathbb{T}}) = \Big\{ z \in PC^1([0,T]_{\mathbb{T}}) : z^{\Delta^2} \in C(J_0), \quad \lim_{t \to t_j^+} z^{\Delta^2}(t) \text{ and } \lim_{t \to t_j^-} z^{\Delta^2}(t) \text{ exist },$$

$$z^{\Delta^2}(t_j^-) = z^{\Delta^2}(t_j), \quad j \in \{1, \ldots, m\} \Big\},$$

endowed with the norms

$$\begin{aligned}
||z|| &= \sup_{t \in [0,T]_{\mathbb{T}}} |z(t)|, \\
||z|| &= \max \Big\{ \sup_{t \in [0,T]_{\mathbb{T}}} |z(t)|, \quad \sup_{t \in [0,T]_{\mathbb{T}}} |z^{\Delta}(t)| \Big\}, \\
||z|| &= \max \Big\{ \sup_{t \in [0,T]_{\mathbb{T}}} |z(t)|, \quad \sup_{t \in [0,T]_{\mathbb{T}}} \left| z^{\Delta}(t) \right|, \quad \sup_{t \in [0,T]_{\mathbb{T}}} \left| z^{\Delta^2}(t) \right| \Big\},
\end{aligned}$$

respectively. Let $z \in PC^2([0,T]_{\mathbb{T}})$ and introduce the operator

$$\begin{aligned}
Tz(t) = &\int_0^T G_1(t, \sigma(s)) f(s, z(s), z^{\Delta}(s), z^{\Delta^2}(s)) \Delta s \\
&+ \sum_{0 < t_j < T} G_1(t, t_j) I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)) \\
&+ \int_0^{\tau_l} G_2(t, s) f(s, z(s), z^{\Delta}(s), z^{\Delta^2}(s)) \Delta s \\
&+ \sum_{0 < t_j < \tau_l} G_2(t, t_j) I_j(z(t_j), z^{\Delta}(t_j), z^{\Delta^2}(t_j)), \quad t \in [0,T]_{\mathbb{T}}.
\end{aligned}$$

Take $A \geq \max \Big\{ A_1, A_2, A_3, A_4 \Big\}$ where

$$\begin{aligned}
A_1 = &\sigma(T) \Big(\sigma(T) \Big(\frac{\sum_{l=1}^{n} |\alpha_l| \tau_l}{\left| 1 - \sum_{l=1}^{n} \alpha_l \right|} \Big) + (\sigma(T))^2 \Big) \left| \frac{\lambda}{1-\lambda} \right| \\
&+ m \Big(\sigma(T) \frac{\sum_{l=1}^{n} |\alpha_l| \tau_l}{\left| 1 - \sum_{l=1}^{n} \alpha_l \right|} + (\sigma(T))^2 \Big) \left| \frac{\lambda}{1-\lambda} \right| + \sigma(T) \frac{1}{\left| 1 - \sum_{l=1}^{n} .\alpha_l \right|} \sum_{l=1}^{n} |\alpha_l| \tau_l^2 \\
&+ \sigma(T) \frac{1}{\left| 1 - \sum_{l=1}^{n} \alpha_l \right|} \sum_{l=1}^{n} |\alpha_l| \sum_{j=1}^{n} (\tau_l + t_j) + 4(\sigma(T))^3 + (\sigma(T))^2,
\end{aligned}$$

$$A_2 = \sigma(T) \frac{\sum_{l=1}^{n} |\alpha_l| \tau_l}{\left| 1 - \sum_{l=1}^{n} \alpha_l \right|} \left| \frac{\lambda}{1-\lambda} \right| + m \frac{\sum_{l=1}^{n} |\alpha_l| \tau_l}{\left| 1 - \sum_{l=1}^{n} \alpha_l \right|} \left| \frac{\lambda}{1-\lambda} \right|$$

$$
\begin{aligned}
&+\frac{1}{\left|1-\sum\limits_{l=1}^{n}\alpha_l\right|}\sum_{l=1}^{n}|\alpha_l|\tau_l^2+\frac{1}{\left|1-\sum\limits_{l=1}^{n}\alpha_l\right|}\sum_{l=1}^{n}|\alpha_l|\sum_{j=1}^{m}(\tau_l+t_j)\\
&+(\sigma(T))^2\left|\frac{\lambda}{1-\lambda}\right|+m\sigma(T)\left|\frac{\lambda}{1-\lambda}\right|+(\sigma(T))^2+m\sigma(T),
\end{aligned}
$$

$$
A_3=\left|\frac{\lambda}{1-\lambda}\right|\sigma(T)+m\left|\frac{\lambda}{1-\lambda}\right|+\sigma(T)+m
$$

and

$$
A_4=1.
$$

Lemma 2.3.2 *Suppose* $(A1)$-$(A5)$ *hold. Then the operator*

$$
T:PC^2([0,T]_{\mathbb{T}})\to PC^2([0,T]_{\mathbb{T}})
$$

is a completely continuous operator.

Proof. Let $D\subset PC^2([0,T]_{\mathbb{T}})$ be a bounded set. Then, there exists a positive constant B such that

$$
\|z\|\le B,\quad z\in D.
$$

Take $z\in D$ arbitrarily. Then

$$
\begin{aligned}
|Tz(t)|=&\Bigg|\int_0^T G_1(t,\sigma(s))f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))\Delta s\\
&+\sum_{0<t_j<T}G_1(t,t_j)I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\\
&+\int_0^{\tau_l}G_2(t,s)f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))\Delta s\\
&+\sum_{0<t_j<\tau_l}G_2(t,t_j)I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\Bigg|\\
\le&\left|\int_0^T G_1(t,\sigma(s))f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))\Delta s\right|\\
&+\left|\sum_{0<t_j<T}G_1(t,t_j)I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right|\\
&+\left|\int_0^{\tau_l}G_2(t,s)f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))\Delta s\right|\\
&+\left|\sum_{0<t_j<\tau_l}G_2(t,t_j)I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right|
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^T |G_1(t,\sigma(s))||f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))|\Delta s\\
&\quad+\sum_{0<t_j<T}|G_1(t,t_j)||I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))|\\
&\quad+\int_0^{\tau_l}|G_2(t,s)||f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))|\Delta s\\
&\quad+\sum_{0<t_j<\tau_l}|G_2(t,t_j)||I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))|\\
&\le \Bigg(\sigma(T)\Bigg(\sigma(T)\Bigg(\frac{\sum\limits_{l=1}^{n}|\alpha_l|\tau_l}{\left|1-\sum\limits_{l=1}^{n}\alpha_l\right|}\Bigg)+(\sigma(T))^2\Bigg)\left|\frac{\lambda}{1-\lambda}\right|\\
&\quad+m\Bigg(\sigma(T)\frac{\sum\limits_{l=1}^{n}|\alpha_l|\tau_l}{\left|1-\sum\limits_{l=1}^{n}\alpha_l\right|}+(\sigma(T))^2\Bigg)\left|\frac{\lambda}{1-\lambda}\right|+\sigma(T)\frac{1}{\left|1-\sum\limits_{l=1}^{n}\alpha_l\right|}\sum_{l=1}^{n}|\alpha_l|\tau_l^2\\
&\quad+\sigma(T)\frac{1}{\left|1-\sum\limits_{l=1}^{n}\alpha_l\right|}\sum_{l=1}^{n}|\alpha_l|\sum_{j=1}^{n}(\tau_l+t_j)+4(\sigma(T))^3+(\sigma(T))^2\Bigg)(3QB+R)\\
&\le A(3QB+R),\quad t\in[0,T]_{\mathbb{T}},
\end{aligned}
$$

and

$$
\begin{aligned}
|(Tz)^{\Delta}(t)| &= \left|\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\frac{\lambda}{1-\lambda}\int_0^T f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s)))\Delta s\right.\\
&\quad+\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right)\\
&\quad+\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s)))\Delta s\\
&\quad+\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j)\\
&\quad+t\frac{\lambda}{1-\lambda}\int_0^T f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s)))\Delta s\\
&\quad+t\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^t (t-\sigma(s))f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s)))\Delta s \\
&+\sum_{0<t_j<t} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(t-t_j)\Big| \\
\le{}& \left|\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\frac{\lambda}{1-\lambda}\int_0^T f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s)))\Delta s\right| \\
&+\left|\left(\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right)\left(\frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right)\right| \\
&+\left|\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\int_0^{\tau_l}(\tau_l-\sigma(s))f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s)))\Delta s\right| \\
&+\left|\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\sum_{l=1}^{n}\alpha_l\sum_{0<t_j<\tau_l} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(\tau_l-t_j)\right| \\
&+\left|t\frac{\lambda}{1-\lambda}\int_0^T f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s)))\Delta s\right| \\
&+\left|t\frac{\lambda}{1-\lambda}\sum_{j=1}^{m} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right| \\
&+\left|\int_0^t (t-\sigma(s))f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s)))\Delta s\right| \\
&+\left|\sum_{0<t_j<t} I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))(t-t_j)\right| \\
\le{}& \left|\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right|\left|\frac{\lambda}{1-\lambda}\right|\int_0^T |f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))|\Delta s \\
&+\left|\frac{\sum\limits_{l=1}^{n}\alpha_l\tau_l}{1-\sum\limits_{l=1}^{n}\alpha_l}\right|\left|\frac{\lambda}{1-\lambda}\right|\sum_{j=1}^{m}|I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))| \\
&+\left|\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\right|\sum_{l=1}^{n}|\alpha_l|\int_0^{\tau_l}(\tau_l-\sigma(s))|f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))|\Delta s
\end{aligned}
$$

$$
\begin{aligned}
&+\left|\frac{1}{1-\sum\limits_{l=1}^{n}\alpha_l}\right|\sum_{l=1}^{n}|\alpha_l|\sum_{0<t_j<\tau_l}|I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))|(\tau_l-t_j)\\
&+t\left|\frac{\lambda}{1-\lambda}\right|\int_0^T|f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))|\Delta s\\
&+t\left|\frac{\lambda}{1-\lambda}\right|\sum_{j=1}^{m}|I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))|\\
&+\int_0^t(t-\sigma(s))|f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))|\Delta s\\
&+\sum_{0<t_j<t}|I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))|(t-t_j)\\
&\le\Bigg(\sigma(T)\frac{\sum\limits_{l=1}^{n}|\alpha_l|\tau_l}{\left|1-\sum\limits_{l=1}^{n}\alpha_l\right|}\left|\frac{\lambda}{1-\lambda}\right|+m\frac{\sum\limits_{l=1}^{n}|\alpha_l|\tau_l}{\left|1-\sum\limits_{l=1}^{n}\alpha_l\right|}\left|\frac{\lambda}{1-\lambda}\right|\\
&\quad+\frac{1}{\left|1-\sum\limits_{l=1}^{n}\alpha_l\right|}\sum_{l=1}^{n}|\alpha_l|\tau_l^2+\frac{1}{\left|1-\sum\limits_{l=1}^{n}\alpha_l\right|}\sum_{l=1}^{n}|\alpha_l|\sum_{j=1}^{m}(\tau_l+t_j)\\
&\quad+(\sigma(T))^2\left|\frac{\lambda}{1-\lambda}\right|+m\sigma(T)\left|\frac{\lambda}{1-\lambda}\right|+(\sigma(T))^2+m\sigma(T)\Bigg)(3QB+R)\\
&\le A(3QB+R),\quad t\in[0,T]_{\mathbb{T}},
\end{aligned}
$$

and

$$
\begin{aligned}
|(Tz)^{\Delta^2}(t)|&=\Bigg|\frac{\lambda}{1-\lambda}\int_0^T f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))\Delta s+\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\\
&\quad+\int_0^t f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))\Delta s+\sum_{0<t_j<t}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\Bigg|\\
&\le\left|\frac{\lambda}{1-\lambda}\int_0^T f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))\Delta s\right|+\left|\frac{\lambda}{1-\lambda}\sum_{j=1}^{m}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right|\\
&\quad+\left|\int_0^t f(s,z(s),z^{\Delta}(s),z^{\Delta^2}(s))\Delta s\right|+\left|\sum_{0<t_j<t}I_j(z(t_j),z^{\Delta}(t_j),z^{\Delta^2}(t_j))\right|\\
&\le\left(\left|\frac{\lambda}{1-\lambda}\right|\sigma(T)+m\left|\frac{\lambda}{1-\lambda}\right|+\sigma(T)+m\right)(3QB+R)\\
&\le A(3QB+R),\quad t\in[0,T]_{\mathbb{T}}.
\end{aligned}
$$

Consequently

$$\|Tz\|\le A(3QB+R).$$

Moreover,

$$\begin{aligned}|(Tz)^{\Delta^3}(t)| &= |-f(t,z(t),z^{\Delta}(t),z^{\Delta^2}(t))|\\ &\leq 3QB+R\\ &\leq A(3QB+R), \quad t\in[0,T]_{\mathbb{T}}.\end{aligned}$$

Hence, by the Arzela-Ascoli theorem, the operator $T : PC^2([0,T]_{\mathbb{T}}) \to PC^2([0,T]_{\mathbb{T}})$ is completely continuous. Thus, the proof is completed. □

Theorem 2.3.3 *Suppose* $(A1)$-$(A5)$ *hold. Suppose that the nonnegative constants A, B, Q and R satisfy*

$$A(3QB+R)\leq B.$$

Then, the BVP (2.1) *has at least one solution.*

Proof. Let

$$S=\{z\in PC^2([0,T]_{\mathbb{T}}) : \|z\|\leq B\}.$$

By Lemma 2.3.1, $T : S \to PC^2([0,T]_{\mathbb{T}})$ is a completely continuous operator, and by the proof of Lemma 2.3.1, it is seen that

$$\|Tz\|\leq A(3QB+R)\leq B, \quad y\in D.$$

Thus, $T : S \to S$. Now, applying Schauder fixed point theorem we conclude that the operator T has a fixed point in S. Thus, the proof is completed. □

2.4 An Example

We provide the following example in order to show that our main result is applicable for dynamic IBVPs.

Example 2.4.1 *Let*

$$\mathbb{T}=\left[0,\frac{1}{32}\right]\cup\left[\frac{1}{16},\frac{1}{8}\right]\cup\left[\frac{1}{4},\frac{1}{3}\right]\cup\left[\frac{1}{2},1\right],$$

where $\left[0,\frac{1}{32}\right]$, $\left[\frac{1}{16},\frac{1}{8}\right]$, $\left[\frac{1}{4},\frac{1}{3}\right]$ *and* $\left[\frac{1}{2},1\right]$ *are the real-valued intervals. Let also,*

$$T=\frac{1}{3}, \quad m=4, \quad n=3,$$

$$t_0 = 0, \quad t_1 = \frac{1}{36}, \quad t_2 = \frac{1}{16}, \quad t_3 = \frac{1}{10}, \quad t_4 = \frac{1}{4}, \quad t_5 = T = \frac{1}{3},$$

$$J = \left[0, \frac{1}{3}\right], \quad J_0 = \left(0, \frac{1}{32}\right) \cup \left(\frac{1}{16}, \frac{1}{8}\right) \cup \left(\frac{1}{4}, \frac{1}{3}\right],$$

$$\tau_1 = \frac{1}{38}, \quad \tau_2 = \frac{3}{32}, \quad \tau_3 = \frac{7}{24}.$$

Consider the BVP

$$\begin{cases} z^{\Delta^3}(t) = \dfrac{z(t) + z^{\Delta}(t) + z^{\Delta^2}(t)}{10^{20}\,(1 + (z(t))^2)\left(1 + (z^{\Delta}(t))^2\right)\left(1 + (z^{\Delta^2}(t))^2\right)}, \quad t \in J_0, \\ z^{\Delta^2}(t_k^+) = z^{\Delta^2}(t_k) + \dfrac{z(t_k)}{\left(1 + \left(z^{\Delta}(t_k)\right)^4\right)\left(1 + \left(z^{\Delta^2}(t_k)\right)^8\right)}, \quad k \in \{1, \ldots, 4\}, \\ z^{\Delta^2}(0) = \dfrac{1}{2} z^{\Delta^2}\left(\dfrac{1}{3}\right), \\ z^{\Delta}(0) = \dfrac{1}{2} z^{\Delta}(\tau_1) + \dfrac{1}{8} z^{\Delta}(\tau_2) + \dfrac{1}{16} z^{\Delta}(\tau_3), \\ z(0) = 0. \end{cases} \tag{2.6}$$

Clearly,

$$f(t, x, y, z) = \frac{x + y + z}{10^{20}(1 + x^2)(1 + y^2)(1 + z^2)}, \quad t \in J, \quad x, y, z \in \mathbb{R},$$

$$I_k(x, y, z) = \frac{x}{10^{40}(1 + y^4)(1 + z^8)}, \quad x, y, z \in \mathbb{R},$$

$$\sigma\left(\frac{1}{3}\right) = \frac{1}{2}, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{8}, \quad \alpha_3 = \frac{1}{16}, \quad \lambda = \frac{1}{2}.$$

Thus, we have

$$|f(t, x, y, z)| \leq \frac{1}{10^{20}}(|x| + |y| + |z|), \tag{2.7}$$

$$|I(x, y, z)| \leq \frac{1}{10^{40}}|x| \tag{2.8}$$

which imply that

$$Q = \frac{1}{10^{20}}, \quad R = 0.$$

If we take

$$A = 10^{10} \quad \textit{and} \quad B = \frac{1}{2},$$

then we obtain

$$A(3QB + R) \ = \ 3AQB \ = \ 3 \cdot 10^{10} \cdot \frac{1}{10^{20}} \cdot \frac{1}{2} \ < \ \frac{1}{2} \ = \ B. \tag{2.9}$$

Therefore, by (2.7)-(2.9) *it follows that all the hypotheses of Theorem 2.3.3 hold. Hence, the IBVP* (2.6) *has at least one solution.*

CHAPTER 3

ODD ORDER MULTI-POINT IMPULSIVE BOUNDARY VALUE PROBLEMS ON TIME SCALES

3.1 Introduction

There is a massive literature about the existence of solutions to IBVPs on time scales. We mentioned have only some of the recent studies, see for example [21, 28] for first order, [4, 25, 30, 32, 33, 34, 36, 38, 39] for second order, and [8, 11, 22, 23] for third order equations, and [29] for fourth order equations. However, there are hardly any papers dealing with m-point odd order IBVPs on time scales. Note that the results in this thesis have been published in the Georgian Mathematical Journal [14].

In [29], Karaca and Fen studied the following fourth order IBVP on time scales

$$\begin{cases} z^{\Delta^4} + b(t)f(t, u(t), z^{\Delta}(t), z^{\Delta\Delta}(t)) & = 0, \\ \Delta z^{\Delta\Delta}|_{t=t_k} & = I_k z(^{\Delta\Delta(t_k)}), \quad k = 1, 2, \ldots, n, \\ \Delta z^{\Delta^3}|_{t=t_k} & = -I_k z(^{\Delta\Delta(t_k)}) \quad k = 1, 2, \ldots, n, \\ a z^{\Delta\Delta(0)} - b \Delta z^{\Delta^3}(0) & = \displaystyle\int_0^1 g_1(s) z^{\Delta\Delta} \Delta s \\ c z^{\Delta\Delta(1)} + d \Delta z^{\Delta^3}(1) & = \displaystyle\int_0^1 g_2(s) z^{\Delta\Delta} \Delta s \\ z(0) = z^{\Delta}(0) = 0, & \end{cases}$$

where $\mathbb{T}$ is an arbitrary time scale, $J = [0,1]_{\mathbb{T}} = [0,1]_{\mathbb{T}} \cap \mathbb{T}$, $0, 1 \in \mathbb{T}$.

The following m-point higher order non-impulsive BVP on time scales

$$\begin{cases} (-1)^n z^{\Delta^{2n}}(t) & = f(t,z(t)), \quad t \in [t_1,t_m] \subset \mathbb{T}, n \in \mathbb{N} \\ z^{\Delta^{2i+1}}(t_1) & = 0, \quad \alpha z^{\Delta^{2i}}(t_m) + \beta z^{\Delta^{2i+1}}(t_m) = \sum_{k=2}^{m-1} z^{\Delta^{2i+1}}(t_k), \end{cases}$$

where $\alpha > 0$ and $\beta > m-2$ are given constants $t_1 \le t_2 \le \ldots \le t_{m-1} \le t_m$, $m \ge 3$ and $0 \le i \le n-1$, is studied by Yaslan [35].

Our aim during this chapter is to partly fill this gap. For this purpose, we consider the following $(2n+1)^{th}$ order periodic boundary value problem (PBVP) with impulse on time scales

$$\begin{cases} z^{\Delta^{2n+1}}(t) & = f(t,z(\sigma(t))), \quad t \in J_0, \\ z^{\Delta^p}(0) & = 0, \quad p \in \{2,3,\ldots,2n\}, \\ z^{\Delta^2}(t_k^+) & = z^{\Delta^2}(t_k) + I_k(z(t_k)), \quad k \in \{1,\ldots,m\}, \\ z^{\Delta}(0) & = \sum_{j=1}^{m} \alpha_j z^{\Delta}(\tau_j), \\ z(t_k^+) & = z(t_k) + J_k(z(t_k)), \quad k \in \{1,\ldots,m\}, \\ z(0) & = z(\sigma(T)), \end{cases} \tag{3.1}$$

where each t_k, $k \in \{1,\ldots,m\}$, is right dense and $J_0 = [0,T]_{\mathbb{T}} \backslash \{t_k\}_{k=1}^m$.

The following hypothesis will be used in the sequel.

(B1) $f \in C([0,T]_{\mathbb{T}} \times \mathbb{R})$,

(B2) $I_k \in C(\mathbb{R})$, $k \in \{1,\ldots,m\}$,

(B3) $J_k \in C(\mathbb{R})$, $k \in \{1,\ldots,m\}$,

(B4) $\alpha_j \in \mathbb{R}$, $j \in \{1,\ldots,m\}$, $\sum_{j=1}^{m} \alpha_j \neq 1$, $q \in \mathbb{N}$, $q \ge 2$,

(B5) $0 \le \tau_1 < \tau_2 < \ldots < \tau_m \le T$,

(B6) $|f(t,z)| \le k_1$, $|I_k(z)| \le k_2$, $|J_k(z)| \le k_3$, $t \in [0,\sigma(T)]_{\mathbb{T}}$, $z \in \mathbb{R}$, $k \in \{1,\ldots,m\}$, with $k_1,k_2,k_3 \in \mathbb{R}^+$.

By using the hypothesis $(B1)-(B6)$, it will be shown that the PBVP (3.1) has at least one solution. Illustrative examples will also be given.

3.2 Integral Representation

For clarity, we first consider dynamic impulsive linear equation

$$\begin{cases} z^{\Delta}(t) + a(t)z^{\sigma}(t) = \eta(t), & t \in J_0, \\ z(t_k^+) = z(t_k) + J_k(z(t_k)), & k \in \{1, \ldots, m\}, \end{cases} \tag{3.2}$$

with the periodicity condition

$$z(0) = z(\sigma(T)), \tag{3.3}$$

where

(B7) $a, h \in C([0, \sigma(T)]_{\mathbb{T}})$, a is regressive on $[0, \sigma(T)]_{\mathbb{T}}$.

Define

$$PC_{rd}([0,\sigma(T)]_{\mathbb{T}}) = \Big\{ z : [0,\sigma(T)]_{\mathbb{T}} \to \mathbb{R}, \quad z \text{ is continuous on } J_0, \quad \text{rd-continuous at } t_k, \lim_{t \to t_k+} z(t) \text{ exist}, \quad k \in \{1, \ldots, m\} \Big\}.$$

$PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$ is a Banach space with the following norm

$$\|z\| = \max_{j \in \{0,\ldots,m\}} \sup_{t \in [t_j, t_{j+1}]_{\mathbb{T}}} |z(t)|.$$

Now, for $r \in \mathbb{N}$, we define another Banach space

$$PC_{rd}^{\Delta^r}([0,\sigma(T)]_{\mathbb{T}}) = \Big\{ z : z^{\Delta^{r-1}} \in PC_{rd}^{\Delta^{r-1}}([0,\sigma(T)]_{\mathbb{T}}), \quad z^{\Delta^r} \text{ exists and it is continuous on } (t_k, t_{k+1}), \lim_{t \to t_k+} z^{\Delta^r}(t) \text{ exists and } \lim_{t \to t_k} z^{\Delta^r}(t) \text{ exists if } t_k \text{ is left-scattered}, \quad k \in \{1, \ldots, m\} \Big\}$$

which is also Banach Space, with the norm

$$\|z\|_{\Delta^r} = \max \Big\{ \|z\|, \sup_{t \in J_0} \Big| z^{\Delta^i}(t) \Big|, \quad i \in \{1, \ldots, r\} \Big\}.$$

Definition 3.2.1 *A function* $z \in PC_{rd}^{\Delta}([0,\sigma(T)]_{\mathbb{T}})$ *satisfying* (3.2)-(3.3) *is said to be a solution of the impulsive PBVP* (3.2)-(3.3).

Lemma 3.2.2 *$e_a(\sigma(T),0) \neq 1$ and suppose (B7) holds. If $z \in PC^{\Delta}_{rd}([0,\sigma(T)]_{\mathbb{T}})$ is a solution of (3.2)-(3.3), then*

$$z(t) = \int_0^{\sigma(T)} \mu(t,s)\eta(s)\Delta s + \sum_{j=1}^{m} \mu(t,t_j)J_j(z(t_j)), \quad t \in [0,\sigma(T)]_{\mathbb{T}}, \tag{3.4}$$

where

$$\mu(t,s) = \begin{cases} e_{\ominus a}(t,s) \quad 0 \le s < t \le \sigma(T) \\ \dfrac{1}{A_0} e_{\ominus a}(\sigma(T),0)e_{\ominus a}(t,s) \quad 0 \le t \le s \le \sigma(T), \end{cases} \tag{3.5}$$

$$A_0 = \frac{1}{1 - e_{\ominus a}(\sigma(T),0)}.$$

Proof. The functions h and e_a are both rd-continuous. So, from the first line in (3.2) we can write

$$[z(t)e_a(t,0)]^{\Delta} = e_a(t,0)\eta(t)$$

which implies that

$$z(t)e_a(t,0) = z(0) + \int_0^t e_a(s,0)\eta(s)\Delta s + \sum_{0<t_j<t} e_a(t_j,0)J_j(z(t_j))$$

or

$$z(t) = e_{\ominus a}(t,0)\Big[z(0) + \int_0^t e_a(s,0)\eta(s)\Delta s + \sum_{0<t_j<t} e_a(t_j,0)J_j(z(t_j))\Big]. \tag{3.6}$$

Applying the boundary condition (3.3), we have

$$z(\sigma(T)) = e_{\ominus a}(\sigma(T),0)\Big[z(0) + \int_0^{\sigma(T)} e_a(s,0)\eta(s)\Delta s + \sum_{0<t_j<\sigma(T)} e_a(t_j,0)J_j(z(t_j))\Big]$$

or

$$\Big[1 - e_{\ominus a}(\sigma(T),0)\Big]z(0) = e_{\ominus a}(\sigma(T),0)\Big[\int_0^{\sigma(T)} e_a(s,0)\eta(s)\Delta s + \sum_{0<t_j<\sigma(T)} e_a(t_j,0)J_j(z(t_j))\Big].$$

Thus,

$$z(0) = A_0\Big[\int_0^{\sigma(T)} e_{\ominus a}(\sigma(T),0)e_a(s,0)\eta(s)\Delta s + \sum_{0<t_j<\sigma(T)} e_{\ominus a}(\sigma(T),0)e_a(t_j,0)J_j(z(t_j))\Big]. \tag{3.7}$$

Using (3.7) into (3.6), we obtain

$$\begin{aligned} z(t) =& A_0\Big[\int_0^{\sigma(T)} e_{\ominus a}(\sigma(T),0)e_{\ominus a}(t,s)\eta(s)\Delta s + \sum_{0<t_j<\sigma(T)} e_{\ominus a}(\sigma(T),0)e_{\ominus a}(t,t_j)J_j(z(t_j))\Big] \\ & + \int_0^t e_{\ominus a}(t,s)\eta(s)\Delta s + \sum_{0<t_j<t} e_{\ominus a}(t,t_j)J_j(z(t_j)). \end{aligned}$$

Then, in view of (3.5), we see that (3.4) holds. □

Lemma 3.2.3 *Suppose* $(B7)$ *holds. If* $z \in PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$ *satisfies* (3.4)*, then* $z \in PC^{\Delta}_{rd}([0,\sigma(T)]_{\mathbb{T}})$ *is a solution of the impulsive PBVP* (3.2)-(3.3).

The proof follows by differentiating (3.4), and setting it is not difficult to see that the boundary condition (3.3) hold. Therefore, we skip the details of the proof.

Theorem 3.2.4 *Suppose* $(B1)$-$(B6)$ *hold. Then the impulsive PBVP* (3.1) *has at least one solution.*

We will prove Theorem 3.2.4 step by step.

Firstly, observe that

$$A = \frac{1}{e_a(\sigma(T),0)-1} \le \mu(t,s) \le \frac{e_a(\sigma(T),0)}{e_a(\sigma(T),0)-1} = B, \quad t,s \in [0,\sigma(T)]_{\mathbb{T}}. \tag{3.8}$$

Define

$$K = k_1(\sigma(T))^{2n} + mk_2\sigma(T) + \frac{\sum\limits_{j=1}^{m}|\alpha_j|}{\left|1-\sum\limits_{j=1}^{m}\alpha_j\right|}\Big(k_1(\sigma(T))^{2n} + mk_2\sigma(T)\Big).$$

Let

$$f_1(t,z(\sigma(t))) = \int_0^t\int_0^{t_1}\dots\int_0^{t_{2n-3}} f(s,z(\sigma(s)))\Delta s\Delta t_{2n-3}\dots\Delta t_1$$

and

$$f_2(t,z(\sigma(t))) = \int_0^t \int_0^{z_1} f_1(s,z(\sigma(s)))\Delta s\Delta z_1 + \sum_{0<t_j<t} I_j(z(t_j))(t-t_j)$$
$$+\frac{1}{1-\sum\limits_{j=1}^{m}\alpha_j}\Big(\sum_{j=1}^{m}\alpha_j\int_0^{\tau_j}\int_0^{z_1} f_1(s,z(\sigma(s)))\Delta s\Delta z_1$$
$$+\sum_{j=1}^{m}\alpha_j\sum_{0<t_j<\tau_j} I_j(z(t_j))(\tau_j-t_j)\Big).$$

Lemma 3.2.5 *If* $z \in PC_{rd}^{\Delta^{2n+1}}([0,\sigma(T)]_{\mathbb{T}})$ *is a solution of* (3.1)*, then*

$$z(t) = \int_0^{\sigma(T)} \mu(t,s)\left[b(s)z(\sigma(s)) + f_2(s,z(\sigma(s)))\right]\Delta s \tag{3.9}$$
$$+\sum_{j=1}^{m}\mu(t,t_j)J_j(z(t_j)), \quad t\in[0,\sigma(T)]_{\mathbb{T}}.$$

Proof.

1. Let $z \in PC_{rd}^{\Delta^{2n+1}}([0,\sigma(T)]_{\mathbb{T}})$ be a solution of (3.1). Then, integrating both sides of the first line of (3.1) $2n-2$ times,from the initial condition $z^{\Delta^p}(0)=0$ see that

$$z^{\Delta^3}(t) = f_1(t,z(\sigma(t))), \quad t\in J_0.$$

Thus, we obtain the following PBVP:

$$\begin{cases} z^{\Delta^3}(t) & = f_1(t,z(\sigma(t))), \quad t\in J_0, \\ z^{\Delta^2}(t_k^+) & = z^{\Delta^2}(t_k) + I_k(z(t_k)), \quad k\in\{1,\dots,m\}, \\ z^{\Delta^2}(0) & = 0, \\ z^{\Delta}(0) & = \sum\limits_{j=1}^{m}\alpha_j z^{\Delta}(\tau_j), \\ z(t_k^+) & = z(t_k) + J_k(z(t_k)), \quad k\in\{1,\dots,m\}, \\ z(0) & = z(\sigma(T)). \end{cases} \tag{3.10}$$

Integration of the first line of yields (3.10) to

$$z^{\Delta^2}(t) = \int_0^t f_1(s,z(\sigma(s)))\Delta s + \sum_{0<t_j<t}\left(z^{\Delta^2}(t_j^+) - z^{\Delta^2}(t_j)\right)$$

$$= \int_0^t f_1(s, z(\sigma(s)))\Delta s + \sum_{0<t_j<t} I_j(z(t_j)), \tag{3.11}$$

and

$$z^{\Delta}(t) = z^{\Delta}(0) + \int_0^t \int_0^{z_1} f_1(s, z(\sigma(s)))\Delta s \Delta z_1 + \sum_{0<t_j<t} I_j(z(t_j))(t - t_j), \tag{3.12}$$

which implies

$$\begin{aligned} \alpha_j z^{\Delta}(\tau_j) =& \alpha_j z^{\Delta}(0) + \alpha_j \int_0^{\tau_j} \int_0^{z_1} f_1(s, z(\sigma(s)))\Delta s \Delta z_1 \\ &+ \alpha_j \sum_{0<t_j<\tau_j} I_j(z(t_j))(\tau_j - t_j), \end{aligned} \tag{3.13}$$

$j \in \{1, \ldots, m\}$. Therefore,

$$\begin{aligned} z^{\Delta}(0) &= \sum_{j=1}^{m} \alpha_j z^{\Delta}(\tau_j) \\ &= z^{\Delta}(0) \sum_{j=1}^{m} \alpha_j + \sum_{j=1}^{m} \alpha_j \int_0^{\tau_j} \int_0^{z_1} f_1(s, z(\sigma(s)))\Delta s \Delta z_1 \\ &\quad + \sum_{j=1}^{m} \alpha_j \sum_{0<t_j<\tau_j} I_j(z(t_j))(\tau_j - t_j), \end{aligned}$$

or

$$\begin{aligned} z^{\Delta}(0) =& \frac{1}{1 - \sum\limits_{j=1}^{m} \alpha_j} \Big(\sum_{j=1}^{m} \alpha_j \int_0^{\tau_j} \int_0^{z_1} f_1(s, z(\sigma(s)))\Delta s \Delta z_1 \\ &+ \sum_{j=1}^{m} \alpha_j \sum_{0<t_j<\tau_j} I_j(z(t_j))(\tau_j - t_j) \Big). \end{aligned}$$

Using it in (3.12), we get

$$\begin{aligned} z^{\Delta}(t) =& \int_0^t \int_0^{z_1} f_1(s, z(\sigma(s)))\Delta s \Delta z_1 + \sum_{0<t_j<t} I_j(z(t_j))(t - t_j) \\ &+ \frac{1}{1 - \sum\limits_{j=1}^{m} \alpha_j} \Big(\sum_{j=1}^{m} \alpha_j \int_0^{\tau_j} \int_0^{z_1} f_1(s, z(\sigma(s)))\Delta s \Delta z_1 \\ &+ \sum_{j=1}^{m} \alpha_j \sum_{0<t_j<\tau_j} I_j(z(t_j))(\tau_j - t_j) \Big) \\ =& f_2(t, z(\sigma(t))), \quad t \in J_0. \end{aligned}$$

Thus, we obtain the first order impulse PBVP,

$$\begin{cases} z^{\Delta}(t) = & f_2(t, z(\sigma(t))), \quad t \in J_0, \\ z(t_k^+) = & z(t_k) + J_k(z(t_k)), \quad k \in \{1, \dots, m\}, \\ z(0) = & z(\sigma(T)) \end{cases} \tag{3.14}$$

which implies that

$$\begin{cases} z^{\Delta}(t) + a(t)z(\sigma(t)) & = a(t)z(\sigma(t)) + f_2(t, z(\sigma(t))), \quad t \in J_0, \\ z(t_k^+) & = z(t_k) + J_k(z(t_k)), \quad k \in \{1, \dots, m\}, \\ z(0) & = z(\sigma(T)). \end{cases}$$

Now, if we apply Lemma 3.2.2, we get the integral representation (3.9).

2. Suppose that z satisfies (3.9). Then, using Lemma 3.2.3, we obtain (3.13) and (3.14). Hence,

$$z^{\Delta}(0) = \sum_{j=1}^{m} \alpha_j z^{\Delta}(\tau_j).$$

Taking delta derivative of the first line in (3.14) we obtain (3.11). Thus,

$$z^{\Delta^2}(t_k) = \int_0^{t_k} f_1(s, z(\sigma(s)))\Delta s + \sum_{0<t_j<t_k} I_j(z(t_j)), \quad k \in \{1, \dots, m\},$$

and hence,

$$z^{\Delta^2}(t_k^+) = z^{\Delta^2}(t_k) + I_k(z(t_k)), \quad k \in \{1, \dots, m\}.$$

In a similar way, we now take delta derivative of (3.11) to get

$$\begin{aligned} z^{\Delta^3}(t) &= f_1(t, z(\sigma(t))) \\ &= \int_0^t \int_0^{z_1} \dots \int_0^{y_{2n-3}} f(s, z(\sigma(s)))\Delta s \Delta y_{2n-3} \dots \Delta z_1, \end{aligned}$$

$t \in J_0$. Hence, $z^{\Delta^3}(0) = 0$.

If we continue in the same way, we have

$$z^{\Delta^{2n+1}}(t) = f(t, z(\sigma(t))), \quad t \in J_0,$$

and

$$z^{\Delta^l}(0) = 0, \quad l \in \{3, \dots, 2n\}.$$

Thus, the proof is completed.

□

3.3 Existence of Solutions

The main result of the present chapter, the existence of solutions for the odd order impulsive PBVP (3.1) will be shown.

Lemma 3.3.1 *Suppose that* $(B1)$-$(B6)$ *hold. Then, for some* $z \in PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$*, we can write*

$$\begin{aligned} |f_1(t,z(\sigma(t)))| &\le k_1(\sigma(T))^{2n-2}, \\ |f_2(t,z(\sigma(t)))| &\le K, \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \end{aligned} \tag{3.15}$$

Proof. From $(B6)$ we have

$$\begin{aligned} |f_1(t,z(\sigma(t)))| &\le \int_0^t \int_0^{z_1} \dots \int_0^{z_{2n-3}} |f(s,z(\sigma(s)))| \Delta s \Delta z_{2n-3} \dots \Delta z_1 \\ &\le k_1(\sigma(T))^{2n-2}, \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \end{aligned}$$

Then using $(B4)$-$(B6)$, we see that

$$\begin{aligned} |f_2(t,z(\sigma(t)))| \le & \int_0^t \int_0^{z_1} |f_1(s,z(\sigma(s)))| \Delta s \Delta z_1 + \sum_{0<t_j<t} |I_j(z(t_j))|(t-t_j) \\ & + \frac{1}{\left|1-\sum\limits_{j=1}^{m} \alpha_j\right|} \times \Big(\sum_{j=1}^{m} |\alpha_j| \int_0^{\tau_j} \int_0^{z_1} |f_1(s,z(\sigma(s)))| \Delta s \Delta z_1 \\ & + \sum_{j=1}^{m} |\alpha_j| \sum_{0<t_j<\tau_j} |I_j(z(t_j))|(\tau_j - t_j) \Big) \\ \le & k_1(\sigma(T))^{2n} + mk_2\sigma(T) + \frac{\sum\limits_{j=1}^{m} |\alpha_j|}{\left|1-\sum\limits_{j=1}^{m} \alpha_j\right|} \Big(k_1(\sigma(T))^{2n} + mk_2\sigma(T) \Big) \\ = & K, \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \end{aligned}$$

Hence, (3.15) holds. This completes the proof. □

Theorem 3.3.2 *Suppose that* $(B1)-(B6)$ *hold, then PBVP* (3.1) *has at least one solution.*

Proof.

For $z \in PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$, we define

$$\Psi z(t) = \int_0^{\sigma(T)} \mu(t,s) f_2(s, z(\sigma(s)))\Delta s + \sum_{j=1}^{m} \mu(t,t_j) J_j(z(t_j)), \quad t \in [0,\sigma(T)]_{\mathbb{T}}.$$

The Schaefer fixed point theorem will be applied to show that the operator Ψz has a fixed point.

(i) The operator $\Psi : PC_{rd}([0,\sigma(T)]_{\mathbb{T}}) \to PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$ *is compact.*

Proof. First, we will prove that $\Psi : PC_{rd}([0,\sigma(T)]_{\mathbb{T}}) \to PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$ is continuous:

Pick a sequence $\{z_n\}_{n\in\mathbb{N}} \subset PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$ such that $z_n \to z \in PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$, as $n \to \infty$. From (3.8) we can write

$$\begin{aligned} |\Psi z_n(t) - \Psi z(t)| \le & \int_0^{\sigma(T)} |\mu(t,s)||f_2(s, z_n(\sigma(s))) - f_2(s, z(\sigma(s)))|\Delta s \\ & + \sum_{j=1}^{m} |\mu(t,t_j)||J_j(y_n(t_j)) - J_j(z(t_j))| \\ \le & B\Big(\int_0^{\sigma(T)} |f_2(s, z_n(\sigma(s))) - f_2(s, z(\sigma(s)))|\Delta s \\ & + \sum_{j=1}^{m} |J_j(z_n(t_j)) - J_j(z(t_j))|\Big), \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \end{aligned}$$

In view of (3.15), the Lebesgue dominated convergence theorem is applicable. Hence,

$$||\Psi z_n - \Psi z|| \to 0, \quad n \to \infty.$$

Now, we need to show that $\Psi \in PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$ maps bounded sets. Take an arbitrary $D \subset PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$ which is bounded. Then, clearly

$$||z|| \le L,$$

where L is some positive constant. Then,

$$\begin{aligned} |\Psi z(t)| \le & \int_0^{\sigma(T)} |\mu(t,s)||f_2(s, z(\sigma(s)))|\Delta s + \sum_{j=1}^{m} |\mu(t,t_j)||J_j(z(t_j))| \\ \le & BK\sigma(T) + Bmk_3, \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \end{aligned} \tag{3.16}$$

Let $t^1, t^2 \in [0,\sigma(T)]_{\mathbb{T}}$. Then

$$|\Psi z(t^1) - \Psi z(t^2)| \leq \int_0^{\sigma(T)} |\mu(t^1,s) - \mu(t^2,s)||f_2(s, z(\sigma(s)))|\Delta s + \sum_{j=1}^{m} |\mu(t^1,t_j) - \mu(t^2,t_j)||J_j(z(t_j))|.$$

Thus,

$$\Psi z(t^1) - \Psi z(t^2) \to 0,\ t^1 \to t^2. \tag{3.17}$$

The desired result follows from (3.16) and (3.17). Therefore, Arzela-Ascoli theorem applies. Hence, the operator

$$\Psi : PC_{rd}([0,\sigma(T)]_{\mathbb{T}}) \to PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$$

is compact. Note that the set

(ii) $\eta(F) = \{z \in S : z = \lambda F z, \quad \lambda \in (0,1)\}$ *is bounded.* Really, we have the following. Consider

$$z(t) = \lambda \Psi z(t), \quad t \in [0,\sigma(T)]_{\mathbb{T}}, \quad \lambda \in (0,1),$$

where $z \in PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$. From (3.15) it is seen that

$$\begin{aligned} |z(t)| &\leq \lambda \int_0^{\sigma(T)} |\mu(t,s)||f_2(s, z(\sigma(s)))|\Delta s + \lambda \sum_{j=1}^{m} |\mu(t,t_j)||J_j(z(t_j))| \\ &\leq \lambda B(K\sigma(T) + mk_3), \quad t \in [0,\sigma(T)]_{\mathbb{T}}. \end{aligned}$$

Thus,

$$||z|| \leq \lambda B(K\sigma(T) + mk_3).$$

Applying Schaefer fixed point theorem, it is seen that the operator Ψ has a fixed point in $PC_{rd}([0,\sigma(T)]_{\mathbb{T}})$. Thus,the proof is completed. □

3.4 An Example

We present the following examples to show that our Theorem 3.3 is applicable to the dynamic impulsive PBVP (3.1).

Example 3.4.1 *Let* $\mathbb{T} = \bigcup_{i=0}^{3}[2i, 2i+1]$*, where the intervals* $[2i, 2i+1]$*,* $i \in \{0, \dots 3\}$ *are real valued. Choose,* $T = 7$*,* $m = 5$*,*

$$t_0 = 0, \quad t_1 = 2, \quad t_2 = 4, \quad t_3 = 6,$$

$$J = [0, 7]_{\mathbb{T}}, \quad J_0 = \bigcup_{i=0}^{3}(2i, 2i+1)$$

$$\tau_1 = \frac{1}{2}, \quad \tau_2 = \frac{5}{2}, \quad \tau_3 = 4, \quad \tau_4 = \frac{9}{2}, \quad \tau_5 = \frac{13}{2},$$

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 1.$$

Consider the impulsive dynamic PBVP

$$\begin{cases} z^{\Delta^5}(t) & = \dfrac{z(\sigma(t))}{1 + (z(\sigma(t)))^2}, \quad t \in J_0, \\ z^{\Delta^4}(0) & = z^{\Delta^3}(0) = z^{\Delta^2}(0) = 0, \\ z^{\Delta^2}(t_k^+) & = z^{\Delta^2}(t_k) + \dfrac{(z(t_k))^2}{1 + (z(t_k))^4}, \quad k \in \{0, \dots, 3\}, \\ z^{\Delta}(0) & = z^{\Delta}\left(\dfrac{1}{2}\right) + z^{\Delta}\left(\dfrac{5}{2}\right) + z^{\Delta}(4) + z^{\Delta}\left(\dfrac{9}{2}\right) + z^{\Delta}\left(\dfrac{13}{2}\right), \\ z(t_k^+) & = z(t_k) + \dfrac{1 + (z(t_k))^2}{1 + (z(t_k))^4}, \quad k \in \{0, \dots, 3\}, \\ z(0) & = z(8). \end{cases}$$

Here

$$f(t, z) = \frac{z}{1 + z^2} \leq 1 = k_1,$$
$$I_k(z) = \frac{z^2}{1 + z^4} \leq 1 = k_2,$$
$$J_k(z) = \frac{1 + z^2}{1 + z^4} \leq 2 = k_3, \quad k \in \{0, \dots, 3\}.$$

Noting that

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 5 \neq 1.$$

It is seen that all hypothesis of Theorem 3.3 hold. Hence, the above PBVP possesses at least one solution.

CHAPTER 4

CONCLUSION

Since time scale calculus enables us to treat together the analysis on continuous and discrete dynamic equations, the unification of these two cases by dynamic equations on time scales has gained importance in recent years. In this thesis, the existence of solutions, which is a classical problem in the theory of differential equations, is considered. By employing the well-known fixed point theorems due to Schauder and Schaefer, it is shown that some classes of nonlinear higher order multi-point (IBVPs) on time scales posseses at least one solutions. In particular, third order and $2n+1$, $n \geq 1$ order IBVPs are studied. Although there is a massive literature on IBVPs on time scales of even-order, there is not much studies on their odd-order counterparts. Thus, this work partially closes the related gap in the literature. The results in Chapter 2 have been accepted for publication in the journal Miskolc Mathematical Notes, and the results in Chapter 3 have been published in the Georgian Mathematical Journal. Since there is not much work on general higher-order IBPVPs on time scales, studying the existence of solutions for IBVPs with arbitrary order will be helpful for closing the related gap in the literature.

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